ESTIMATION OF PARAMETERS IN RANDOM AMPLITUDE CHIRP SIGNAL

SWAGATA NANDI[1] AND DEBASIS KUNDU[2]

Abstract. In this paper, we consider the random amplitude chirp signal model observed with independent and identically distributed additive error as proposed by Besson et al. [2]. The random amplitudes are also independent and identically distributed random variables with non-zero mean and with some finite higher order moments. We consider the estimators proposed by Besson et al. [2] and study their theoretical properties. It is observed that the estimators are strongly consistent and asymptotically normally distributed. We propose a new multi-component random amplitude chirp model, which can be more useful in practice. Numerical experiments are conducted to see the small sample performances. It is observed that the biases and mean squared errors are quite small..

1. Introduction

In this paper we mainly consider the estimation of the unknown parameters of a complex valued chirp signal model with random time-varying amplitude. Mathematically, the model can be expressed as follows:

$$y(t) = \alpha(t)e^{i(\theta_1^0 t + \theta_2^0 t^2)} + e(t), \quad t = 1, \ldots, n. \tag{1}$$

Here $y(t) = y_R(t) + iy_I(t)$ are the complex-valued signals at n equi-distant time points; the amplitude $\{\alpha(t)\}$ is a sequence of real-valued random variables with non-zero mean, μ_α, and its specific structure is given in Assumption 1; θ_1^0 and θ_2^0 are the frequency and chirp rate, respectively and $0 < \theta_1^0, \theta_2^0 < \pi$. The additive error $\{e(t)\}$ is a sequence of complex-valued random variables with zero mean and finite fourth moment. The random amplitude $\alpha(t)$ is like a multiplicative error. The problem here is to estimate the unknown parameters θ_1^0 and θ_2^0 based on the measurements $\{y(1), \ldots, y(n)\}$.

It should be mentioned that although, all physical signals are real valued, it might be advantageous from an analytical, a notational or an algorithmic point of view to work with

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2000 *Mathematics Subject Classification.* 62J02; 62E20; 62C05.

Key words and phrases. Random Amplitude Chirp Signal; Non-linear Least Squares; Multiplicative Error; Asymptotic Distribution;

signals in their analytic form which is complex valued, see for example Gabor [7]. For a real valued continuous signal, its analytic form can be easily obtained using the Hilbert transformation. Therefore, it is quite natural to work with complex valued signals and develop the necessary properties associated with the complex model and use them to the corresponding real model. Due to this reason, in most of the signal processing literature it is observed that the complex valued models are being used for analytical derivation or for algorithmic development, although they can be easily used for analyzing any real valued signal, see for example Stoica and Moses [13].

Model (1) is known as the random amplitude chirp signal model and it was first introduced by Besson et al. [2]. It has been mentioned by the authors that *This kind of signal arises in many applications of signal processing, one of the most important being the radar problem. For instance, consider a radar illuminating a target. Then, the transmitted signal will be affected by two different phenomena. First, it will undergo a phase shift induced by the distance and relative motion between the target and the receiver. Assuming this motion is continuous and differentiable, the phase shift can be adequately modeled as* $\phi(t) = a_0 + a_1 t + a_2 t^2$, *where the parameters* a_1 *and* a_2 *are either related to speed and acceleration or range and speed, depending on what the radar is intended for and on the kind of waveforms transmitted. The second phenomenon to be accounted for is amplitude distortion caused either by target fluctuation or scattering of the medium (e.g. fading).*

Note that Besson et al. [2] originally considered the following model

$$y(t) = \alpha(t) e^{i(\theta_0^0 + \theta_1^0 t + \theta_2^0 t^2)} + e(t), \quad t = 1, \ldots, n, \tag{2}$$

where $\alpha(t)$, $e(t)$, θ_1^0, θ_2^0 are same as in (1), but in model (2), there is a phase term θ_0, which has not been considered in model (1). It can be easily observed that when the mean of the multiplicative error, μ_α, is unknown and a phase term θ_0^0, which is also unknown, is present, then both are not identifiable. Therefore, if both are present one has to be known. So a separate phase term has not been considered in our model (1), and it is basically included in $\alpha(t)$ without loss of any generality. Hence, (1) and (2) are equivalent models.

Model (1) can be seen as a generalized version of the chirp model, where $\alpha(t)$ is a constant function of t. An extensive amount of work has been done on different aspects of a chirp model because of its wide scale applicability, see for example, Abatzoglou [1], Farquharson et al. [5], Gini et al. [8], Grover et al. [9] and the references cited therein. Model (1) can also be seen as a generalized version of the random amplitude sinusoidal model when $\theta_2^0 = 0$. The random amplitude sinusoidal model also has received a considerable amount of attention in the signal processing literature, see for example Francos and Friedlander [6], Besson and Stoica [3], Zhou and Ginnakis [14] etc. Finally, if $\theta_2^0 = 0$ and $\alpha(t)$ is constant, then Model (1) becomes the sinusoidal model, which is one of the most studied models in the statistical signal processing literature, see for example, the monograph by Kundu and Nandi [10].

The main aim of this paper is to consider the estimation of the parameters θ_1^0 and θ_2^0 and study their theoretical properties under suitable assumptions on $\alpha(t)$ and $e(t)$. Besson et al. [2] proposed a new estimation procedure of θ_1^0 and θ_2^0 and showed that the performances of their estimators are quite satisfactory. The details about the procedure are provided in Section 2. They provided some heuristic argument regarding the properties of these estimators and obtained the approximate theoretical variances of these estimators. In this paper we have shown that under a reasonable set of assumptions on the model parameters and on the error random variables, the proposed estimators are strongly consistent and they are asymptotically normally distributed. Finally we propose a multicomponent random amplitude chirp model along the same line as the multicomponent chirp model of Djurić and Kay [4], and we extend the results in this case. We perform some extensive simulation experiments to see how the estimators behave for different sample sizes and for different error variances. **The performances are quite satisfactory.** *The performances of the estimators in simulation study is as per the theoretical results obtained in this paper.*

In Section 2, we provide the necessary assumptions and establish the consistency and the asymptotic normality properties of the estimators proposed by Besson et al. [2] in case of single component model. We propose a multicomponent random amplitude chirp model in Section 3 and extend our results of Section 2 in this case. Extensive numerical experiments and the analysis of a real dataset are presented in Section 4 and Section 5, respectively. Finally, we conclude the paper in Section 6.

2. Estimation of Parameters of Random Amplitude Single Chirp Model

In this section, we discuss the problem of estimation of unknown parameters, namely, the frequency and chirp rate present in model (1). The following assumptions on the random amplitude, the additive error and the true values of the parameters are required.

