Exact Inference on Multiple exponential populations under A Joint Type-II Progressive Censoring Scheme

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Abstract

Recently Mondal and Kundu [13] introduced a Type-II progressive censoring scheme for two populations. In this article we extend the above scheme for more than two populations. The aim of this paper is to study the statistical inference under the multi sample Type-II progressive censoring scheme, when the underlying distributions are exponential. We derive the maximum likelihood estimators (MLEs) of the unknown parameters when they exist and find out their exact distributions. The stochastic monotonicity of the MLEs has been established and this property can be used to construct exact confidence intervals of the parameters via pivoting the cumulative distribution functions of the MLEs. The distributional properties of the ordered failure times are also obtained. The Bayesian analysis of the unknown model parameters has been provided. We assume a very flexible gamma-Dirichlet prior and it turns out to be a conjugate prior also, when the sample sizes are equal. The performances of the different methods have been examined by extensive Monte Carlo simulations. We analyze two data sets for illustrative purpose. Finally we conclude the paper with some open problems.

Key Words and Phrases: Type-II censoring scheme; progressive censoring scheme; joint progressive censoring scheme; maximum likelihood estimator; confidence interval; bootstrap confidence interval; simulation algorithm; conjugate prior.

AMS Subject Classifications: 62N01, 62N02, 62F10.

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1 Introduction

In any life testing experiment censoring is inevitable, mainly to optimize cost and time. In the literature an extensive amount of work has been done on different censoring schemes, specially on single sample. Recently, Balakrishnan and Rasouli [5] introduced a Type-II censoring scheme for two samples and provided the inference procedures of the unknown parameters when the lifetime distributions are exponential. Ashour and Abo-Kasem [1] considered Weibull populations under the joint Type-II censoring scheme proposed by Balakrishnan and Rasouli [5]. Balakrishnan and Su [6] extended the results of Balakrishnan and Rasouli [5] for multi sample cases.

Rasouli and Balakrishnan [20] introduced a joint Type-II progressive censoring scheme and provided the likelihood and Bayesian inferences on two exponential populations under the proposed scheme. From now onwards we call this scheme as a JPC-1 scheme. Parsi et al. [17] studied likelihood inference under the JPC-1 scheme when the underlying distributions are Weibull. Mondal and Kundu [14] provided order restricted inference of two Weibull populations under the JPC-1 scheme. Balakrishnan et al. [7] extended the results of Rasouli and Balakrishnan [20] for more than two populations.

Recently, Mondal and Kundu [13] introduced a new joint Type-II progressive censoring scheme. It is observed that the scheme proposed by Mondal and Kundu [13] can be implemented quite easily in practice and analytically it is more tractable than the scheme proposed by Rasouli and Balakrishnan [20]. From now onwards the joint Type-II progressive censoring scheme proposed by Mondal and Kundu [13] will be named as a JPC-2 scheme. The JPC-2 scheme is a generalization of the self reallocated design (SRD), originally proposed by Srivastava [21]. Mondal and Kundu [13] studied the inference on two exponential distributions under the JPC-2 scheme. In this paper we extend the results of Mondal and

Kundu [13] for more than two populations.

The results under the two sample JPC-2 scheme are generalized for multi sample cases when the underlying distributions are exponential. In this paper we derive maximum likelihood estimators (MLEs) of the unknown parameters whenever they exist and provide their exact distributions. Under the multi sample JPC-2 scheme, we obtain the distributional properties of the censored order statistics and these results can be used for interval estimation and for generating samples for the simulation experiment. Under the JPC-2 scheme stochastic monotonicity of the MLEs can be established, hence exact confidence intervals can be obtained via pivoting the cumulative distribution function (CDF) of the MLEs. Since, exact confidence intervals are difficult to compute numerically, we propose to use bootstrap confidence intervals of the parameters.

We further consider the Bayesian inference of the model parameters. We consider a very flexible gamma-Dirichlet prior as a joint prior of the parameters. It turns out to be a conjugate prior when the sample sizes are equal and in that case the Bayes estimators based on the squared error loss function can be derived explicitly. Otherwise, we rely on the importance sample technique to compute the Bayes estimates and the associated credible intervals. We perform extensive simulation experiment to compare different methods and the analyses of two data sets have been performed for illustrative purposes.

The main contribution of the present manuscript is two fold. First of all, we have extended the classical inference of the two sample case of Mondal and Kundu [13] to multi sample case. The main theoretical results of two sample case are based on Lemma 1 and Theorem 1 of Mondal and Kundu [13]. In the present manuscript those two results have been extended in Lemma 3.2.1 and Theorem 3.2.1, respectively. Lemma 3.2.2 of the present manuscript is purely a new result and that provides more insight about the behavior of the joint CDF of the MLEs. But the major contribution of this present manuscript is the Bayesian inference under a fairly general set of priors. Since, the construction of the exact confidence intervals are quite complicated in practice, the Bayesian inference seems to be a natural choice, in which finite sample inference can be obtained quite conveniently. The Bayesian inference was not developed for the two sample case, and the present development can be easily applied for the two sample case also.

Rest of the paper is organized as follows. In Section 2 we provide notations and briefly discuss the JPC-2 scheme. In Section 3 the MLEs are derived along with their exact distributions and also we provide distributional properties of the censored order statistics. We construct exact and bootstrap confidence interval in Section 4. In Section 5 we provide the Bayes estimates and the associated credible intervals. In Section 6 simulation experiments are performed along with real data analyses. Finally, we conclude the paper in Section 7.

2 Notation, Model Description and Model assumption

2.1 Notation

CDF : Cumulative distribution function.

$\overset{d}{=}$: equality in distribution.

HPD : Highest posterior density.

i.i.d : Independent and identically distributed.

MGF : Moment generated function.

MLE : Maximum likelihood estimator.

PDF: Probability density function.

$\forall h$: Means for $h = 1, \ldots, H$.

$D(a_1, a_2, \ldots, a_H)$: Multivariate Dirichlet distribution with PDF:

$$\frac{1}{B(a_1, a_2, \ldots, a_H)} \prod_{h=1}^{H} x_h^{a_h - 1}; \text{ where } \quad x_h > 0, \ \forall h, \ \sum_{h=1}^{H} x_h = 1,$$

$$B(a_1, a_2, \ldots, a_H) = \frac{\prod_{h=1}^{H} \Gamma(a_h)}{\Gamma(a)} \text{ and } \quad a_h > 0, \ \forall h, \ a = \sum_{h=1}^{H} a_h.$$

$Exp(\theta)$: Exponential distribution with PDF:

$$\frac{1}{\theta} e^{-\frac{x}{\theta}}; \quad x > 0, \ \lambda > 0.$$

$GA(\beta, \lambda)$: Gamma distribution with PDF:

$$\frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta - 1} e^{-\lambda x}; \quad x > 0, \ \lambda, \beta > 0.$$

$Mult(k, p_1, p_2, \ldots, p_H)$: Multinomial distribution with probability mass function

$$P(\mathbf{K} = \mathbf{r}) = P(K_1 = r_1, K_2 = r_2, \ldots, K_H = r_H) = \frac{k!}{r_1! r_2! \cdots r_H!} p_1^{r_1} \times p_2^{r_2} \times \cdots \times p_H^{r_H};$$

$\sum_{h=1}^{H} r_h = k$, $r_h \in \{1, 2, \ldots, k\}$, $\sum_{h=1}^{H} p_h = 1$, $p_h \in [0, 1]$, $\forall h$.

Through out the paper it is assumed that the 'small' letter is the sample version of the 'capital' letter, and it should be clear from the context.

2.2 Model Description and Model assumption

Suppose we have H different populations. We draw independent samples of size n_h from the population h and call it as the sample h (Sam-h), $\forall h$. Let $k \leq min\{n_1, \ldots, n_H\}$ be the number of failures to be observed in the experiment and $R_1, \ldots, R_{k-1}$ are non-negative integers satisfying $\sum_{i=1}^{k-1} (R_i + 1) < min\{n_1, \ldots, n_H\}$.

Under the JPC-2 scheme all the samples are put on a test simultaneously. Suppose the first failure occurs at W_1 from Sam-h_1, then we remove R_1 units randomly from the remaining $(n_{h_1} - 1)$ units of Sam-h_1 and we remove randomly $R_1 + 1$ units each from the remaining $(H-1)$ samples at W_1. Next, suppose the second failure occurs from Sam-h_2 at W_2. We remove R_2 units from the remaining $(n_{h_2} - R_1 - 2)$ surviving units of Sam-h_2 and $R_2 + 1$ units each from the remaining surviving units from the rest of the $(H-1)$ samples. We continue the experiment until the k-th failure occurs. At the k-th failure time point W_k, we stop the experiment, see Mondal and Kundu [13] for details.

Under the JPC-2 scheme, along with $W_1, \ldots W_k$, we introduce another set of random variables $Z_{ih}, i = 1, \ldots, k; \forall h$, where $Z_{ih} = 1$, if i-th failure occurs from Sam-h and 0, otherwise. Let K_h denote total number of failures coming from Sam-h. Under the JPC-2 scheme, the data set consists of $(\mathbf{W}, \mathbf{Z})$, where

W	**Z**
W_1	$(Z_{11}, \ldots, Z_{1H})$
$\vdots$	
W_k	$(Z_{k1}, \ldots, Z_{kH})$.

Here $\sum_{h=1}^{H} Z_{ih} = 1$, $\sum_{i=1}^{k} Z_{ih} = K_h$ and $\sum_{h=1}^{H}\sum_{i=1}^{k} Z_{ih} = \sum_{h=1}^{H} K_h = k$. In Figure 1 we provide a schematic diagram of the JPC-2.

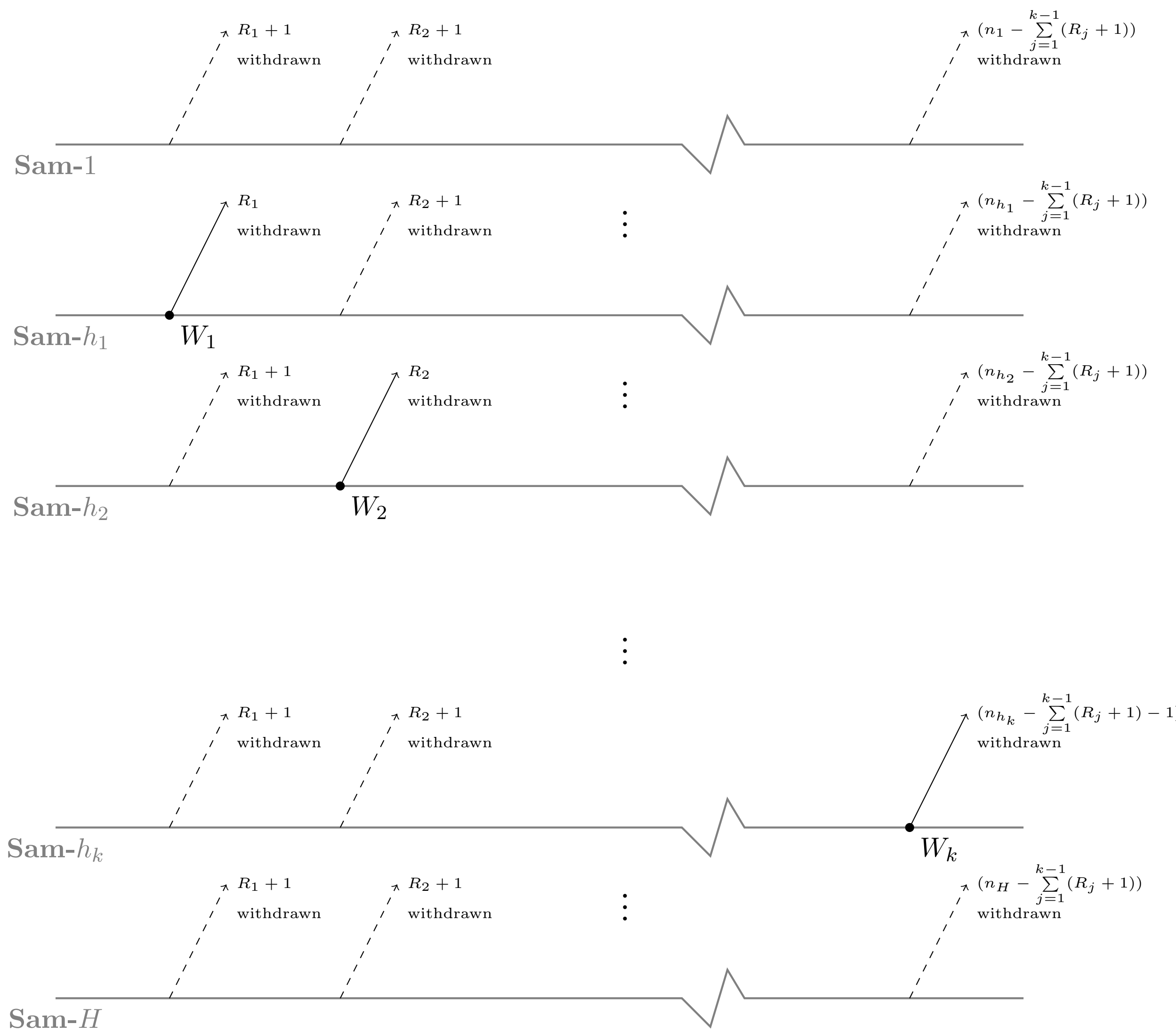


Figure 1: A schematic diagram of JPC-2 scheme

3 Some Distributional Properties

In this section, we derive the MLEs of the unknown parameters whenever they exist. We obtain the exact distributions of the MLEs from their joint and marginal MGFs. The distributional properties of the ordered failure times are also studied.