Assumption 1. *The multiplicative error $\{\alpha(t)\}$ is a sequence of independent and identically distributed (i.i.d.) real-valued random variables with mean μ_α, variance σ_α^2, $\mu_\alpha \neq 0$ and $\sigma_\alpha^2 > 0$. The fourth moment of $\{\alpha(t)\}$ exists.*

Assumption 2. *The additive error $\{e(t)\}$ is a sequence of complex-valued i.i.d. random variables with mean zero and variance σ^2. Write $e(t) = e_R(t) + ie_I(t)$, then $\{e_R(t)\}$ and $\{e_I(t)\}$ are i.i.d. $(0, \frac{\sigma^2}{2})$, have finite fourth moment γ and are independently distributed.*

Assumption 3. *$\{e(t)\}$ is assumed to be independent of $\{\alpha(t)\}$.*

Assumption 4. *(θ_1^0, θ_2^0) is an interior point of its parameter space $[0, \pi] \times [0, \pi]$, that is, the space of admissible parameter values.*

Note that in Assumptions 1 and 2 the existence of the fourth order moments has been assumed. The existence of fourth moment of the additive error has been assumed for constant amplitude chirp model also, see for example Nandi and Kundu [12]. Moreover, often the errors are assumed to be Gaussian, and in that case all the moments are finite, hence Assumptions 1 and 2 hold true.

We consider the same estimators as proposed by Besson et al. [2]. We do not assume any specific distribution of $\{e(t)\}$ like Besson et al. [2], where it is assumed that $\{e(t)\}$ is complex circular Gaussian process. Our aim is to estimate the parameters θ_1 and θ_2 given a sample of size n from model (1). Write $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and $\boldsymbol{\theta}^0$ be the true value of $\boldsymbol{\theta}$. Define

$$Q(\boldsymbol{\theta}) = \frac{1}{n}\left|\sum_{t=1}^{n} y^2(t)e^{-i2(\theta_1 t+\theta_2 t^2)}\right|^2. \tag{3}$$

Note that $Q(\boldsymbol{\theta})$ is the periodogram function of $y^2(t)$ with exponent replaced by twice the usual periodogram exponent. The unknown parameters θ_1 and θ_2 are estimated by maximizing $Q(\boldsymbol{\theta})$. Let $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2)$ be the maximizer of $Q(\boldsymbol{\theta})$, then

$$\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}). \tag{4}$$

If we write $y^2(t) = z(t)$, and $2\theta_1 = \beta_1$ and $2\theta_2 = \beta_2$, then $Q(\boldsymbol{\theta})$ is nothing but the usual periodogram function for chirp model. The explicit expressions for real and imaginary parts of $z(t)$, say $z_R(t)$ and $z_I(t)$ are given in Appendix A. Note that the maximization of $Q(\boldsymbol{\theta})$, as given in (4), can be obtained by any two-dimensional optimization method over a bounded region namely $[0, \pi] \times [0, \pi]$. We have used downhill-simplex method for this optimization.

Under Assumptions 1-4, we prove the strong consistency of the proposed estimators and derive the asymptotic distribution. To prove the consistency, we need that the random amplitude $\alpha(t)$ has non-zero mean and finite variance. The existence of fourth moment is required to develop the asymptotic distribution. We state the results in the following theorems. Theorem 2.1 is proved in Appendix A and Theorem 2.2 is in Appendix B.

Theorem 2.1. *Under Assumptions 1-4, $\widehat{\theta}_1$ and $\widehat{\theta}_2$ defined in* (4)*, are strongly consistent estimators of θ_1^0 and θ_2^0, respectively.*

Theorem 2.2. *Under Assumptions 1-4, as $n \to \infty$,*

$$(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} \xrightarrow{d} \mathcal{N}_2(\mathbf{0}, 4(\sigma_\alpha^2 + \mu_\alpha^2)^2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Sigma}^{-1})$$

where with $C_\alpha = 8(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + \frac{1}{2}\gamma + \frac{1}{8}\sigma^4$,

$$\mathbf{D} = \begin{pmatrix} n^{-\frac{3}{2}} & 0 \\ 0 & n^{-\frac{5}{2}} \end{pmatrix}, \quad \boldsymbol{\Sigma} = \frac{2(\sigma_\alpha^2 + \mu_\alpha^2)^2}{3} \begin{pmatrix} 1 & 1 \\ 1 & \frac{16}{15} \end{pmatrix}, \quad \boldsymbol{\Gamma} = C_\alpha \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

and $\xrightarrow{d}$ denotes convergence in distribution.

Remark 1. It can be shown explicitly that the asymptotic variances of $\widehat{\boldsymbol{\theta}}_1$ and $\widehat{\boldsymbol{\theta}}_2$ are

$$\begin{aligned} \mathrm{Var}(\widehat{\boldsymbol{\theta}}_1) &= \frac{93}{n^3(\sigma_\alpha^2 + \mu_\alpha^2)^2}\Big[8(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + \frac{1}{2}\gamma + \frac{1}{8}\sigma^4\Big] \\ \mathrm{Var}(\widehat{\boldsymbol{\theta}}_2) &= \frac{135}{n^5(\sigma_\alpha^2 + \mu_\alpha^2)^2}\Big[8(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + \frac{1}{2}\gamma + \frac{1}{8}\sigma^4\Big]. \end{aligned}$$

In case the sequence of additive error is Gaussian with mean zero and variance σ^2, then the fourth moment is $3\sigma^4$ and asymptotic variances reduce to

$$\begin{aligned} \mathrm{Var}(\widehat{\boldsymbol{\theta}}_1) &= \frac{93}{n^3(\sigma_\alpha^2 + \mu_\alpha^2)^2}\Big[8(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + \frac{13}{8}\sigma^4\Big] \\ \mathrm{Var}(\widehat{\boldsymbol{\theta}}_2) &= \frac{135}{n^5(\sigma_\alpha^2 + \mu_\alpha^2)^2}\Big[8(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + \frac{13}{8}\sigma^4\Big]. \end{aligned}$$

The asymptotic variances of $\widehat{\theta}_1$ and $\widehat{\theta}_2$, obtained by Besson et al. [2] are

$$\mathrm{Var}(\widehat{\theta}_1) = \frac{96}{n^3}\frac{(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + .5\sigma^4}{(\sigma_\alpha^2 + \mu_\alpha^2)^2}, \quad \mathrm{Var}(\widehat{\theta}_2) = \frac{90}{n^5}\frac{(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + .5\sigma^4}{(\sigma_\alpha^2 + \mu_\alpha^2)^2}$$

Remark 2. The asymptotic variances of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ depend on the mean μ_α and variance σ_α^2 of the random amplitude and variance σ^2 and fourth moment γ of the additive error. According to Theorem 2.2, $\widehat{\theta}_1 = O_p(n^{-\frac{3}{2}})$ and $\widehat{\theta}_2 = O_p(n^{-\frac{5}{2}})$, where $O_p(.)$ denotes bounded in probability as $n \to \infty$. Therefore, for a given sample size, the chirp rate θ_2 can be estimated more accurately than the frequency θ_1.