3.1 Maximum Likelihood Estimators

We assume that all the H populations are exponentially distributed. The sample from the h-th population $X_{1,h}, X_{2,h}, \ldots, X_{n_h,h}$ are i.i.d random variables from $Exp(\theta_h)$, $\forall h$.

For a given sampling scheme $n_1, n_2, \ldots, n_H, k, R_1, R_2, \ldots, R_{k-1}$ and for the sample $(\mathbf{w}, \mathbf{z})$ the likelihood function can be written as

$$L(\theta_1, \ldots, \theta_H | \mathbf{w}, \mathbf{z}) = C\Big(\prod_{h=1}^{H} \frac{1}{\theta_h^{k_h}}\Big) e^{-\sum\limits_{h=1}^{H} \frac{A_h(w)}{\theta_h}}, \tag{1}$$

where

$$\begin{aligned} C &= \prod_{i=1}^{k}\Big(\sum_{h=1}^{H}(n_h - \sum_{j=1}^{i-1}(R_j+1))z_{ih}\Big), \\ A_h(w) &= \sum_{i=1}^{k-1}(R_i+1)w_i + (n_h - \sum_{i=1}^{k-1}(R_i+1))w_k, \ \forall h. \end{aligned}$$

From (1), the MLE of θ_h is obtained as

$$\widehat{\theta}_h = \frac{A_h(w)}{k_h}, \quad \forall h.$$

It is clear that MLEs exist only when $K_h > 0, \ \forall h$. Hence $(\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_H)$ are conditional MLEs conditioning on the event $\mathcal{M} = \{\mathbf{K} = (K_1, \ldots, K_H) : \prod\limits_{h=1}^{H} K_h \neq 0\}$. We have used the following usual convention that $\sum\limits_{i=1}^{0} a_i = 0$, for any arbitrary a_i.

3.2 Exact Distribution of MLEs

We derive the exact distributions of MLEs from their joint and marginal MGFs. The following lemma is needed for further development.

Lemma 3.2.1 *Let* $\mathbf{K} = (K_1, \ldots, K_H)$, $\mathbf{r} = (r_1, \ldots, r_H)$ *with* $r_h \in \{0, 1, \ldots, k\}$ *and* $\sum\limits_{h=1}^{H} r_h = k$. *Here* K_h *denotes the total number of failures from Sam-h,* $\forall h$, *as mentioned before.*

Then,

$$P(\mathbf{K}=\mathbf{r}) = \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \frac{\prod_{i=1}^{k}\Big(\sum_{h=1}^{H}(n_h - \sum_{j=1}^{i-1}(R_j+1))z_{ih}\Big)}{\Big(\prod_{h=1}^{H}\theta_h^{r_h}\Big)\Big(\prod_{i=1}^{k}\sum_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}, \tag{2}$$

where

$$\begin{aligned} Q_{(\mathbf{r})} &= \Big\{\mathbf{u}_{k\times H} : \mathbf{u} = \begin{pmatrix} u_{11},\ldots,u_{1H} \\ \vdots \\ u_{k1},\ldots,u_{kH}\end{pmatrix}; u_{ih}\in\{0,1\}; \\ &\quad \sum_{i=1}^{k} u_{ih} = r_h,\ \forall h; \sum_{h=1}^{H} u_{ih} = 1,\ \textit{for } i=1,\ldots,k\Big\}. \end{aligned}$$

Proof: See in the Appendix A.

Special Case: From (2) it is clear that when $n_1 = n_2 = \cdots = n_H$,

$$P(\mathbf{K}=\mathbf{r}) = \frac{k!}{r_1!r_2!\cdots r_H!} \quad p_1^{r_1}\times p_2^{r_2}\times\cdots\times p_H^{r_H},$$

where

$$p_h = \frac{\frac{1}{\theta_h}}{\sum_{l=1}^{H}\frac{1}{\theta_l}}.$$

Based on the conditioning on the event $\mathcal{M}$, the joint MGF is given below.

Theorem 3.2.1 *Let $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1,\ldots,\widehat{\theta}_H)$ and $\mathbf{t} = (t_1,\ldots,t_H)$. Then the joint MGF of $\widehat{\boldsymbol{\theta}}$ conditioning on the event $\mathcal{M}$, is obtained as*

$$M_{\widehat{\boldsymbol{\theta}}}(\mathbf{t}) = \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \prod_{i=1}^{k}\Big(1 - \frac{\alpha_{i,1}}{r_1}t_1 - \frac{\alpha_{i,2}}{r_2}t_2 - \cdots - \frac{\alpha_{i,H}}{r_H}t_H\Big)^{-1} \tag{3}$$

where

$$P_{\mathbf{r}}^* = \frac{P(\mathbf{K}=\mathbf{r})}{P(\mathcal{M})}, \alpha_{i,h} = \frac{(n_h - \sum_{j=1}^{i-1}(R_j+1))}{\sum_{l=1}^{H}\frac{(n_l-\sum_{j=1}^{i-1}(R_j+1))}{\theta_l}},$$

$$S^* = \{\mathbf{r} = (r_1,r_2,\ldots,r_H) : \prod_{h=1}^{H} r_h \neq 0, \sum_{h=1}^{H} r_h = k\}.$$

PROOF: See in the Appendix A.

From the joint MGF, the joint CDF of $\widehat{\boldsymbol{\theta}}$ can be derived and we state the result in the following lemma.

Lemma 3.2.2 *The joint CDF of $\widehat{\boldsymbol{\theta}}$ can be obtained as*

$$P(\widehat{\boldsymbol{\theta}} \leq x) = \sum_{\mathbf{r} \in S^*} P^*_{\mathbf{r}} \times P(\mathbf{G}(\mathbf{r}) \leq x),$$

where $\mathbf{G}(\mathbf{r})$ is explicitly defined in Appendix A.

PROOF: See in Appendix A.

Note that $\mathbf{G}(\mathbf{r})$ has a singular distribution function. Therefore, the joint PDF of $\widehat{\boldsymbol{\theta}}$ will not exist. We derive the marginal MGFs and hence marginal PDFs.

Corollary 3.2.1 *The marginal MGF of $\widehat{\theta}_h$, conditioning on the event $\mathcal{M}$ is given as*

$$M_{\widehat{\theta}_h}(t) \quad = \quad \sum_{\mathbf{r} \in S^*} P^*_{\mathbf{r}} \prod_{i=1}^{k} \left(1 - \frac{\alpha_{i,h}}{r_h} t\right)^{-1}. \tag{4}$$

SPECIAL CASE: When $n_1 = n_2 = \ldots = n_H$, $\alpha_{i,h} = (\sum\limits_{l=1}^{H} \frac{1}{\theta_l})^{-1} = \alpha$ (say), $\forall i = 1, \ldots, k$ and $\forall h$. Therefore, the term $\prod\limits_{i=1}^{k} \left(1 - \frac{\alpha_{i,h}}{r_h} t\right)^{-1}$ is turned out to be $\left(1 - \frac{\alpha}{r_h} t\right)^{-k}$.

Theorem 3.2.2 *The marginal PDF of $\widehat{\theta}_h$, conditioning on the event $\mathcal{M}$, is given as*

$$f_{\widehat{\theta}_h}(t) = \sum_{\mathbf{r} \in S^*} P^*_{\mathbf{r}} \times g_{Y_{h,r_h}}(t). \tag{5}$$

If $n_1 = \cdots = n_H$,

$$g_{Y_{h,r_h}}(t) = \frac{r_h}{\alpha^k \Gamma(k)} t^{k-1} e^{-\frac{r_h}{\alpha} t}; t > 0,$$

and otherwise

$$g_{Y_{h,r_h}}(t) = (\prod_{i=1}^{k} \frac{r_h}{\alpha_{i,h}}) \times \sum_{i=1}^{k} \frac{e^{-\frac{r_h}{\alpha_{i,h}}t}}{\prod_{j=1,j\neq i}^{k}\left(\frac{r_h}{\alpha_{j,h}} - \frac{r_h}{\alpha_{i,h}}\right)}; t > 0.$$

Remark: $g_{Y_{h,r_h}}(\cdot)$ is the PDF of Y_{h,r_h}. When $n_1 = n_2 = \ldots = n_H$, Y_{h,r_h} follows $GA(k, \frac{\alpha}{r_h})$. Therefore, for equal sample sizes, the distributions of the MLEs are the mixture of the gamma distributions. Otherwise, all $\alpha_{i,h}$ are distinct and $Y_{h,r_h} = \sum_{i=1}^{k} U_{h,r_h}^{(i)}$ where $U_{h,r_h}^{(i)} \sim Exp(\frac{\alpha_{i,h}}{r_h})$ and they are independently distributed.

Then, $P(\widehat{\theta}_h > t|\mathcal{M})$ is obtained as

$$P(\widehat{\theta}_h > t|\mathcal{M}) = \begin{cases} \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times \int_t^\infty \frac{r_h}{\alpha^k\Gamma(k)} x^{k-1} e^{-\frac{r_h}{\alpha}x} dx; & t > 0 \text{ if } n_1 = \ldots = n_H \\ \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times (\prod_{i=1}^{k} \frac{r_h}{\alpha_{i,h}}) \times \sum_{i=1}^{k} \frac{(\frac{\alpha_{i,h}}{r_h}) e^{-\frac{r_h t}{\alpha_{i,h}}}}{\prod_{j=1,j\neq i}^{k}(\frac{r_h}{\alpha_{j,h}} - \frac{r_h}{\alpha_{i,h}})}; & t > 0 \text{ otherwise.} \end{cases}$$

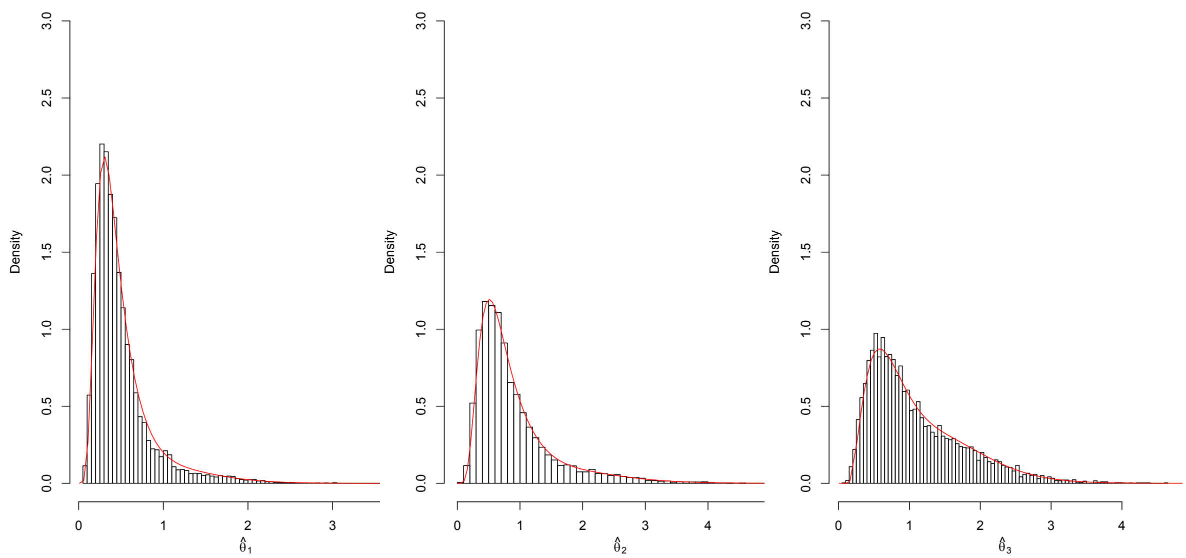
Corollary 3.2.2 *When* $n_1 = n_2 = \ldots = n_H$*,*

$$\begin{aligned} E(\widehat{\theta}_h) &= \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times \frac{k\alpha}{r_h}, \\ E(\widehat{\theta}_h^2) &= \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times \frac{k(k+1)\alpha^2}{r_h^2}. \\ E(\widehat{\theta}_{h_1}, \widehat{\theta}_{h_2}) &= \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times \frac{k(k+1)\alpha^2}{r_{h_1} r_{h_2}}. \end{aligned}$$

Otherwise,

$$\begin{aligned} E(\widehat{\theta}_h) &= \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times \Big(\sum_{i=1}^{k} \frac{\alpha_{i,h}}{r_h}\Big), \\ E(\widehat{\theta}_h^2) &= \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times \Big(2\sum_{i=1}^{k} (\frac{\alpha_{i,h}}{r_h})^2 + \sum_{i=1,i\neq j}^{k} \frac{\alpha_{i,h}\alpha_{j,h}}{{r_h}^2}\Big). \\ E(\widehat{\theta}_{h_1}, \widehat{\theta}_{h_2}) &= \sum_{\mathbf{r}\in S^*} P_{\mathbf{r}}^* \times \Big(2\sum_{i=1}^{k} (\frac{\alpha_{i,h_1}\alpha_{i,h_2}}{r_{h_1} r_{h_2}}) + \sum_{i=1,i\neq j}^{k} \frac{\alpha_{i,h_1}\alpha_{j,h_2}}{r_{h_1} r_{h_2}}\Big). \end{aligned}$$

The PDFs of the MLEs are not quite simple. To have some idea about the shape of the PDFs, in Figure 2 we plot the histograms of the MLEs along their exact PDFs. The histograms computed based on 10,000 generations. It is observed that the PDFs are unimodal and the mode of the PDFs are close to the true values of the respective parameters. We use the notation $R = (0_{(7)})$ implies $R_1 = \cdots, R_7 = 0$.



(a) Histogram of $\hat{\theta}_1$ along its PDF

(b) Histogram of $\hat{\theta}_2$ along its PDF

(c) Histogram of $\hat{\theta}_3$ along its PDF

Figure 2: Histogram of the MLEs along with PDFs with $\theta_1 = 0.4, \theta_2 = 0.7, \theta_3 = 1, n_1 = 11, n_2 = 15, n_3 = 13, k = 8, R = (0_{(7)})$

3.3 Distributional properties of the censored order statistics

In a progressive censoring scheme, the ordered failure times are termed as the progressively censored order statistics, see Balakrishnan [2], Balakrishnan and Cramer [3]. The distributional properties of the progressively censored order statistics can be applied to develop different inferential procedures. Similarly, in the JPC-2 scheme, we term the ordered failure times as the censored order statistics. Under the JPC-2 scheme for two exponential samples we obtained the distributional properties of the censored order statistics, see in Mondal and Kundu [13]. Here we extend those results for the general case.

Theorem 3.3.1 *The censored order statistics $W_1 \leq W_2 \leq \ldots \leq W_k$ have the following distributional properties.*

$$W_i \overset{d}{=} \sum_{j=1}^{i} V_j$$

where $V_j \sim Exp(\frac{1}{E_j})$ and $E_j = \sum_{h=1}^{H} \frac{(n_h - \sum_{s=1}^{j-1}(R_s+1))}{\theta_h}$.

Proof: See in Appendix A.

This result can used for constructing confidence intervals and also to generate sample for a JPC-2 scheme.

4 Construction of Confidence Intervals

4.1 Exact Confidence Intervals

The exact confidence interval of a real valued parameter θ can be obtained using the CDF of a statistic T only if $P_\theta(T > t)$ is an increasing function of θ for any fixed t see for example Casella and Berger [9], Lehmann and Romano [12]. Once this monotonicity assumption is satisfied, a $100(1-\gamma)\%$ confidence interval of θ can be obtained by solving $P_\theta(T > t) = \gamma_1$ and $P_\theta(T > t) = 1-\gamma_2$ where $\gamma_1 + \gamma_2 = \gamma$. If the solutions exist, the monotonicity assumption guarantees the uniqueness of the solutions of the above two equations.

In the JPC-2 scheme for exponential populations, the CDF of $\widehat{\theta}_h$ can be obtained in explicit form $\forall h$. As discussed above, the exact confidence interval can be obtained via pivoting the CDF of $\widehat{\theta}_h$ if the stochastic monotonicity property is satisfied. The following lemma provides the necessary assumption of the monotonicity of $P_{\theta_h}(\widehat{\theta}_h > t)$, $\forall h$.

Lemma 4.1.1 *$P_{\theta_h}(\widehat{\theta}_h > t|\mathcal{M})$ is an increasing function of θ_h for fixed t and θ_j where $j = 1, 2, \ldots, H$ and $j \neq h$.*

Proof: See in the Appendix A.

Using the assumption that $P_{\theta_h}(\hat{\theta}_h > t)$ is a strictly increasing function of θ_h, keeping θ_j constant for all $j \neq h$, an equal tailed $100(1-\gamma)\%$ exact confidence interval of θ_h can be constructed as $(\theta_{hL}, \theta_{hU})$ solving the following two nonlinear equations $\forall h$;

$$P_{\theta_{hL}}(\widehat{\theta}_h > \widehat{\theta}_{h_{obs}}|\mathcal{M}) = \frac{\gamma}{2}, \tag{6}$$

$$P_{\theta_{hU}}(\widehat{\theta}_h > \hat{\theta}_{hobs}|\mathcal{M}) = 1 - \frac{\gamma}{2}. \tag{7}$$

These two non linear equations can be solved by various numerical methods like, bisection method or Newton-Raphson method. While solving (6) and (7), we replace θ_j's by $\widehat{\theta}_j$ for all $j \neq h$.

4.2 Bootstrap Confidence Intervals

It has been observed in the previous section that numerically the exact confidence intervals are not easy to compute. As an alternative we can use bootstrap confidence intervals in this case. The following steps are used to construct a $100(1-\gamma)\%$ percentile bootstrap confidence interval of θ_h, $\forall h$. Here $[x]$ denotes the largest integer less than or equal to x.

Step 1: Given the original data, compute $\widehat{\theta}_h$, $\forall h$.

Step 2: Generate a bootstrap sample $(\mathbf{W}^*, \mathbf{Z}^*)$ based on $n_1, \ldots, n_H$, k, $(R_1, \ldots R_{k-1})$, $\widehat{\theta}_1, \ldots, \widehat{\theta}_H$.

Step 3: Compute $\widehat{\theta}^*_h$ based on the bootstrap sample $\forall h$.

Step 4: Repeat Step 1-Step 3 say B times and obtain $\{\widehat{\theta}^*_{h1}, \ldots, \widehat{\theta}^*_{hB}\}$, $\forall h$. Sort $\widehat{\theta}^*_{hj}$ in ascending order to obtain $(\widehat{\theta}^*_{h(1)}, \ldots, \widehat{\theta}^*_{h(B)})$, $\forall h$.

Step 5: A $100(1-\gamma)\%$ bootstrap-p confidence interval of θ_h, $\forall h$, can be obtained as $\big(\widehat{\theta}^*_{h([\frac{\gamma}{2}B])}, \widehat{\theta}^*_{h([(1-\frac{\gamma}{2})B])}\big)$.

5 Bayesian Analysis

In the previous section we have seen that exact confidence interval of the unknown parameters cannot be obtained explicitly. Moreover, for any function of the parameters, it may not be possible to construct the exact confidence interval. Hence, Bayesian analysis seems to be a natural choice. Here we make the following re-parameterization $\lambda_h = \frac{1}{\theta_h}$, $\forall h$ and we denote $\boldsymbol{\lambda} = (\lambda_1, \lambda_2 \ldots, \lambda_H)$.

5.1 Prior Assumption

Based on the idea of Pena and Gupta [18] it is assumed that

$$\Big(\frac{\lambda_1}{\sum\limits_{h=1}^{H}\lambda_h}, \frac{\lambda_2}{\sum\limits_{h=1}^{H}\lambda_h}, \ldots \frac{\lambda_H}{\sum\limits_{h=1}^{H}\lambda_h}\Big) \sim D(a_1, a_2, \ldots, a_H), \quad \sum_{h=1}^{H}\lambda_h \sim GA(a_0, b_0)$$

and they are independently distributed.

The joint PDF of $\boldsymbol{\lambda}$ is obtained as

$$\pi(\boldsymbol{\lambda}|a_0, b_0, a_1, a_2, \ldots, a_H) \;\propto\; \Big(\prod_{h=1}^{H}\lambda_h^{a_h-1}\Big)\Big(\sum_{h=1}^{H}\lambda_h\Big)^{(a_0-\sum\limits_{h=1}^{H}a_h)} e^{-b_0\sum\limits_{h=1}^{H}\lambda_h}, \tag{8}$$

where the normalizing constant is $\frac{b_0^{a_0}}{\Gamma(a_0)B(a_1,a_2,\ldots,a_H)}$.

We call this prior as the gamma-Dirichlet ($GD(a_0, b_0, a_1, a_2, \ldots, a_H)$) prior. When all the sample sizes are equal, it is reduced to a conjugate prior. The hyper parameters namely a_0, b_0, a_1, a_2,..., a_H play important roles on the shape of the prior densities and on the mutual dependence of $\lambda_h's$. When $\sum\limits_{h=1}^{H}a_h = a_0$, λ_hs are mutually independent. Moreover, any two

λ_{h_1} and λ_{h_2} for $h_1 \neq h_2$ and $h_1, h_2 \in \{1, 2, \ldots, H\}$ are positively correlated, if $\sum\limits_{h=1}^{H} a_h > a_0$ and they are negatively correlated, if $\sum\limits_{h=1}^{H} a_h < a_0$. The following result will be used later.

RESULT 1: Under the prior $GD(a_0, b_0, a_1, a_2, \ldots, a_H)$, $\forall h$ and for all h_1, h_2 such that $h_1 \neq h_2$,

$$
\begin{aligned}
E(\lambda_h) &= \frac{a_0}{b_0} \times \frac{a_h}{(\sum\limits_{l=1}^{H} a_l)}, \\
E(\lambda_h^2) &= \frac{a_0(a_0+1)}{b_0^2} \times \frac{a_h(a_h+1)}{(\sum\limits_{l=1}^{H} a_l)(\sum\limits_{l=1}^{H} a_l + 1)}, \\
E(\lambda_{h_1}\lambda_{h_2}) &= \frac{a_0(a_0+1)}{b_0^2} \times \frac{a_{h_1}a_{h_2}}{(\sum\limits_{h=1}^{H} a_h)(\sum\limits_{h=1}^{H} a_h + 1)}. \\
cov(\lambda_{h_1}\lambda_{h_2}) &= \frac{a_0}{b_0^2} \times \frac{a_{h_1}a_{h_2}(\sum\limits_{h=1}^{H} a_h - a_0)}{(\sum\limits_{h=1}^{H} a_h)^2(\sum\limits_{h=1}^{H} a_h + 1)}.
\end{aligned}
$$

PROOF: See in Appendix B.