Remark 3. The random amplitude sinusoidal signal (complex) model $y(t) = \alpha(t)e^{i\theta_1^0 t} + e(t)$ is a special case of model (1). In this case, the effective frequency does not change over time and is constant since the chirp rate $\theta_2^0 = 0$. The unknown frequency can be estimated by maximizing a similar function as $Q(\boldsymbol{\theta})$, defined in (3). The consistency and asymptotic normality of the estimator follow in the same way.

Remark 4. Random amplitude generalized chirp is a complex-valued model of the form $y(t) = \alpha(t)e^{i(\theta_1^0 t+\theta_2^0 t^2+\cdots+\theta_q^0 t^q)} + e(t)$. In this case, the chirp rate is not linear and the change in frequency is governed by the term $\theta_2^0 t^2 + \cdots + \theta_q^0 t^q$. The parameters are estimated using a periodogram like function of the squared signal. Under similar assumptions on $\alpha(t)$, $e(t)$ and true values of the parameter as in model (1), the consistency and asymptotic normality can be obtained.

3. Multi-component Model

In this section, we consider a general model, where instead of a single frequency and chirp rate pair, p such pairs are present. The model can be written as

$$y(t) = \sum_{k=1}^{p} \alpha_k(t)e^{i(\theta_{1k}^0 t+\theta_{2k}^0)} + e(t); \qquad t = 1, \ldots, n. \tag{5}$$

The sequence of additive errors $\{e(t)\}$ is complex-valued and satisfies Assumption 2 similar to the single component model. The sequences of multiplicative errors, $\{\alpha_1(t)\} \ldots \{\alpha_p(t)\}$ are sequences of i.i.d. errors and satisfy the following assumptions.

Assumption 5. *The multiplicative error corresponding to k-th component $\{\alpha_k(t)\}$ is a sequence of i.i.d. real-valued random variables with mean $\mu_{k\alpha}$, variance $\sigma_{k\alpha}^2$, and finite fourth moment, $\mu_{k\alpha} \neq 0$ and $\sigma_{k\alpha}^2 > 0$, $k = 1, \ldots, p$. Additionally, $\{\alpha_j(t)\}$ and $\{\alpha_k(t)\}$ for $j \neq k$ are independent.*

Assumption 6. *$\{e(t)\}$ is assumed to be independent of $\{\alpha_1(t)\}, \ldots, \{\alpha_p(t)\}$.*

Assumption 7. *$\{(\theta_{11}^0, \theta_{21}^0),\ (\theta_{12}^0, \theta_{22}^0), \ldots, (\theta_{1p}^0, \theta_{2p}^0)\}$ is an interior point of its parameter space and $(\theta_{1j}^0, \theta_{2j}^0) \neq (\theta_{1k}^0, \theta_{2k}^0)$ for $j \neq k$, $j, k = 1, \ldots, p$.*

The unknown parameters for multicomponent models are estimated by maximizing $Q(\boldsymbol{\theta})$ locally. We write $\boldsymbol{\theta}_k = (\theta_{1k}, \theta_{2k})$ and let $\boldsymbol{\theta}_k^0$ be the true value of $\boldsymbol{\theta}_k$. We use the notation N_k as a neighborhood of $\boldsymbol{\theta}_k^0$ such that for $j \neq k$, $\boldsymbol{\theta}_j^0 \notin N_k$. Estimate $\boldsymbol{\theta}_k$ as

$$\widehat{\boldsymbol{\theta}}_k = \arg \max_{(\theta_1,\theta_2)\in N_k} \frac{1}{n} \left| \sum_{t=1}^{n} y^2(t)e^{-i2(\theta_1 t+\theta_2 t^2)} \right|^2 = \arg \max_{(\theta_1,\theta_2)\in N_k} Q_p(\boldsymbol{\theta}),$$

where $y(t)$ is given in (5). Write $y^2(t) = z^p(t) = z_R^p(t) + iz_I^p(t)$; p is to denote a p-component model. The exact expression of $z_R^p(t)$ and $z_I^p(t)$ are given in Appendix C. Then, $\widehat{\boldsymbol{\theta}}_k$ is obtained as the argument of the maximum of

$$\begin{aligned} Q_p(\boldsymbol{\theta}) &= \frac{1}{n}\left[\sum_{t=1}^{n}\left\{z_R^p(t)\cos(2(\theta_1 t+\theta_2 t^2)) + z_I^p(t)\sin(2(\theta_1 t+\theta_2 t^2))\right\}\right]^2 + \\ & \frac{1}{n}\left[\sum_{t=1}^{n}\left\{-z_R^p(t)\sin(2(\theta_1 t+\theta_2 t^2)) + z_I^p(t)\cos(2(\theta_1 t+\theta_2 t^2))\right\}\right]^2 \end{aligned}$$

in N_k. Note that in this case to compute the estimators of the unknown parameters, we need to solve p two-dimensional optimization problems over bounded region and we have used the downhill-simplex method in simulation. Now we have the following results.

THEOREM 3.1. *Under Assumptions 2, and 5-7, $\widehat{\boldsymbol{\theta}}_k$, which maximizes $Q_p(\boldsymbol{\theta})$ in N_k, is a strongly consistent estimator of $\boldsymbol{\theta}_k^0$.*

THEOREM 3.2. *Under Assumptions 2, and 5-7, as $n \to \infty$*

$$(\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k^0)\mathbf{D}^{-1} \xrightarrow{d} \mathcal{N}_2(\mathbf{0}, 4(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2)^2 \boldsymbol{\Sigma}_k^{-1}\boldsymbol{\Gamma}_k\boldsymbol{\Sigma}_k^{-1})$$

where with $C_{k\alpha} = 8(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2)\sigma^2 + \frac{1}{2}\gamma + \frac{1}{8}\sigma^4$; $\mathbf{D}$ is same as defined in previous section and

$$\boldsymbol{\Sigma}_k = \frac{2(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2)^2}{3}\begin{pmatrix} 1 & 1 \\ 1 & \frac{16}{15} \end{pmatrix}, \quad \boldsymbol{\Gamma}_k = C_{k\alpha}\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

THEOREM 3.3. *Under Assumptions 2, and 5-7, $(\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k^0)\mathbf{D}^{-1}$ and $(\widehat{\boldsymbol{\theta}}_j - \boldsymbol{\theta}_j^0)\mathbf{D}^{-1}$ for $k \neq j$ are asymptotically independently distributed.*

The proof of Theorem 3.1 is given in Appendix C. The proof of Theorem 3.2 goes along the same line as the proof of Theorem 2.2 once we have equivalent results of Lemma 5 (see Appendix A) for the multicomponent model (5) using $z_R^p(t)$ and $z_I^p(t)$ instead of $z_R(t)$ and $z_I(t)$, respectively. A lemma, similar to Lemma 5, has been stated and proved in Appendix C as Lemma 6. Therefore, the proof of Theorem 3.2 is omitted to avoid repetition. Theorem 3.3 is proved in Appendix D.

4. Numerical Experiments

In this section, we present results of numerical experiments based on simulation. We consider two models, the first one is a random amplitude single chirp model and is given by

$$\text{Model 1}: \quad y(t) = \alpha(t)e^{i(\theta_1^0 t + \theta_2^0 t^2)} + e(t), \quad t = 1, \ldots, n \tag{6}$$

with parameter values $\theta_1^0 = 1.0$, $\theta_2^0 = 0.15$ and $\alpha(t) \sim \mathcal{N}(5, \sigma_\alpha^2)$ and $e_R(t) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$ and $e_I(t) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$ and $\{e_R(t)\}$ and $\{e_I(t)\}$ are independently distributed. The second model is a two-component random amplitude chirp model ($p = 2$ in (5))

$$\text{Model 2}: \quad y(t) = \alpha_1(t) e^{i(\theta_{11}^0 t + \theta_{21}^0 t^2)} + \alpha_2(t) e^{i(\theta_{12}^0 t + \theta_{22}^0 t^2)} + e(t), \quad t = 1, \ldots, n \tag{7}$$

with parameter values $\theta_{11}^0 = 1.0$, $\theta_{21}^0 = 0.1$, $\theta_{12}^0 = 2.0$, $\theta_{22}^0 = 0.2$. The random amplitudes $\alpha_1(t) \sim \mathcal{N}(6, \sigma_{1\alpha}^2)$ and $\alpha_2(t) \sim \mathcal{N}(5, \sigma_{2\alpha}^2)$ and $e_R(t)$ and $e_I(t)$ are same as single chirp model and $\alpha_1(t)$ and $\alpha_2(t)$ are independently distributed. We have taken the value of the frequency/chirp rate to be much smaller than the initial frequency as it represents the rate of change and is comparatively small in general. We would like to see how the estimator discussed in the paper performs for different values of sample size n, additive error variance σ^2 and multiplicative error variance σ_α^2 ($\sigma_{1\alpha}^2$ and $\sigma_{2\alpha}^2$ for two component model). That is, in case of model (6), we are interested in estimators of θ_1^0 and θ_2^0 and in case model (7), θ_{11}^0, θ_{21}^0, θ_{12}^0 and θ_{22}^0. For simulation, we consider $n = 100, 200$ and 500; $\sigma^2 = 0.01, 0.1, 0.5$ and 1.0; $\sigma_\alpha^2 = 0.5, 1.0$ and 2.0; $\sigma_{1\alpha}^2, \sigma_{2\alpha}^2 = 0.5, 1.0$ and 2.0.

In order to generate a sample of size n for each combination of σ^2 and σ_α^2 in case of model (6), we first generate $e(t)$ and $\alpha(t)$ and then a sample $y(1), \ldots, y(n)$ using the true values of the parameters. The parameters θ_1 and θ_2 are estimated by maximizing $Q(\boldsymbol{\theta})$, defined in (3). Similarly, in case of model (7), we first generate $e(t)$, $\alpha_1(t)$ and $\alpha_2(t)$ using σ^2, $\sigma_{1\alpha}^2$ and $\sigma_{2\alpha}^2$, respectively, and then obtain the sample $y(1), \ldots, y(n)$ based on the true parameter values. The parameters are estimated by maximizing $Q_p(\boldsymbol{\theta}), p = 2$ locally, that is, component-wise. The maximization of the criterion function $Q(\boldsymbol{\theta})$ in case of model (6) and $Q_p(\boldsymbol{\theta})$ in case of model (7) has been carried out using downhill simplex method. For both the models, the true values of the parameters have been used as the initial estimators. *We have checked the algorithm from different initial estimators and results are similar to the case when the algorithm starts from the true values. Therefore, we have reported results with initial estimators as the true values.* A three-dimensional plot of $Q(\boldsymbol{\theta})$ reveals that the local maximizers are very close to the true values of the parameters. Therefore, the true values have been used as the initial estimators. We observe that the process of estimation of parameters for model (6) involves a two-dimensional optimization, whereas for model (7), it involves two, two-dimensional optimizations. In case of a p-component model, it requires p, two-dimensional optimizations.

We replicate the process of data generation and estimation of parameters 5000 times and calculated the average estimate and mean squared error (MSE) for each parameter estimates over these 5000 replications. The average estimates are very close to the true parameter values in all the cases considered here, thus biases are negligible and they are not reported. The MSE and asymptotic variances (ASYM) for the frequency parameter θ_1 in log 10 scale are presented in Figure 1 and for chirp rate parameter θ_2 are given in Figure 2 in case of single chirp model (6). Three plots in each figure corresponds to $\sigma_\alpha^2 = .5$, 1.0 and 2.0. The ASYMs of the proposed estimators based on Theorems 2.2 and Cramer Rao bounds (CRB) proposed by Besson et al [2] are also plotted in Figures 1-2 for comparison. The results for two-component model (7) are presented in Figures 3-6. In case of model (7), CRBs are not available, therefore, asymptotic variances obtained from Theorem 3.2 are compared in Figures 3-6.

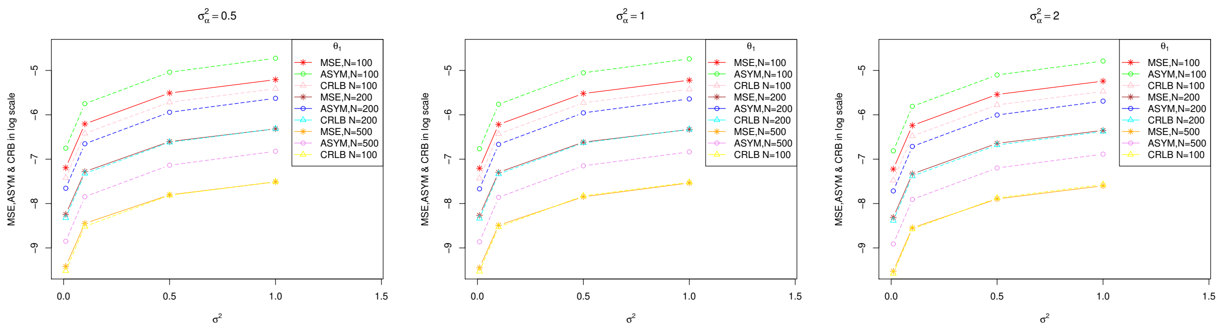

FIGURE 1. Model 1: MSE, ASYM and CRB of θ_1 in log scale for different samples sizes and additive error variances σ^2.