5.2 Posterior Analysis

Based on the prior assumption above, we derive Bayes estimator and credible interval (CRI) of parameters λ_h, $\forall h$. Based on the likelihood equation in section 3 and the given prior the joint posterior density function of $\boldsymbol{\lambda}$ can be written as

$$
\pi(\boldsymbol{\lambda}|data) \propto (\prod_{h=1}^{H} \lambda_h^{a_h+k_h-1})(\sum_{h=1}^{H} \lambda_h)^{(a_0 - \sum\limits_{h=1}^{H} a_h)} e^{-\sum\limits_{h=1}^{H}(b_0+A_h(w))\lambda_h} \tag{9}
$$

5.2.1 Special case

When $n_1 = n_2 = \cdots = n_H$, all $A_h(w)$ are equal and we denote $A_1(w) = \cdots, A_H(w) = B(w)$. The joint posterior density of $\boldsymbol{\lambda}$ is obtained as

$$\pi(\boldsymbol{\lambda}|data) \sim GD(a_0 + k, b_0 + B(w), a_1 + k_1, a_2 + k_2, \cdots, a_H + k_H).$$

Hence, in this case it turns out to be a conjugate prior. Therefore, the Bayes estimate of λ_h based on the squared error loss function can be obtained as

$$\widehat{\lambda}_{hB} = E(\lambda_h|data) = \frac{(a_0 + k)}{(b_0 + B(w))} \times \frac{(a_h + k_h)}{(\sum_{l=1}^{H} a_l + k)}.$$

The posterior variance and covariance are

$$V(\lambda_h|data) = \frac{(a_0 + k)(a_h + k_h)}{(b_0 + B(w))^2(\sum_{l=1}^{H} a_l + k)} \times \Big[\frac{(a_0 + k + 1)(a_h + k_h + 1)}{(\sum_{l=1}^{H} a_l + k + 1)} - \frac{(a_0 + k)(a_h + k_h)}{(\sum_{l=1}^{H} a_l + k)}\Big]$$

and

$$cov(\lambda_{h_1}, \lambda_{h_2}|data) = \frac{(a_0 + k)}{(b_0 + B(w))^2} \times \frac{(a_{h_1} + k_{h_1})(a_{h_2} + k_{h_2})(\sum_{h=1}^{H} a_h - a_0)}{(\sum_{h=1}^{H} a_h + k)^2(\sum_{h=1}^{H} a_h + k + 1)},$$

respectively. It is possible to provide a joint credible set of $\boldsymbol{\lambda}$ based on the following lemma.

Lemma 5.2.1 $\boldsymbol{\lambda} \sim GD(a_0 + k, b_0 + B(w), a_1 + k_1, a_2 + k_2, \ldots, a_H + k_H)$, *if and only if*

$$\Big(\frac{\lambda_1}{\sum_{h=1}^{H} \lambda_h}, \frac{\lambda_2}{\sum_{h=1}^{H} \lambda_h}, \ldots \frac{\lambda_H}{\sum_{h=1}^{H} \lambda_h}\Big) \sim D(a_1 + k_1, a_2 + k_2, \ldots, a_H + k_H),$$

$$\sum_{h=1}^{H} \lambda_h \sim GA(a_0 + k, b_0 + B(w)) \quad \textit{and they are independently distributed.}$$

Proof: See in Appendix B.

Suppose we want to construct a $100(1-\gamma)\%$ joint credible set of $\boldsymbol{\lambda}$. Let $\gamma_1, \gamma_2, \ldots, \gamma_H$ are such that $\prod_{h=1}^{H}(1-\gamma_h) = (1-\gamma)$. Based on Lemma 4, a $100(1-\gamma)\%$ joint credible set of $\boldsymbol{\lambda}$ can be constructed as,

$$\begin{aligned} C(\gamma) &= \{\boldsymbol{\lambda} : \lambda_h > 0,\ \forall h; D_1 \leq \sum_{h=1}^{H} \lambda_h \leq D_2; \\ &\qquad L_1(h) \leq \frac{\lambda_h}{\sum_{l=1}^{H} \lambda_l} \leq L_2(h); h = 1, 2, \ldots, H-1\}, \end{aligned}$$

where $D_1, D_2, L_1(h), L_2(h)$ are such that

$$\begin{aligned} P(L_1(h) \leq \frac{\lambda_h}{\sum_{l=1}^{H} \lambda_l} \leq L_2(h)) &= 1-\gamma_h; \quad h = 1, 2, \ldots, H-1, \\ P(D_1 \leq \sum_{h=1}^{H} \lambda_h \leq D_2) &= 1-\gamma_H. \end{aligned}$$

For any function of $\boldsymbol{\lambda}$ say $g(\boldsymbol{\lambda})$, the posterior density function may not be in explicit form. Therefore, the Bayes estimator of $g(\boldsymbol{\lambda})$ cannot be derived in closed form in that case. Also it is not possible to derive HPD credible intervals of any $g(\boldsymbol{\lambda})$ by the above method. In this case we can follow the procedure suggested in Kundu et al. [11] to compute the Bayes estimate and the associated credible intervals.

5.2.2 General case

If the samples sizes are not equal, the joint posterior density function cannot be obtained in a standard form. We propose to use importance sampling technique to compute Bayes estimates and the associated credible intervals of the unknown parameters.

Let $min\{A_1(w), A_2(w), \ldots, A_h(w)\} = A(w)$. Observe that, (9) can be written as

$$\pi(\boldsymbol{\lambda}|data) \propto \pi_1^*(\boldsymbol{\lambda}|data) \times u(\boldsymbol{\lambda}) \tag{10}$$

where

$$\pi_1^*(\boldsymbol{\lambda}|data) \sim GD(a_0 + k, b_0 + A(w), a_1 + k_1, a_2 + k_2, \ldots, a_H + k_H),$$
$$u(\boldsymbol{\lambda}) = e^{-\sum_{h=1}^{H}(A_h(w)-A(w))\lambda_h}.$$

If the Bayes estimator of any function of $g(\boldsymbol{\lambda})$ exists, the following algorithm can be used to compute the Bayes estimator of $g(\boldsymbol{\lambda})$ and to construct the associated HPD credible intervals of $g(\boldsymbol{\lambda})$.

Algorithm :

Step 1: First generate $\boldsymbol{\lambda}$ from $\pi_1^*(\boldsymbol{\lambda}|data)$ following the method in Kundu and Pradhan [10].

Step 2: Repeat the process say N times to generate $\{(\lambda_{1j}, \ldots, \lambda_{Hj}); j = 1, \ldots, N\}$.

Step 3: Compute $(g_1, \ldots, g_N)$ and $(u_1, \ldots, u_N)$ where $g_j = g(\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{Hj})$ and $u_j = u(\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{Hj})$.

Step 4: Bayes estimator of $g(\boldsymbol{\lambda})$ can be approximated as

$$\frac{\sum_{j=1}^{N} u_j g_j}{\sum_{j=1}^{N} u_j} = \sum_{j=1}^{N} \eta_j g_j,$$

where $\eta_j = \frac{u_j}{\sum_{l=1}^{N} u_l}$.

Step 5: To compute a $100(1-\gamma)\%$ credible interval of $g(\boldsymbol{\lambda})$, arrange g_j in ascending order to obtain $(g_{(1)}, \ldots g_{(N)})$ and record the corresponding η_j as $(\eta_{(1)}, \ldots, \eta_{(N)})$. A $100(1-\gamma)\%$ credible interval can be obtained as $(g_{(j_1)}, g_{(j_2)})$ where j_1, j_2 are such that

$$j_1 < j_2, \quad j_1, j_2 \in \{1, \ldots, N\} \quad \text{and} \quad \sum_{i=j_1}^{j_2} \eta_i \leq 1 - \gamma \leq \sum_{i=j_1}^{j_2+1} \eta_i. \tag{11}$$

The $100(1-\gamma)\%$ highest posterior density (HPD) credible interval can be obtained as

$(g_{(j_1^*)}, g_{(j_2^*)})$, such that $g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)}$ and j_1^*, j_2^* satisfying (11) for all j_1, j_2 satisfying (11).

6 Simulation study and Data Analysis

6.1 Simulation study

In this section we perform some simulation experiments to see how the proposed methods work for different sample sizes and for different parameter values. We consider five exponential populations with mean $\theta_1 = 0.3$, $\theta_2 = 0.4$ $\theta_3 = 0.5$, $\theta_4 = 0.6$ and $\theta_5 = .7$. Different sample sizes $(n_1, n_2, n_3, n_4, n_5)$, different effective sample size k and different choices of $R_1, \ldots, R_{k-1}$ are considered. We use the following notation to denote a particular JPC-2 censoring scheme, for fixed $k = 10$ and $R = (5, 0_{(18)})$ indicates $R_1 = 5$ and $R_2 = \cdots = R_{19} = 0$.

In each case, we compute the MLEs of the parameters. In Table 5 we record the average estimates (AEs) and the corresponding mean squared errors (MSEs) of the MLEs based on $10,000$ replications. We also construct 90% exact and bootstrap confidence intervals. In Table 6 we report the average length (AL) and the corresponding coverage percentage (CP) of those intervals based on 1000 replications. For bootstrap interval estimation, for each replication we have used 1000 re-samplings.

In Bayesian analysis we compute the Bayes estimates based on the squared error loss function and the associated HPD and symmetric credible intervals both for informative and non-informative priors. For informative prior we set $a_0 = 1, b_0 = 0.091, a_1 = 5, a_2 = 3.7, a_3 = 3, a_4 = 2.5, a_5 = 2.142$. These values are chosen by equating the prior expectations with the true values of the parameters so that prior variance of θ_hs exist for $h = 1, 2, 3, 4, 5$. In case of non-informative priors, we have assumed that $a_0 = b_0 = 0$ and $a_1 = a_2 = a_3 = a_4 = a_5 = 2.005$. Note that we have chosen $a_h > 2$ for $h = 1, 2, 3, 4, 5$, as it ensures the existence of the posterior variance of θ_h for $h = 1, 2, 3, 4, 5$.

In Table 5 we compute the average Bayes estimates (BE) and corresponding MSEs based on 1000 samples both for informative and non-informative priors. We record the AL and the CP of the 90% HPD and symmetric credible intervals both for informative and non-informative priors in Table 7. These figures are computed based on 1000 replications.

It is observed that both MLEs and Bayes estimates are biased estimates, and the biases are positive. Though for the small effective sample size k, the MLEs perform better in terms of the average bias and the MSE than the non-informative prior based Bayes estimators, for moderate to large values of k, they perform more or less similar. Another important point is that, though for $\theta_1, \theta_2, \theta_3, \theta_4$, the informative prior based Bayes estimators perform better than the non-informative prior based Bayes estimators and the MLEs in terms of the average bias and the MSEs, but for θ_5 which is the largest among five parameters, the performance of informative prior based Bayes estimators is sometimes worse than the other two estimators.

In interval estimation the bootstrap intervals are performing better than the exact confidence intervals in terms of the average lengths. For exact intervals the CPs always exceed the nominal levels. But to find out the exact confidence intervals, we need to solve two non-linear equations (6) and (7). Depending upon the choice of sample sizes, effective sample size, the censoring scheme and the maximum likelihood estimates, the solution of these two equations may not exist. Again for moderate to large values of sample sizes, when all the sample sizes are not equal, solving these two equations becomes computationally challenging. But the bootstrap confidence intervals have no such issues and can be derived conveniently.

Both for informative and non-informative priors, HPD credible intervals provide shorter lengths than the corresponding symmetric CRIs. All the cases CPs are very close to the nominal level. From these extensive simulation experiments, the effect of the hyper parameters are also quite clear. It is observed that in case of the informative priors the biases, MSEs and the length of the credible intervals are also smaller compared to the non-informative priors.