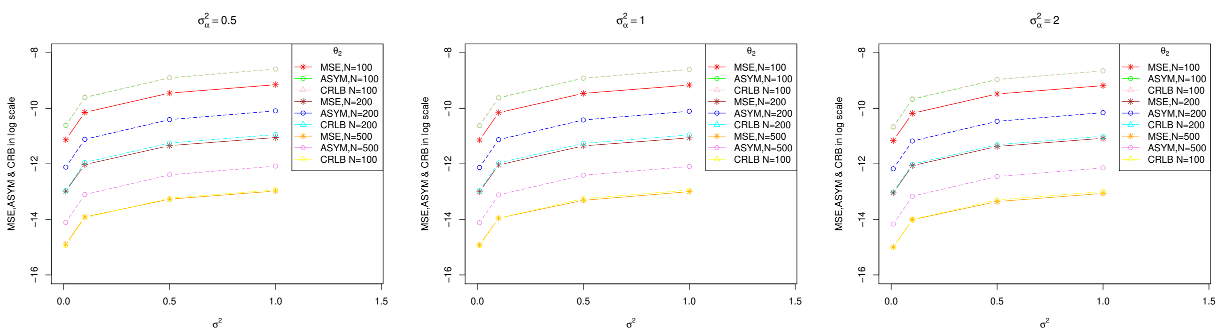

FIGURE 2. Model 1: MSE, ASYM and CRB of θ_2 in log scale for different samples sizes and additive error variances σ^2.

The important findings of the above numerical experiments are as follows;

- As the sample size increases, the performance of the estimator improves. The biases and MSEs decreases as the sample size increases.
- MSE increases as the additive error variance σ^2 increases.

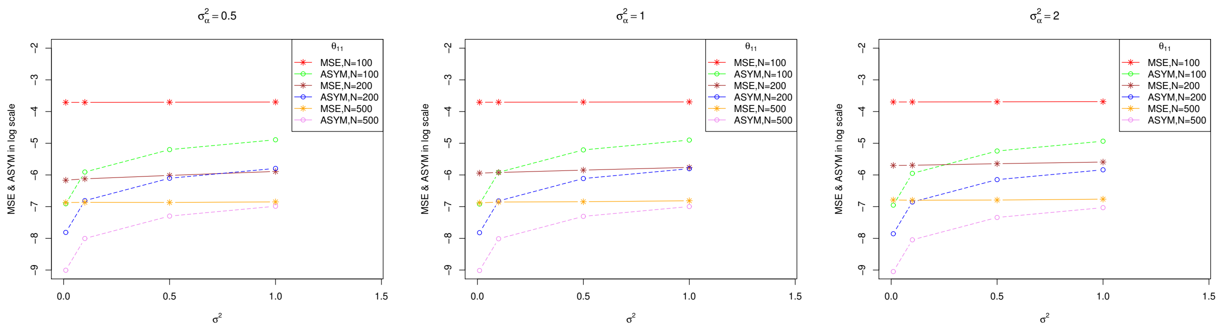

FIGURE 3. Model 2: MSE and ASYM of θ_{11} in log scale for different samples sizes and additive error variances σ^2.

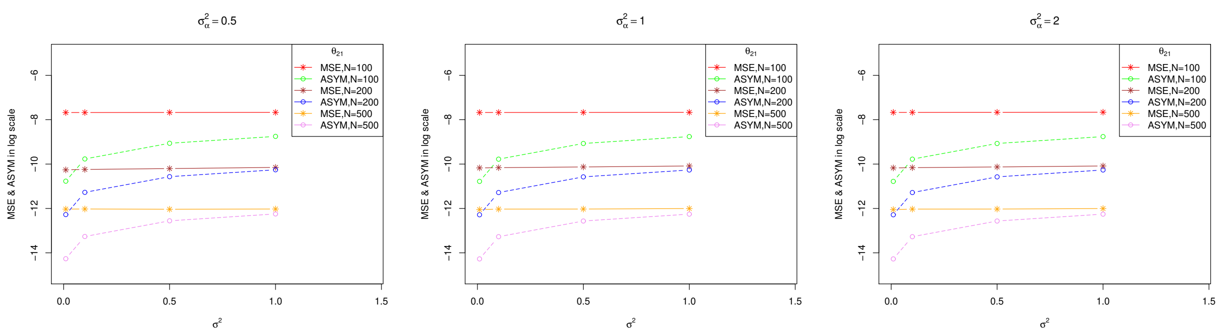

FIGURE 4. Model 2: MSE and ASYM of θ_{21} in log scale for different samples sizes and additive error variances σ^2.

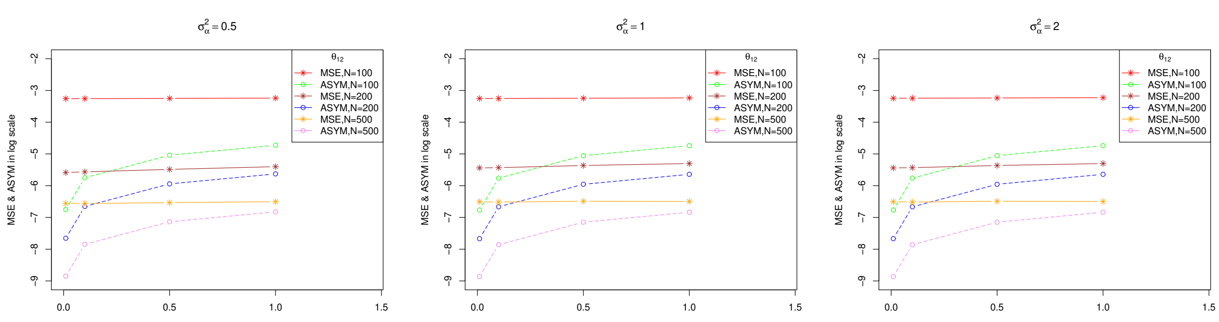

FIGURE 5. Model 2: MSE and ASYM of θ_{12} in log scale for different samples sizes and additive error variances σ^2.

- MSE decreases as the variance of random amplitude, σ_α^2 increases in case of model (6). The same is also observed in case of (7) as $\sigma_{1\alpha}^2$ and $\sigma_{2\alpha}^2$ increase. The explanation behind this is that the asymptotic variances decrease as σ_α^2 increases.

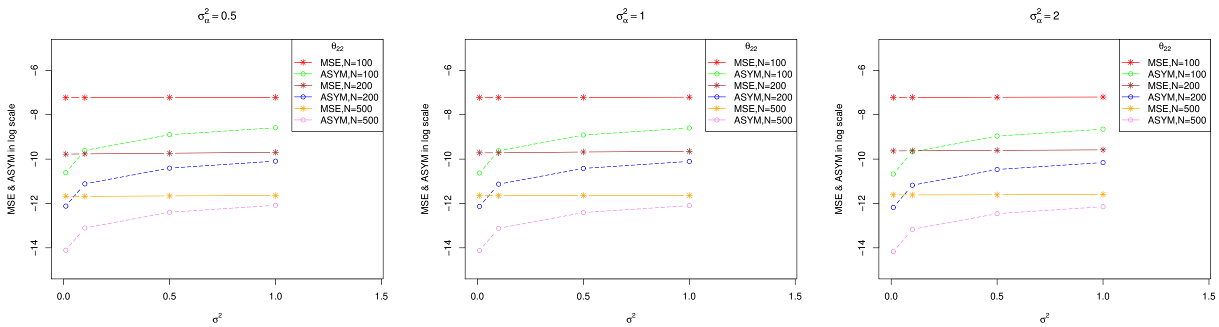

FIGURE 6. Model 2: MSE and ASYM of θ_{22} in log scale for different samples sizes and additive error variances σ^2.