6.2 Data Analysis

In this section we present the analyses of two real data sets for illustrative purposes.

Example 1. Nelson [15] (Chapter-1, Table 1.1) provided the data, containing the times to breakdown of an insulating fluid between electrodes recorded at seven different voltages. For illustrative purposes we choose the breakdown times at voltage 32 KV, 34 KV, 36 KV and 38 KV. The data are presented below for easy reference.

Data set 1 (Breakdown at 32 KV): 0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, 215.10.

Data set 2 (Breakdown at 34 KV): 0.19, 0.78,0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52,33.91, 36.71, 72.89.

Data set 3 (Breakdown at 36 KV): 0.35, 0.59 , 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71,2.90, 3.67, 3.99, 5.35, 13.77, 25.50.

Data set 4 (Breakdown at 38 KV): 0.09, 0.39, 0.47, 0.73, 0.74, 1.13, 1.40, 2.38.

Here $H = 4$ and $n_1 = 15, n_2 = 19, n_3 = 15, n_4 = 8$. It is observed that the exponential distribution fits the data sets quite well.

We apply a JPC-2 scheme with $k = 8$ and $R_1 = \ldots, R_{k-1} = 0$ on the given data and the observed sample is presented below.

Table 1: JPC-2 sample (example 1)

W	Z
0.09	(0, 0,0,1)
0.19	(0, 1,0,0)
0.27	(1, 0,0,0)
0.35	(0, 0,1,0)
0.40	(1, 0,0,0)
0.47	(0, 0,0,1)
0.69	(1, 0,0,0)
0.74	(0, 0,0,1)

Based on the censored sample we compute MLEs and 90% exact and bootstrap confidence intervals. The Bayesian analysis is performed based on non-informative prior, with $a_0 = b_0 = 0$ and $a_1 = a_2 = a_3 = a_4 = 1.005$ which guarantees the existence of posterior mean and variance of each θ_h. The Bayes estimates based on square error loss function and the associated 90% HPD and symmetric credible intervals are recorded in Table 2.

In this case, as only one observation comes from each of the Data sets 2 and 3, no solution exist for equation 7 at least when $\gamma = 0.1$, i.e. for 90% confidence coefficient. We only can derive the lower limits. For bootstrap intervals we always can derive confidence intervals. In the Bayesian set up HPD credible intervals are shorter than the symmetric credible intervals.

Table 2: Results based on JPC-2 censored data (example 1). Here 'SNF' indicates that solution not found.

parameter	MLE	90% Exact CI	90% Percentile Bootstrap CI
θ_1	2.793	(0.374, 2.812)	(1.321, 9.685)
θ_2	11.340	(1.992, SNF)	(3.095, 17.602)
θ_3	8.340	(1.545, SNF)	(2.288, 13.121)
θ_4	1.006	(0.257, 2.319)	(0.489, 3.862)

parameter	BE	90% HPD CRI	90% Symmetric CRI
θ_1	3.388	(0.816, 6.538)	(1.420, 12.979)
θ_2	12.763	(1.542, 20.194)	(3.043, 48.427)
θ_3	8.487	(1.436, 18.821)	(2.343, 31.094)
θ_4	2.269	(0.653, 4.700)	(0.768, 5.840)

Example 2:

Nelson [15] (Chapter 10 , Table 4.1) presented times to breakdown in minutes of an insulating fluid subjected to high voltage stress. The breakdown times were presented in six groups and each group is of size 10. For our analysis purpose we choose group 1, 2 and 3 and exponential distribution fits the data sets quite well.

Group 1: 0.31, 0.66, 1.54, 1.70, 1.82, 1.89, 2.17, 2.24, 4.03, 9.99.

Group 2: 0.00, 0.18, 0.55, 0.66, 0.71, 1.30, 1.63, 2.17, 2.75, 10.60.

Group 3: 0.49, 0.64 ,0.82 ,0.93 ,1.08, 1.99, 2.06, 2.15, 2.57, 4.75.

To generate a JPC-2 censored sample from the data we set $k = 10$ and $R_1 = \ldots, R_{k-1} = 0$. The censored sample is presented below.

Table 3: JPC-2 sample (example 2)

W	**Z**
0.00	(0, 1, 0)
0.18	(0, 1, 0)
0.31	(1, 0, 0)
0.55	(0, 1, 0)
0.64	(0, 0, 1)
0.66	(1, 0, 0)
0.66	(0, 1, 0)
0.71	(0, 1, 0)
0.93	(0, 0, 1)
1.08	(0, 0, 1)

Based on the data we compute MLEs, 90% exact and bootstrap confidence intervals. We derive Bayes estimates based on the squared error loss function and 90% credible intervals based on non-informative prior. Here we set hyper-parameters as $a_0 = b_{=}0$ and $a_1 = a_2 = a_3 = 0$ as each $K_h > 2$, which guarantees the existence of posterior mean and variance of each θ_h for $h = 1, 2, 3$. We report all the results in Table 4. Bootstrap intervals are smaller than the exact intervals whereas HPD credible intervals are shorter than symmetric credible intervals.

Table 4: Results based on data generated by Scheme 2

parameter	MLE	90% Exact CI	90% Percentile Bootstrap CI
θ_1	2.860	(1.030, 25.319)	(1.110, 7.659)
θ_2	1.144	(0.559, 2.700)	(0.550, 2.694)
θ_3	1.906	(0.788, 6.918)	(0.802, 6.133)

parameter	BE	90% HPD CRI	90% Symmetric CRI
θ_1	5.929	(0.747, 10.994)	(1.241, 16.136)
θ_2	1.404	(0.482, 2.461)	(0.609, 2.995)
θ_3	2.765	(0.646, 5.292)	(0.899, 6.868)

7 Conclusion

In this article we extend the new two sample Type-II progressive censoring scheme (JPC-2) introduce by Mondal and Kundu [13] for more than two exponential populations. We provide both the classical and the Bayesian inferences of the unknown parameters. The performances of both the estimators are quite satisfactory. In this paper we consider only one parameter exponential distributions. For other cases like two-parameter Weibull or two-parameter generalized exponential distributions also the method can be developed. It may be mentioned that although in case of exponential distribution it is possible to derive the exact distribution of the MLEs, it may not be possible to develop the exact inference in case of Weibull or generalized exponential distributions. For multi-sample study it is quite natural to study the order restricted inference. It will be interesting to develop the inference for two-parameter exponential distribution also for multi-sample case. More works are needed in those directions.

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Appendix A

Proof of lemma 3.2.1:

If $d_1, \ldots, d_k > 0$, then

$$\int_0^\infty \int_{w_1}^\infty \cdots \int_{w_{k-1}}^\infty e^{-\sum_{i=1}^{k} d_i w_i} dw_k \ldots dw_2 dw_1 = \frac{1}{d_k(d_k+d_{k-1})\cdots(d_k+d_{k-1}+\cdots+d_1)}. \tag{12}$$

$$\begin{aligned} P(\mathbf{K}=\mathbf{r}) &= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \int_0^\infty \int_{w_1}^\infty \cdots \int_{w_{k-1}}^\infty L(\theta_1,\theta_2,\ldots,\theta_H|\mathbf{w},\mathbf{z}) dw_k \ldots dw_2 dw_1 \\ &= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} C\Big(\prod_{h=1}^{H} \frac{1}{\theta_h^{r_h}}\Big) \int_0^\infty \int_{w_1}^\infty \cdots \int_{w_{k-1}}^\infty e^{-\sum_{h=1}^{H} \frac{A_h(w)}{\theta_h}} dw_k \ldots dw_2 dw_1 \\ &= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} C\Big(\prod_{h=1}^{H} \frac{1}{\theta_h^{r_h}}\Big) \int_0^\infty \int_{w_1}^\infty \cdots \int_{w_{k-1}}^\infty e^{-\sum_{i=1}^{k} a_i w_i} dw_k \ldots dw_2 dw_1, \end{aligned}$$

where $a_i = \sum_{h=1}^{H} \frac{(R_i+1)}{\theta_h}$ for $i = 1, \ldots, k-1$ and $a_k = \sum_{h=1}^{H} \frac{(n_h - \sum_{i=1}^{k-1}(R_i+1))}{\theta_h}$.

Using(12)

$$
\begin{aligned}
P(\mathbf{K}=\mathbf{r}) &= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\frac{1}{a_k(a_k+a_{k-1})\cdots(a_k+a_{k-1}+\cdots+a_1)}\\
&= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\frac{1}{\Big(\prod_{i=1}^{k}\sum\limits_{h=1}^{H}\frac{(n_h-\sum\limits_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}.
\end{aligned}
$$

Substituting $C=\prod_{i=1}^{k}\Big(\sum\limits_{h=1}^{H}(n_h-\sum\limits_{j=1}^{i-1}(R_j+1))z_{ih}\Big)$,

$$
P(\mathbf{K}=\mathbf{r}) = \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \frac{\prod_{i=1}^{k}\Big(\sum\limits_{h=1}^{H}(n_h-\sum\limits_{j=1}^{i-1}(R_j+1))z_{ih}\Big)}{\Big(\prod_{h=1}^{H}\theta_h^{r_h}\Big)\Big(\prod_{i=1}^{k}\sum\limits_{h=1}^{H}\frac{(n_h-\sum\limits_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}.
$$

Proof of Theorem 3.2.1:

$$
\begin{aligned}
M_{\widehat{\boldsymbol{\theta}}}(\mathbf{t})) &= E\big(e^{\mathbf{t}\widehat{\boldsymbol{\theta}}}|\mathcal{M}\big)\\
&= \sum_{\mathbf{r}\in S^*} E(e^{\mathbf{t}\widehat{\theta}}|\mathbf{K}=\mathbf{r})P(\mathbf{K}=\mathbf{r}|\mathcal{M})\\
&= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}}\sum_{\mathbf{r}\in S^*} E(e^{\mathbf{t}\widehat{\theta}}|\mathbf{Z}=\mathbf{z},\mathbf{K}=\mathbf{r})P(\mathbf{Z}=\mathbf{z}|\mathbf{K}=\mathbf{r})P(\mathbf{K}=\mathbf{r}|\mathcal{M})\\
&= \frac{1}{P(\mathcal{M})}\sum_{\mathbf{z}\in Q_{(\mathbf{r})}}\sum_{\mathbf{r}\in S^*} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\int\limits_{0}^{\infty}\int\limits_{w_1}^{\infty}\cdots\int\limits_{w_{k-1}}^{\infty} e^{-\sum\limits_{i=1}^{k} b_i w_i}\,dw_k\ldots dw_2 dw_1
\end{aligned}
$$

where

$$
\begin{aligned}
b_i &= \sum_{h=1}^{H}\frac{(R_i+1)}{\theta_h}-t_1\frac{(R_i+1)}{r_1}-t_2\frac{(R_i+1)}{r_2}-\cdots-t_H\frac{(R_i+1)}{r_H}\quad\forall\quad i=1,\ldots,k-1,\\
b_k &= \sum_{h=1}^{H}\frac{(n_h-\sum\limits_{i=1}^{k-1}(R_i+1))}{\theta_h}-t_1\frac{(n_1-\sum\limits_{i=1}^{k-1}(R_i+1))}{r_1}-t_2\frac{(n_2-\sum\limits_{i=1}^{k-1}(R_i+1))}{r_2}\\
&\quad -\cdots-t_H\frac{(n_H-\sum\limits_{i=1}^{k-1}(R_i+1))}{r_H}.
\end{aligned}
$$

Using(12) we obtain

$$\begin{aligned}
M_{\widehat{\boldsymbol{\theta}}}(\mathbf{t})) &= \frac{1}{P(\mathcal{M})}\sum_{\mathbf{z}\in Q_{(\mathbf{r})}}\sum_{\mathbf{r}\in S^*} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\frac{1}{b_k(b_k+b_{k-1})\cdots(b_k+b_{k-1}+\cdots+b_1)}\\
&= \frac{1}{P(\mathcal{M})}\sum_{\mathbf{z}\in Q_{(\mathbf{r})}}\sum_{\mathbf{r}\in S^*} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\times\\
&\quad \frac{1}{\prod_{i=1}^{k}\Big(\sum\limits_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}-t_1\frac{(n_1-\sum_{j=1}^{i-1}(R_j+1))}{r_1}\cdots-t_H\frac{(n_H-\sum_{j=1}^{i-1}(R_j+1))}{r_H}\Big)}\\
&= \frac{1}{P(\mathcal{M})}\sum_{\mathbf{z}\in Q_{(\mathbf{r})}}\sum_{\mathbf{r}\in S^*} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\frac{1}{\prod_{i=1}^{k}\Big(\sum\limits_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}\times\\
&\quad \frac{1}{\prod_{i=1}^{k}\Big(1-t_1\frac{(n_1-\sum_{j=1}^{i-1}(R_j+1))}{r_1\Big(\sum_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}-\cdots-t_H\frac{(n_H-\sum_{j=1}^{i-1}(R_j+1))}{r_H\Big(\sum_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}\Big)}
\end{aligned}$$

Substituting $C=\prod_{i=1}^{k}\Big(\sum\limits_{h=1}^{H}(n_h-\sum\limits_{j=1}^{i-1}(R_j+1))z_{ih}\Big)$,

$$\begin{aligned}
M_{\widehat{\boldsymbol{\theta}}}(\mathbf{t}))) &= \frac{1}{P(\mathcal{M})}\sum_{\mathbf{z}\in Q_{(\mathbf{r})}}\sum_{\mathbf{r}\in S^*}\prod_{i=1}^{k}\Big(\sum_{h=1}^{H}(n_h-\sum_{j=1}^{i-1}(R_j+1))z_{ih}\Big)\times\\
&\quad \Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\frac{1}{\prod_{i=1}^{k}\Big(\sum\limits_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}\times\\
&\quad \frac{1}{\prod_{i=1}^{k}\Big(1-t_1\frac{(n_1-\sum_{j=1}^{i-1}(R_j+1))}{r_1\Big(\sum_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}-\cdots-t_H\frac{(n_H-\sum_{j=1}^{i-1}(R_j+1))}{r_H\Big(\sum_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}\Big)}\\
&= \frac{1}{P(\mathcal{M})}\sum_{\mathbf{r}\in S^*}P(\mathbf{K}=\mathbf{r})\prod_{i=1}^{k}\big(1-t_1\frac{\alpha_{i,1}}{r_1}-\cdots-t_H\frac{\alpha_{i,H}}{r_H}\big)^{-1}.
\end{aligned}$$

Proof of Lemma 3.2.2:

Let $\mathbf{V}$= $(a_1U, \ldots, a_HU)$ where $U \sim Exp(1)$ and $\mathbf{t}$ =$(t_1, \ldots, t_H)$, $\mathbf{x} = (x_1, \ldots, x_H)$. Then the MGF of $\mathbf{V}$ is obtained as

$$M_{\mathbf{V}}(\mathbf{t}) = (1 - a_1t_1 - \ldots - a_Ht_H)^{-1}$$

and the CDF is

$$F_{\mathbf{V}}(\mathbf{x}) = 1 - e^{-\min\limits_{1\le h\le H}(\frac{x_h}{a_h})}.$$

Clearly, $\mathbf{V}$ follows a singular distribution and we denote it by $F(a_1, \ldots, a_H)$.

Let $\mathbf{V}_i(\mathbf{r}) \sim F(\frac{\alpha_{i,1}}{r_1}, \cdots, \frac{\alpha_{i,H}}{r_H})$ independently for $i = 1, \ldots, k$ and $\forall\ \mathbf{r} = (r_1, \ldots, r_H) \in S^*$. We define

$$\mathbf{G}(\mathbf{r}) = \sum_{i=1}^{k} \mathbf{V}_i(\mathbf{r}). \tag{13}$$

Then the MGF of $\mathbf{G}(\mathbf{r})$ is

$$\prod_{i=1}^{k} \big(1 - \frac{\alpha_{i,1}}{r_1}t_1 - \frac{\alpha_{i,2}}{r_2}t_2 - \cdots - \frac{\alpha_{i,H}}{r_H}t_H\big)^{-1}$$

Therefore, the result follows.

The joint CDF of $\mathbf{G}(\mathbf{r})$ can be obtained as

$$\begin{aligned}
& P(\mathbf{G}(\mathbf{r}) < \mathbf{x}) \\
= \ & P\Big(\sum_{i=1}^{k} \boldsymbol{V}_i(\boldsymbol{r}) \le \boldsymbol{x}\Big) \\
= \ & P\Big(\sum_{i=1}^{k} \frac{\alpha_{i,1}}{r_1}U_i \le x_1, \ldots, \sum_{i=1}^{k} \frac{\alpha_{i,H}}{r_H}U_i \le x_H\Big) \\
= \ & \int_0^{\min\limits_h(\frac{x_h}{\frac{\alpha_{k,h}}{r_h}})} P\Big(\sum_{i=1}^{k-1} \frac{\alpha_{i,1}}{r_1}U_i \le x_1 - \frac{\alpha_{k,1}}{r_1}u_k, \ldots, \sum_{i=1}^{k-1} \frac{\alpha_{i,H}}{r_H}U_i \le x_H - \frac{\alpha_{k,H}}{r_H}u_k\Big)e^{-u_k}du_k
\end{aligned}$$

$$= \int_0^{\min_h(\frac{x_h}{\frac{\alpha_{k,h}}{r_h}})} \int_0^{\min_h(\frac{x_h - \frac{\alpha_{k,h}}{r_h} u_k}{\frac{\alpha_{k-1,h}}{r_h}})} P\Big(\sum_{i=1}^{k-2} \frac{\alpha_{i,1}}{r_1} U_i \leq x_1 - \frac{\alpha_{k,1}}{r_1} u_k - \frac{\alpha_{k-1,1}}{r_1} u_{k-1}, , \ldots, \sum_{i=1}^{k-2} \frac{\alpha_{i,H}}{r_H} U_i \leq x_H - \frac{\alpha_{k,H}}{r_H} u_k - \frac{\alpha_{k-1,H}}{r_H} u_{k-1}\Big) e^{-u_{k-1}} e^{-u_k} du_{k-1} du_k$$

$$\vdots$$

$$= \int_0^{\min_h(\frac{x_h}{\frac{\alpha_{k,h}}{r_h}})} \int_0^{\min_h(\frac{x_h - \frac{\alpha_{k,h}}{r_h} u_k}{\frac{\alpha_{k-1,h}}{r_h}})} \ldots \int_0^{\min_h(\frac{x_h - \sum_{i=3}^{k} \frac{\alpha_{i,h}}{r_h} u_i}{\frac{\alpha_{2,h}}{r_h}})} P\Big(\frac{\alpha_{1,1}}{r_1} U_1 \leq x_1 - \sum_{i=2}^{k} \frac{\alpha_{i,1}}{r_1} u_i, \ldots, \frac{\alpha_{1,H}}{r_H} U_1 \leq x_H - \sum_{i=2}^{k} \frac{\alpha_{i,H}}{r_H} u_i\Big) e^{-u_2} \cdots e^{-u_k} du_2 \cdots du_k$$

$$= \int_0^{\min_h(\frac{x_h}{\frac{\alpha_{k,h}}{r_h}})} \int_0^{\min_h(\frac{x_h - \frac{\alpha_{k,h}}{r_h} u_k}{\frac{\alpha_{k-1,h}}{r_h}})} \ldots \int_0^{\min_h(\frac{x_h - \sum_{i=3}^{k} \frac{\alpha_{i,h}}{r_h} u_i}{\frac{\alpha_{2,h}}{r_h}})} \Big(1 - e^{-\min_h(\frac{x_h - \sum_{i=2}^{k} \frac{\alpha_{i,h} u_i}{r_h}}{\frac{\alpha_{1,h}}{r_h}})}\Big) e^{-u_2} \cdots e^{-u_k} du_2 \cdots du_k.$$

Proof of Theorem 3.3.1:

$$
\begin{aligned}
E(e^{tW_j}) &= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \sum_{\mathbf{r}\in S} E(e^{tW_j}|\mathbf{Z}=\mathbf{z},\mathbf{K}=\mathbf{r})P(\mathbf{Z}=\mathbf{z}|\mathbf{K}=\mathbf{r}) \\
&= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \sum_{\mathbf{r}\in S} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\int_0^{\infty}\int_{w_1}^{\infty}\cdots\int_{w_{k-1}}^{\infty} e^{tw_j}e^{-\sum_{h=1}^{H}\frac{A_h(w)}{\theta_h}}\,dw_k\ldots dw_2dw_1 \\
&= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \sum_{\mathbf{r}\in S} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\int_0^{\infty}\int_{w_1}^{\infty}\cdots\int_{w_{k-1}}^{\infty} e^{-\Big\{\sum_{i=1,i\neq j}^{k} a_iw_i+a'_jw_j\Big\}}\,dw_k\ldots dw_2dw_1 \\
&= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \sum_{\mathbf{r}\in S} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\times \\
&\quad \frac{1}{a_k}\frac{1}{(a_k+a_{k-1})}\cdots\frac{1}{(a_k+a_{k-1}+\cdots+a_{j+1})}\frac{1}{(a_k+a_{k-1}+\cdots+a_{j+1}+a'_j)}\cdots\times \\
&\quad \frac{1}{(a_k+a_{k-1}+\cdots+a_{j+1}+a'_j+a_{j-1}+\cdots+a_1)} \\
&= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \sum_{\mathbf{r}\in S} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\frac{1}{a_k(a_k+a_{k-1})\cdots(a_k+a_{k-1}+\cdots+a_{j+1}+a_j)}\cdots\times \\
&\quad \frac{1}{(a_k+a_{k-1}+\cdots+a_{j+1}+a_j+a_{j-1}+\cdots a_1)}\times \\
&\quad \frac{(a_k+a_{k-1}+\cdots+a_{j+1}+a_j)\cdots(a_k+a_{k-1}+\cdots+a_{j+1}+a_j+a_{j-1}+\cdots+a_1)}{(a_k+a_{k-1}+\cdots+a_{j+1}+a'_j)\cdots(a_k+a_{k-1}+\cdots+a_{j+1}+a'_j+a_{j-1}+\cdots+a_1)} \\
&= \sum_{\mathbf{z}\in Q_{(\mathbf{r})}} \sum_{\mathbf{r}\in S} C\Big(\prod_{h=1}^{H}\frac{1}{\theta_h^{r_h}}\Big)\frac{1}{\prod_{i=1}^{k}\Big(\sum_{h=1}^{H}\frac{(n_h-\sum_{j=1}^{i-1}(R_j+1))}{\theta_h}\Big)}\times \\
&\quad \frac{(a_k+a_{k-1}+\cdots+a_{j+1}+a_j)\cdots(a_k+a_{k-1}+\cdots+a_{j+1}+a_j+a_{j-1}+\cdots+a_1)}{(a_k+a_{k-1}+\cdots+a_{j+1}+a'_j)\cdots(a_k+a_{k-1}+\cdots+a_{j+1}+a'_j+a_{j-1}+\cdots+a_1)} \\
&= \sum_{\mathbf{r}\in S} P(\mathbf{K}=\mathbf{r})\frac{(a_k+a_{k-1}+\cdots+a_j)\cdots(a_k+a_{k-1}+\cdots+a_j+a_{j-1}+\cdots+a_1)}{(a_k+a_{k-1}+\cdots+a_j-t)\cdots(a_k+a_{k-1}+\cdots+a_j-t+a_{j-1}+\cdots+a_1)} \\
&= \frac{1}{\big(1-\frac{t}{(a_k+a_{k-1}+\cdots+a_j)}\big)\cdots\big(1-\frac{t}{(a_k+a_{k-1}+\cdots+a_j+a_{j-1}+\cdots+a_1)}\big)} \\
&= \frac{1}{(1-\frac{t}{\sum_{h=1}^{H}\frac{(n_h-\sum_{i=1}^{j-1}(R_i+1))}{\theta_h}})}\cdots\frac{1}{(1-\frac{t}{\sum_{h=1}^{H}\frac{n_h)}{\theta_h}})} \\
&= \prod_{l=1}^{j}\Bigg(1-\frac{t}{\sum_{h=1}^{H}\frac{(n_h-\sum_{i=1}^{l-1}(R_i+1))}{\theta_h}}\Bigg)^{-1}.
\end{aligned}
$$

Here $S = \{\mathbf{r} = (r_1, r_2, \dots, r_H) : r_h \in \{0, 1, \dots, k\}, \forall h; \sum_{h=1}^{H} r_h = k\}$ is the support of $\mathbf{K} = (K_1, K_2, \dots, K_H)$. and $a_j' = a_j - t$.