- The performances of the estimator are not reasonably good if $E[\alpha^2(t)] = (\sigma_\alpha^2 + \mu_\alpha^2)$ is small. The means of random amplitudes are kept fixed and variances were varied. For the two component model (7), if $(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2)$ is small for a particular component, say kth then the corresponding estimators behave badly for small samples in terms of biases as well as MSEs.
- MSEs are very close to the asymptotic variances and Cramer Rao bounds in most of the cases considered and they become closer as the sample size increases.
- The signal to noise ratio (SNR) in single chirp case is $(\mu_\alpha^2 + \sigma_\alpha^2)/\sigma^2$. MSE and ASYM are plotted against SNR when the sample size is 500 in Figure 7. As SNR increases, MSE decreases and quite close to asymptotic variances.

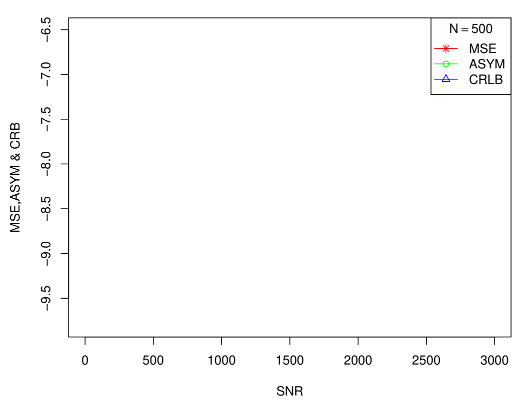

FIGURE 7. MSE and ASYM of θ_1 in log 10 scale against SNR when $N = 500$ in case of Model 1.

5. Concluding Remarks

In this paper, we discuss an equivalent estimator of the non-linear least squares estimator for the frequency and chirp rate parameters of random amplitude chirp signal. It has been shown that the estimators are strongly consistent and asymptotically normally distributed. We have proposed a multi-component random amplitude model and use the same estimator locally. It is implemented step by step. Numerical experiments based on simulation are reported to see the small sample performances. A real small duration speech dataset has been analyzed using multi-component random amplitude chirp model and the performances are quite satisfactory in terms of biases and MSEs.

Acknowledgements

The authors would like to thank three unknown reviewers and the associate editor for their constructive comments which have helped to improve the manuscript significantly. Part of the work of the second author has been funded by the grant no. MTR/2018/000179 of the Science and Engineering Research Board, Government of India.

Appendix A

Write $z(t) = y^2(t) = z_R(t) + iz_I(t)$, then

$$\begin{aligned} z_R(t) &= \alpha^2(t)\cos(2(\theta_1^0 t + \theta_2^0 t^2)) + (e_R^2(t) - e_I^2(t)) \\ &\quad +2\alpha(t)e_R(t)\cos(\theta_1^0 t + \theta_2^0 t^2) - 2\alpha(t)e_I(t)\sin(\theta_1^0 t + \theta_2^0 t^2), \qquad (8) \\ z_I(t) &= \alpha^2(t)\sin(2(\theta_1^0 t + \theta_2^0 t^2)) + 2\alpha(t)e_I(t)\cos(\theta_1^0 t + \theta_2^0 t^2) \\ &\quad +2\alpha(t)e_R(t)\sin(\theta_1^0 t + \theta_2^0 t^2) + 2e_R(t)e_I(t). \qquad (9) \end{aligned}$$

Lemma 1. *(Lahiri et al. [11]) If $(\theta_1, \theta_2) \in (0,\pi)\times(0,\pi)$, then except for a countable number of points, the following results are true.*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \cos(\theta_1 t + \theta_2 t^2) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sin(\theta_1 t + \theta_2 t^2) = 0,$$

$$\lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \cos^2(\theta_1 t + \theta_2 t^2) = \lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \sin^2(\theta_1 t + \theta_2 t^2) = \frac{1}{2(k+1)},$$

$$\lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \cos(\theta_1 t + \theta_2 t^2)\sin(\theta_1 t + \theta_2 t^2) = 0, \quad k = 0,1,2.$$

Lemma 2. *Let $\widehat{\boldsymbol\theta} = (\widehat\theta_1, \widehat\theta_2)$ be an estimate of $\boldsymbol\theta^0 = (\theta_1^0, \theta_2^0)$ that maximizes $Q(\boldsymbol\theta)$, defined in* (3) *and for any $\epsilon > 0$, let $S_\epsilon = \big\{\boldsymbol\theta : |\boldsymbol\theta - \boldsymbol\theta^0| > \epsilon\big\}$ for some fixed $\boldsymbol\theta^0 \in (0,\pi)\times(0,\pi)$. If for any $\epsilon > 0$,*

$$\overline{\lim}_{n\to\infty} \sup_{S_\epsilon} \frac{1}{n}\big[Q(\boldsymbol\theta) - Q(\boldsymbol\theta^0)\big] \le 0, \quad a.s. \tag{10}$$

then as $n\to\infty$, $\widehat{\boldsymbol\theta} \to \boldsymbol\theta^0$ a.s., that is, $\widehat\theta_1 \to \theta_1^0$ and $\widehat\theta_2 \to \theta_2^0$ a.s. Here, $\overline{\lim}$ denotes the limit supremum.

Proof of Lemma 2: We write $\widehat{\boldsymbol\theta}$ by $\widehat{\boldsymbol\theta}_n$ and $Q(\boldsymbol\theta)$ by $Q_n(\boldsymbol\theta)$ to emphasize that these quantities depend on n. Suppose (10) is true but $\widehat{\boldsymbol\theta}_n$ does not converges to $\boldsymbol\theta^0$ as $n\to\infty$. Then, there exists an $\epsilon > 0$ and a subsequence $\{n_k\}$ of $\{n\}$ such that $|\widehat{\boldsymbol\theta}_{n_k} - \boldsymbol\theta^0| > \epsilon$ for $k = 1,2,\ldots$. Therefore, $\widehat{\boldsymbol\theta}_{n_k} \in S_\epsilon$ for all $k = 1,2,\ldots$. By definition, $\widehat{\boldsymbol\theta}_{n_k}$ is the estimator of $\boldsymbol\theta^0$ that maximizes $Q_n(\boldsymbol\theta)$ when $n = n_k$. This implies that

$$Q_{n_k}(\widehat{\boldsymbol\theta}_{n_k}) \ge Q_{n_k}(\boldsymbol\theta^0) \Rightarrow \frac{1}{n_k}\Big[Q_{n_k}(\widehat{\boldsymbol\theta}_{n_k}) - Q_{n_k}(\boldsymbol\theta^0)\Big] \ge 0.$$