Proof of lemma 4.1.1 :

$P_{\theta_h}(\widehat{\theta}_h > t|\mathcal{M})$ can be decomposed as

$$P_{\theta_h}(\widehat{\theta}_h > t|\mathcal{M}) = \sum_{r_h=1}^{k-(H-1)} P_{\theta_h}(\widehat{\theta}_h > t|K_h = r_h,)P_{\theta_h}(K_h = r_h|\mathcal{M})$$

To prove Lemma 4, using three Monotonicity Lemmas of Balakrishnan and Illiopoulos [4] it is enough to prove that

(M1) $P_{\theta_h}(\widehat{\theta}_h > t|K_h = r_h)$ is increasing in θ_h, $\forall t, r \in \{1, \dots k-1\}$;

(M2) $P_{\theta_h}(\widehat{\theta}_h > t|K_h = r_h)$ is decreasing in r_h, $\forall t, \theta_h > 0$;

(M3) The conditional distribution of K_h is stochastically decreasing in θ_h.

To prove (M1), we compute the MGF of $\widehat{\theta}_h|K_h = r_h$ conditioned on $\mathcal{M}$, which is obtained as

$$E(e^{t\widehat{\theta}_h}|K_h = r_h, \mathcal{M}) = \prod_{i=1}^{k} (1 - \alpha_{i,h}t)^{-1}$$

Therefore conditioned on $K_h = r_h$ and $\mathcal{M}$, $\widehat{\theta}_h$ can be expressed as $\widehat{\theta}_h = \sum_{i=1}^{k} V_{i,h}$ where $V_{i,h} \sim Exp(\alpha_{i,h})$ independently. As $\alpha_{i,h}$ is increasing with θ_h and $V_{i,h}$'s are independently distributed exponential random variables with mean $\alpha_{i,h}$, sum of $V_{i,h}$ is stochastically increasing in θ_h, therefore (M1) follows.

Now to prove (M2), we notice

$$\widehat{\theta}_h|\{K_h = r_h\} \stackrel{d}{=} \frac{\sum_{i=1}^{k-1} (R_i + 1)w_i + (n_h - \sum_{i=1}^{k-1} (R_i + 1))w_k}{r_h}$$

$$\widehat{\theta}_1|\{K_h = r_h + 1\} \stackrel{d}{=} \frac{\sum_{i=1}^{k-1} (R_i + 1)w_i + (n_h - \sum_{i=1}^{k-1} (R_i + 1))w_k}{r_h + 1}.$$

Hence for all t and for $\theta_h > 0$, $P_{\theta_h}(\widehat{\theta}_h > t | K_h = r_h)$ is decreasing in r_h.

(M3) can be proved showing that K_h has monotone likelihood ratio property with respect to θ_h. For $\theta_h < {\theta_h}'$

$$\frac{P_{\theta_h}(K_h = r_h|\mathcal{M})}{P_{{\theta_h}'}(K_h = r_h|\mathcal{M})} \propto (\frac{{\theta_h}'}{\theta_h})^{r_h} \uparrow r_h.$$

Appendix B

Proof of lemma 5.2.1:

If part:

Let $X_1, X_2, \ldots X_n$ be non-negative random variables such that

$$\sum_{i=1}^{n} X_i \sim GA(a_0, b_0),$$

$$(\frac{X_1}{\sum_{i=1}^{n} X_i}, \frac{X_1}{\sum_{i=1}^{n} X_i}, \ldots, \frac{X_n}{\sum_{i=1}^{n} X_i}) \sim D(a_1, a_2, \ldots, a_n) \quad \text{and they are independently distributed.}$$

We use the following notations,

$$T = \sum_{i=1}^{n} X_i; \; Y_i = \frac{X_i}{\sum_{i=1}^{n} X_i} \quad \text{for} \quad i = 1, \ldots, n-1.$$

Then the joint distribution of $T, Y_1, \ldots, Y_{n-1}$ can be written as

$$f_{Y_1,\ldots,Y_{n-1},T}(t, y_1, \cdots, y_{n-1}) \propto \prod_{i=1}^{n-1} y_i^{a_i - 1}(1 - \sum_{i=1}^{n-1} y_i)^{a_n - 1} t^{a_0 - 1} e^{-b_0 t}.$$

Now the X_i's can be written as

$$X_i = TY_i \quad \text{for} \quad i = 1, \ldots, n-1, \quad \text{and} \quad X_n = T(1 - \sum_{i=1}^{n-1} Y_i).$$

To derive the joint distribution of $(X_1, \ldots, X_n)$, we find out the Jacobian J where

$$J^{-1} = \begin{bmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \ldots & \frac{\partial X_1}{\partial Y_{n-1}} & \frac{\partial X_1}{\partial T} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \ldots & \frac{\partial X_2}{\partial Y_{n-1}} & \frac{\partial X_2}{\partial T} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \frac{\partial X_n}{\partial Y_1} & \frac{\partial X_n}{\partial Y_2} & \ldots & \frac{\partial X_n}{\partial Y_{n-1}} & \frac{\partial X_n}{\partial T} \end{bmatrix} = \begin{bmatrix} T & 0 & \ldots & 0 & Y_1 \\ 0 & T & \ldots & 0 & Y_2 \\ \vdots & & & & \\ -T & -T & & -T & (1 - \sum_{i=1}^{n-1} Y_i) \end{bmatrix}.$$

Therefore, the determinant of J, $|J|$ is $T^{-(n-1)} = (\sum_{i=1}^{n} X_i)^{-(n-1)}$. The joint PDF of $(X_1, \ldots, X_n)$ can be obtained as

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &\propto \prod_{i=1}^{n} (\frac{x_i}{\sum_{j=1}^{n} x_j})^{a_i-1} (\sum_{j=1}^{n} x_j)^{a_0-1} e^{b_0 \sum_{j=1}^{n} x_j} (\sum_{i=1}^{n} x_i)^{-(n-1)} \\ &= \prod_{i=1}^{n} {x_i}^{a_i-1} (\sum_{j=1}^{n} x_j)^{a_0 - \sum_{j=1}^{n} a_j} e^{b_0 \sum_{j=1}^{n} x_j}. \end{aligned}$$

Therefore $(X_1, \ldots, X_n) \sim D(a_1, \ldots, a_n)$.

Only if part:

Let $(X_1, \ldots, X_n) \sim GD(a_0, b_0, a_1, \ldots, a_n)$, then the joint PDF of $(X_1, \ldots, X_n)$ is obtained as

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) \propto \prod_{i=1}^{n} x_i^{a_i-1} (\sum_{i=1}^{n} x_i)^{(a_0 - \sum_{i=1}^{n} a_i)} e^{-b_0 \sum_{i=1}^{n} x_i}.$$

Let use the transformation

$$Y_1 = \frac{X_1}{\sum_{i=1}^{n} X_i}, Y_2 = \frac{X_2}{\sum_{i=1}^{n} X_i}, \ldots, Y_{n-1} = \frac{X_{n-1}}{\sum_{i=1}^{n} X_i}, Y_n = \sum_{i=1}^{n} X_i.$$

Then X_i's can be written as

$$X_i = Y_i Y_n \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{and} \quad X_n = Y_n(1 - \sum_{i=1}^{n-1} Y_i).$$

The Jacobian of the transformation $(X_1, \ldots, X_n) \to (Y_1, \ldots, Y_n)$, is

$$J = \begin{bmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \ldots & \frac{\partial X_1}{\partial Y_{n-1}} & \frac{\partial X_1}{\partial Y_n} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \ldots & \frac{\partial X_2}{\partial Y_{n-1}} & \frac{\partial X_2}{\partial Y_n} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \frac{\partial X_n}{\partial Y_1} & \frac{\partial X_n}{\partial Y_2} & \ldots & \frac{\partial X_n}{\partial Y_{n-1}} & \frac{\partial X_n}{\partial Y_n} \end{bmatrix} = \begin{bmatrix} Y_n & 0 & \ldots & 0 & Y_1 \\ 0 & Y_n & \ldots & 0 & Y_2 \\ \vdots & & & & \\ -Y_n & -Y_n & & -Y_n & (1 - \sum_{i=1}^{n-1} Y_i) \end{bmatrix}.$$

The determinant of J is Y_n^{n-1}. The joint PDF of $(Y_1, \ldots, Y_n)$ is obtained as

$$f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n) \propto \prod_{i=1}^{n-1} y_i^{a_i-1} (1 - \sum_{i=1}^{n-1} y_i)^{a_n-1} y_n^{a_0-1} e^{-b_0 y_n}.$$

It is clear that $(Y_1, Y_2, \dots, Y_{n-1})$ and Y_n are independently distributed. Also

$$(Y_1, Y_2, \dots, Y_{n-1}, (1 - \sum_{i=1}^{n-1} Y_i)) \sim D(a_1, a_2, \dots, a_{n-1}, a_n) \quad \text{and} \quad Y_n \sim GA(a_0, b_0).$$

Therefore, $\left(\frac{X_1}{\sum_{i=1}^{n} X_i}, \frac{X_2}{\sum_{i=1}^{n} X_i}, \dots, \frac{X_n}{\sum_{i=1}^{n} X_i}\right) \sim D(a_1, a_2, \dots, a_n)$ and $\sum_{i=1}^{n} X_i \sim GA(a_0, b_0)$ are independently distributed.

Proof of Result 1:

Using Lemma 5, we can obtain,

$$E(\lambda_h) = E(\frac{\lambda_h}{\sum_{l=1}^{H} \lambda_l} \sum_{l=1}^{H} \lambda_l) = E(\sum_{l=1}^{H} \lambda_l) E(\frac{\lambda_h}{\sum_{l=1}^{H} \lambda_l}) = \frac{a_0}{b_0} \frac{a_h}{(\sum_{l=1}^{H} a_l)}$$

$$\begin{aligned} E(\lambda_{h_1}\lambda_{h_2}) = E\big(\frac{\lambda_{h_1}\lambda_{h_2}}{\sum_{l=1}^{H} \lambda_l \sum_{l=1}^{H} \lambda_l} (\sum_{l=1}^{H} \lambda_l)^2\big) &= E((\sum_{l=1}^{H} \lambda_l)^2) E(\frac{\lambda_{h_1}\lambda_{h_2}}{\sum_{l=1}^{H} \lambda_l \sum_{l=1}^{H} \lambda_l}) \\ &= \frac{a_0(a_0+1)}{b_0^2} \frac{a_{h_1} a_{h_2}}{(\sum_{h=1}^{H} a_h)(\sum_{h=1}^{H} a_h + 1)}. \end{aligned}$$

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Table 5: AE and MSE of the MLEs and BE and MSE of Bayes Estimators of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ based on informative prior (IP) and non-informative prior (NIP) with $\theta_1 = 0.3, \theta_2 = 0.4, \theta_3 = 0.5, \theta_4 = 0.6, \theta_5 = 0.7$

sample sizes	censoring scheme	Parameter	MLE		Bayes IP		Bayes NIP	
			AE	MSE	BE	MSE	BE	MSE
$n_1 = 15, n_2 = 16, n_3 = 15, n_4 = 15, n_5 = 16$	k=8,R=(2, $0_{(6)}$)	θ_1	0.453	0.092	0.394	0.032	0.626	0.283
		θ_2	0.547	0.113	0.557	0.078	0.773	0.434
		θ_3	0.548	0.080	0.779	0.212	0.914	0.547
		θ_4	0.571	0.077	1.034	0.448	1.015	0.579
		θ_5	0.645	0.090	1.336	0.990	1.034	0.512
$n_1 = 15, n_2 = 15, n_3 = 15, n_4 = 15, n_5 = 15$	k=8,R=(2, $0_{(6)}$)	θ_1	0.463	0.101	0.385	0.027	0.629	0.275
		θ_2	0.525	0.097	0.557	0.078	0.781	0.392
		θ_3	0.565	0.088	0.766	0.192	0.899	0.548
		θ_4	0.590	0.084	1.007	0.395	0.999	0.567
		θ_5	0.631	0.091	1.309	0.872	1.109	0.599
$n_1 = 25, n_2 = 25, n_3 = 25, n_4 = 25, n_5 = 25$	k=20,R=(5, $0_{(18)}$)	θ_1	0.351	0.041	0.353	0.014	0.436	0.069
		θ_2	0.505	0.127	0.497	0.041	0.562	0.123
		θ_3	0.645	0.210	0.667	0.108	0.724	0.253
		θ_4	0.777	0.288	0.863	0.256	0.868	0.349
		θ_5	0.881	0.344	1.106	0.585	0.969	0.446
	k=20,R=($0_{(18)}$, 5)	θ_1	0.352	0.039	0.354	0.014	0.438	0.056
		θ_2	0.504	0.124	0.494	0.040	0.589	0.138
		θ_3	0.657	0.231	0.670	0.111	0.708	0.232
		θ_4	0.776	0.288	0.893	0.283	0.851	0.357
		θ_5	0.875	0.332	1.093	0.555	0.960	0.425
$n_1 = 25, n_2 = 26, n_3 = 27, n_4 = 28, n_5 = 29$	k=20,R=(5, $0_{(18)}$)	θ_1	0.371	0.064	0.359	0.016	0.462	0.070
		θ_2	0.510	0.133	0.502	0.043	0.598	0.135
		θ_3	0.645	0.221	0.661	0.112	0.716	0.212
		θ_4	0.773	0.316	0.831	0.226	0.824	0.307
		θ_5	0.906	0.417	0.998	0.386	0.925	0.349
	k=20, R=($0_{(18)}$, 5)	θ_1	0.364	0.054	0.350	0.014	0.475	0.100
		θ_2	0.505	0.132	0.489	0.042	0.577	0.136
		θ_3	0.650	0.220	0.662	0.101	0.711	0.223
		θ_4	0.780	0.312	0.874	0.280	0.815	0.273
		θ_5	0.909	0.415	1.023	0.410	0.922	0.424
$n_1 = 30, n_2 = 30, n_3 = 30, n_4 = 30, n_5 = 30$	k=25,R=(5, $0_{(23)}$)	θ_1	0.341	0.030	0.393	0.011	0.406	0.034
		θ_2	0.482	0.094	0.489	0.036	0.552	0.094
		θ_3	0.635	0.198	0.640	0.088	0.676	0.153
		θ_4	0.782	0.317	0.812	0.198	0.798	0.251
		θ_5	0.904	0.408	1.051	0.552	0.923	0.375
	k=25,R=($0_{(23)}$, 5)	θ_1	0.338	0.032	0.346	0.013	0.412	0.046
		θ_2	0.486	0.103	0.490	0.039	0.543	0.112
		θ_3	0.637	0.210	0.643	0.090	0.680	0.107
		θ_4	0.774	0.309	0.825	0.215	0.809	0.278
		θ_5	0.901	0.405	1.062	0.523	0.941	0.412

Table 6: AL and CP of the 90% Exact and Percentile Bootstrap CI of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ with $\theta_1 = 0.3, \theta_2 = 0.4, \theta_3 = 0.5, \theta_4 = 0.6, \theta_5 = 0.7$

sample sizes	censoring scheme	Parameter	90% Bootstrap CI		90% Exact CI	
			AL	CP	AL	CP
$n_1 = 15, n_2 = 16, n_3 = 15, n_4 = 15, n_5 = 16$	k=8,R=(2, $0_{(6)}$)	θ_1	0.847	94.1%	1.967	94.0%
		θ_2	0.952	97.2%	2.121	96.1%
		θ_3	0.873	95.1%	2.412	96.5%
		θ_4	0.875	89.9%	2.609	94.8%
		θ_5	0.976	88.4%	2.251	93.3%
$n_1 = 15, n_2 = 15, n_3 = 15, n_4 = 15, n_5 = 15$	k=8,R=(2, $0_{(6)}$)	θ_1	0.850	94.1%	1.748	95.6%
		θ_2	0.873	97.2%	1.971	94.7%
		θ_3	0.881	95.2%	1.992	93.9%
		θ_4	0.887	91.8%	2.170	91.1%
		θ_5	0.889	86.5%	2.175	95.8%
$n_1 = 25, n_2 = 25, n_3 = 25, n_4 = 25, n_5 = 25$	k=20,R=(5, $0_{(18)}$)	θ_1	0.716	91.0%	0.771	92.7%
		θ_2	1.105	89.6%	1.601	96.0%
		θ_3	1.417	88.2%	2.072	92.1%
		θ_4	1.525	91.7%	2.909	95.1%
		θ_5	1.627	95.0%	3.344	94.2%
	k=20,R=($0_{(18)}$, 5)	θ_1	0.718	91.2%	0.686	92.0%
		θ_2	1.096	90.3%	1.646	94.6%
		θ_3	1.337	91.1%	2.243	94.6%
		θ_4	1.527	93.4%	3.061	95.1%
		θ_5	1.597	95.9%	2.641	90.5%
$n_1 = 30, n_2 = 30, n_3 = 30, n_4 = 30, n_5 = 30$	k=25,R=(5, $0_{(23)}$)	θ_1	0.608	89.3%	0.531	91.0%
		θ_2	1.039	88.1%	0.925	94.8%
		θ_3	1.316	89.9%	1.422	95.7%
		θ_4	1.590	90.1%	2.288	94.1%
		θ_5	1.748	89.9%	2.100	93.7%
	k=25,R=($0_{(23)}$, 5)	θ_1	0.620	88.1%	0.518	93.0%
		θ_2	1.019	90.7%	0.923	94.9%
		θ_3	1.365	90.0%	0.923	94.9%
		θ_4	1.583	904%	2.013	96.5%
		θ_5	1.723	90.3%	2.761	94.6%

Table 7: AL and CP of the 90% HPD credible interval (H-CRI) and Symmetric credible interval (S-CRI) of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ based on informative prior (IP) and non-informative prior (NIP) with $\theta_1 = 0.3, \theta_2 = 0.4, \theta_3 = 0.5, \theta_4 = 0.6, \theta_5 = 0.7$

sample sizes	censoring scheme	Parameter	90% H-CRI				90% S-CRI			
			IP		NIP		IP		NIP	
			AL	CP	AL	CP	AL	CP	AL	CP
$n_1 = 15, n_2 = 16, n_3 = 15, n_4 = 15, n_5 = 16$	k=8,R=(2, $0_{(6)}$)	θ_1	0.527	96.8%	1.067	95.4%	0.626	97.2%	1.336	88.9%
		θ_2	0.845	97.4%	1.221	96.2%	1.003	97.9%	1.645	94.9%
		θ_3	1.277	97.8%	1.634	95.0%	1.601	96.8%	2.251	95.6%
		θ_4	1.750	98.0%	1.822	93.2%	2.361	98.4%	2.537	96.4%
		θ_5	2.271	97.8%	1.814	94.2%	2.178	97.9%	2.516	95.4%
$n_1 = 15, n_2 = 15, n_3 = 15, n_4 = 15, n_5 = 15$	k=8,R=(2, $0_{(6)}$)	θ_1	0.536	97.0%	1.030	97.4%	0.614	95.9%	1.330	87.6%
		θ_2	0.820	96.5%	1.315	96.6%	1.004	96.3%	1.784	93.3%
		θ_3	1.246	97.6%	1.530	97.0%	1.547	94.4%	2.137	94.9%
		θ_4	1.692	97.1%	1.723	95.2%	2.251	97.3%	2.497	97.5%
		θ_5	2.281	97.2%	1.876	92.2%	2.231	97.6%	2.632	96.0%
$n_1 = 25, n_2 = 25, n_3 = 25, n_4 = 25, n_5 = 25$	k=20,R=(5, $0_{(18)}$)	θ_1	0.376	96.9%	0.530	93.6%	0.421	92.8%	0.611	89.4%
		θ_2	0.591	95.5%	0.774	96.4%	0.676	94.4%	0.948	92.3%
		θ_3	0.886	94.7%	1.009	93.0%	1.035	95.4%	1.319	93.2%
		θ_4	1.222	95.7%	1.281	95.0%	1.538	93.9%	1.628	93.9%
		θ_5	1.780	95.1%	1.436	92.8%	2.113	95.4%	1.866	94.2%
	k=20,R=($0_{(18)}$, 5)	θ_1	0.369	93.2%	0.516	94.8%	0.408	95.6%	0.603	88.4%
		θ_2	0.580	96.2%	0.783	94.8%	0.690	93.8%	0.905	93.8%
		θ_3	0.879	97.0%	1.009	96.2%	1.045	94.4%	1.283	92.9%
		θ_4	1.206	96.2%	1.147	92.4%	1.506	95.2%	1.564	94.8%
		θ_5	1.668	96.2%	1.507	91.4%	2.219	95.0%	1.905	95.2%
$n_1 = 25, n_2 = 26, n_3 = 27, n_4 = 28, n_5 = 29$	k=20, R=(5, $0_{(18)}$)	θ_1	0.393	95.6%	0.582	95.0%	0.444	95.6%	0.744	89.2%
		θ_2	0.599	94.4%	0.845	94.6%	0.716	96.0%	1.032	91.4%
		θ_3	0.826	96.2%	0.939	95.0%	1.026	94.9%	1.228	93.2%
		θ_4	1.164	96.8%	1.162	91.6%	1.429	93.5%	1.503	94.3%
		θ_5	1.365	95.0%	1.280	89.8%	1.807	94.2%	1.692	93.8%
	k=20, R=($0_{(18)}$, 5)	θ_1	0.388	94.6%	0.590	93.8%	0.426	96.2%	0.712	87.1%
		θ_2	0.575	93.8%	0.826	93.8%	0.707	94.8%	0.982	93.2%
		θ_3	0.890	97.6%	0.991	94.0%	1.033	95.7%	1.237	94.8%
		θ_4	1.186	96.0%	1.208	93.6%	1.391	94.2%	1.498	94.2%
		θ_5	1.591	95.4%	1.300	91.8%	1.942	94.2%	1.779	93.9%
$n_1 = 30, n_2 = 30, n_3 = 30, n_4 = 30, n_5 = 30$	k=25, R=(5, $0_{(23)}$)	θ_1	0.333	94.2%	0.456	94.8%	0.368	93.8%	0.522	88.2%
		θ_2	0.533	95.0%	0.674	94.6%	0.605	94.4%	0.760	92.3%
		θ_3	0.790	95.8%	0.898	93.9%	0.904	93.5%	1.091	92.7%
		θ_4	1.098	96.2%	1.073	91.8%	1.331	93.6%	1.342	95.3%
		θ_5	1.534	94.8%	1.359	90.8%	1.823	94.3%	1.654	92.9%
	k=25, R=($0_{(23)}$, 5)	θ_1	0.337	94.6%	0.443	96.6%	0.358	95.2%	0.504	89.6%
		θ_2	0.543	96.1%	0.663	94.6%	0.609	92.4%	0.759	92.5%
		θ_3	0.784	96.1%	0.927	96.4%	0.918	93.1%	1.143	90.9%
		θ_4	1.093	95.8%	1.125	93.4%	1.317	95.5%	1.373	94.0%
		θ_5	1.510	95.4%	1.335	92.0%	1.887	94.3%	1.667	93.5%