Therefore, $\overline{\lim}_{n\to\infty} \sup_{\boldsymbol\theta_{n_k}\in S_\epsilon} \frac{1}{n_k}\big[Q_{n_k}(\widehat{\boldsymbol\theta}_{n_k}) - Q_{n_k}(\boldsymbol\theta^0)\big] \ge 0$, which contradicts inequality (10). Hence, the result follows. ∎

Lemma 3. *(Nandi and Kundu [12]) Let $\{e(t)\}$ be a sequence of i.i.d. real-valued random variables with mean zero and finite variance $\sigma^2 > 0$, then as $n\to\infty$,*

$$\sup_{a,b}\left|\frac{1}{n}\sum_{t=1}^{n} e(t)\cos(at)\cos(bt^2)\right| \xrightarrow{a.s.} 0.$$

The result is true for all combinations of sine and cosine functions.

Corollary of Lemma 3:

$$\sup_{a,b}\left|\frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k e(t)\cos(at)\cos(bt^2)\right| \xrightarrow{a.s.} 0,$$

The result is true for all combinations of sine and cosine functions.

Lemma 4. *(Grover et al. [9]) If $(\beta_1, \beta_2) \in (0,\pi)\times(0,\pi)$, then except for a countable number of points, the following results hold.*

$$\lim_{n\to\infty}\frac{1}{n^{\frac{2m+1}{2}}}\sum_{t=1}^{n} t^m\cos(\beta_1 t+\beta_2 t^2) = \lim_{n\to\infty}\frac{1}{n^{\frac{2m+1}{2}}}\sum_{t=1}^{n} t^m\sin(\beta_1 t+\beta_2 t^2) = 0, \quad m = 0,1,2$$

Lemma 5. *Under Assumptions 1-3, the following results are true for model* (1)*.*

$$\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_R(t)\cos(2\theta_1^0 t+2\theta_2^0 t^2) \xrightarrow{a.s} \frac{1}{2(m+1)}(\sigma_\alpha^2+\mu_\alpha^2), \tag{11}$$

$$\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_I(t)\cos(2\theta_1^0 t+2\theta_2^0 t^2) \xrightarrow{a.s} 0, \tag{12}$$

$$\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_R(t)\sin(2\theta_1^0 t+2\theta_2^0 t^2) \xrightarrow{a.s} 0, \tag{13}$$

$$\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_I(t)\sin(2\theta_1^0 t+2\theta_2^0 t^2) \xrightarrow{a.s} \frac{1}{2(m+1)}(\sigma_\alpha^2+\mu_\alpha^2), \tag{14}$$

for $m = 0,1,\ldots,4$.

Proof of Lemma 5: We first note that $E[e_R(t)e_I(t)] = 0$ and $\text{Var}[e_R(t)e_I(t)] = \frac{\sigma^4}{2}$ and so $\{e_R(t)e_I(t)\} \overset{i.i.d.}{\sim} (0, \frac{\sigma^4}{2})$. Similarly, under Assumptions 1-3, we can show that

$$\{e_R^2(t)-e_I^2(t)\} \overset{i.i.d.}{\sim} (0, 2\gamma-\frac{\sigma^4}{2}), \quad \{\alpha(t)e_R(t)\} \overset{i.i.d.}{\sim} (0, (\sigma_\alpha^2+\mu_\alpha^2)\frac{\sigma^2}{2}),$$
$$\{\alpha(t)e_I(t)\} \overset{i.i.d.}{\sim} (0, (\sigma_\alpha^2+\mu_\alpha^2)\frac{\sigma^2}{2}). \tag{15}$$

$$
\begin{aligned}
& \frac{1}{n^{m+1}} \sum_{t=1}^{n} t^m z_R(t) \cos(2\theta_1^0 t + 2\theta_2^0 t^2) \\
= \quad & \frac{1}{n^{m+1}} \sum_{t=1}^{n} t^m \alpha^2(t) \cos^2(2\theta_1^0 t + 2\theta_2^0 t^2) + \frac{1}{n^{m+1}} \sum_{t=1}^{n} t^m (e_R^2(t) - e_I^2(t)) \cos(2\theta_1^0 t + 2\theta_2^0 t^2) \\
& \quad + \frac{2}{n^{m+1}} \sum_{t=1}^{n} t^m \alpha(t) e_R(t) \cos(\theta_1^0 t + \theta_2^0 t^2) \cos(2\theta_1^0 t + 2\theta_2^0 t^2) \\
& \quad + \frac{2}{n^{m+1}} \sum_{t=1}^{n} t^m \alpha(t) e_I(t) \sin(\theta_1^0 t + \theta_2^0 t^2) \cos(2\theta_1^0 t + 2\theta_2^0 t^2).
\end{aligned}
$$

The second term converges to zero as $n \to \infty$ using Corollary of Lemma 3 as $(e_R^2(t) - e_I^2(t))$ is a mean zero and finite variance process. Similarly, the third and fourth terms also converge to zero as $n \to \infty$ using (15). Now the first term can be written as

$$
\begin{aligned}
& \frac{1}{n^{m+1}} \sum_{t=1}^{n} t^m \alpha^2(t) \cos^2(2\theta_1^0 t + 2\theta_2^0 t^2) \\
= \quad & \frac{1}{n^{m+1}} \left[\sum_{t=1}^{n} t^m \{\alpha^2(t) - E[\alpha^2(t)]\} \cos^2(2\theta_1^0 t + 2\theta_2^0 t^2) + \sum_{t=1}^{n} t^m E[\alpha^2(t)] \cos^2(2\theta_1^0 t + 2\theta_2^0 t^2) \right] \\
\xrightarrow{a.s.} \quad & 0 + \frac{1}{2(m+1)} E[\alpha^2(t)] \\
= \quad & \frac{1}{2(m+1)} (\sigma_\alpha^2 + \mu_\alpha^2),
\end{aligned}
$$

using Lemmas 1 and 5. Note that here we have used the fact that the fourth moment of $\alpha(t)$ exists. In a similar way, (12), (13) and (14) can be proved. ∎

Proof of Theorem 2.1: Expanding $Q(\boldsymbol{\theta})$ and using $y^2(t) = z(t) = z_R(t) + i z_I(t)$

$$
\begin{aligned}
\frac{1}{n}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] \quad = \quad & \Big[\frac{1}{n} \sum_{t=1}^{n} \Big\{ z_R(t) \cos(2\theta_1 t + 2\theta_2 t^2) + z_I(t) \sin(2\theta_1 t + 2\theta_2 t^2) \Big\}\Big]^2 \\
& + \Big[\frac{1}{n} \sum_{t=1}^{n} \Big\{ -z_R(t) \sin(2\theta_1 t + 2\theta_2 t^2) + z_I(t) \cos(2\theta_1 t + 2\theta_2 t^2) \Big\}\Big]^2 \\
& - \Big[\frac{1}{n} \sum_{t=1}^{n} \Big\{ z_R(t) \cos(2\theta_1^0 t + 2\theta_2^0 t^2) + z_I(t) \sin(2\theta_1^0 t + 2\theta_2^0 t^2) \Big\}\Big]^2 \\
& - \Big[\frac{1}{n} \sum_{t=1}^{n} \Big\{ -z_R(t) \sin(2\theta_1^0 t + 2\theta_2^0 t^2) + z_I(t) \cos(2\theta_1^0 t + 2\theta_2^0 t^2) \Big\}\Big]^2 \\
= \quad & S_1 + S_2 + S_3 + S_4.
\end{aligned}
$$

Using Lemma 5 with $m = 0$, we have

$$\frac{1}{n}\sum_{t=1}^{n} z_R(t)\cos(2\theta_1^0 t + 2\theta_2^0 t^2) \xrightarrow{a.s.} \frac{1}{2}(\sigma_\alpha^2 + \mu_\alpha^2),$$
$$\frac{1}{n}\sum_{t=1}^{n} z_I(t)\cos(2\theta_1^0 t + 2\theta_2^0 t^2) \xrightarrow{a.s.} 0,$$
$$\frac{1}{n}\sum_{t=1}^{n} z_R(t)\sin(2\theta_1^0 t + 2\theta_2^0 t^2) \xrightarrow{a.s.} 0,$$
$$\frac{1}{n}\sum_{t=1}^{n} z_I(t)\sin(2\theta_1^0 t + 2\theta_2^0 t^2) \xrightarrow{a.s.} \frac{1}{2}(\sigma_\alpha^2 + \mu_\alpha^2).$$

Then

$$\lim_{n\to\infty} S_3 = -(\sigma_\alpha^2 + \mu_\alpha^2)^2 \quad \text{and} \quad \lim_{n\to\infty} S_4 = 0.$$

Now

$$\begin{aligned}
& \overline{\lim}_{n\to\infty} \sup_{S_\epsilon} S_1 \\
= \quad & \overline{\lim}_{n\to\infty} \sup_{S_\epsilon} \Big[\frac{1}{n}\sum_{t=1}^{n}\Big\{z_R(t)\cos(2\theta_1 t + 2\theta_2 t^2) + z_I(t)\sin(2\theta_1 t + 2\theta_2 t^2)\Big\}\Big]^2 \\
= \quad & \overline{\lim}_{n\to\infty} \sup_{S_\epsilon} \Big[\frac{1}{n}\sum_{t=1}^{n}\Big\{\alpha^2(t)\cos[2(\theta_1^0 - \theta_1)t + 2(\theta_2^0 - \theta_2)t^2] + 2e_R^2(t)e_I^2(t)\sin(2\theta_1 t + 2\theta_2 t^2) \\
& (e_R^2(t) - e_I^2(t))\cos(2\theta_1 t + 2\theta_2 t^2) + 2\alpha(t)e_R(t)\cos[(2\theta_1^0 - \theta_1)t + (2\theta_2^0 - \theta_2)t^2] \\
& +2\alpha(t)e_I(t)\sin[(2\theta_1^0 - \theta_1)t + (2\theta_2^0 - \theta_2)t^2]\Big\}\Big]^2 \\
= \quad & \overline{\lim}_{n\to\infty} \sup_{|\boldsymbol{\theta}^0 - \boldsymbol{\theta}| > \epsilon} \Big[\frac{1}{n}\sum_{t=1}^{n}\Big\{(\alpha^2(t) - (\sigma_\alpha^2 + \mu_\alpha^2))\cos[2(\theta_1^0 - \theta_1)t + 2(\theta_2^0 - \theta_2)t^2] \\
& +2\alpha(t)e_R(t)\cos[(2\theta_1^0 - \theta_1)t + (2\theta_2^0 - \theta_2)t^2] + 2\alpha(t)e_I(t)\sin[(2\theta_1^0 - \theta_1)t + (2\theta_2^0 - \theta_2)t^2] \\
& +(\sigma_\alpha^2 + \mu_\alpha^2)\cos[2(\theta_1^0 - \theta_1)t + 2(\theta_2^0 - \theta_2)t^2]\Big\}\Big]^2
\end{aligned}$$

(The second and third terms are independent of $\boldsymbol{\theta}^0$ and they vanish using Lemma 3)

$\xrightarrow{a.s.} 0,$

using Lemma 1 and Lemma 3. Similarly, we can show that $\overline{\lim}_{n\to\infty} \sup_{S_\epsilon} S_2 \xrightarrow{a.s.} 0$. Therefore,

$$\overline{\lim}_{n\to\infty} \sup_{S_\epsilon} \frac{1}{n}\big[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\big] = \overline{\lim}_{n\to\infty} \sup_{S_\epsilon}\Big[S_1 + S_2 + S_3 + S_4\Big] \to -(\sigma_\alpha^2 + \mu_\alpha^2)^2 < 0 \ \text{ a.s.}$$

and using Lemma 2, $\widehat{\theta}_1$ and $\widehat{\theta}_2$ which maximize $Q(\boldsymbol{\theta})$ are strongly consistent estimators of θ_1^0 and θ_2^0. ∎

Appendix B

In this Appendix, we prove Theorem 2.2 which states the asymptotic distribution of the unknown parameters of model (1). The first order derivatives of $Q(\boldsymbol{\theta})$ with respect to θ_k, $k = 1, 2$ are as follows;

$$\frac{\partial Q(\boldsymbol{\theta})}{\partial \theta_k} = \frac{2}{n} f_1(\boldsymbol{\theta}) g_1(k;\boldsymbol{\theta}) + \frac{2}{n} f_2(\boldsymbol{\theta}) g_2(k;\boldsymbol{\theta}), \tag{16}$$

where

$$\begin{aligned}
f_1(\boldsymbol{\theta}) &= \sum_{t=1}^{n} z_R(t)\cos(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t)\sin(2\theta_1 t + 2\theta_2 t^2),\\
g_1(k;\boldsymbol{\theta}) &= \sum_{t=1}^{n} z_I(t) 2t^k \cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_R(t) 2t^k \sin(2\theta_1 t + 2\theta_2 t^2),\\
f_2(\boldsymbol{\theta}) &= \sum_{t=1}^{n} z_I(t)\cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_R(t)\sin(2\theta_1 t + 2\theta_2 t^2),\\
g_2(k;\boldsymbol{\theta}) &= -\sum_{t=1}^{n} z_I(t) 2t^k \sin(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_R(t) 2t^k \cos(2\theta_1 t + 2\theta_2 t^2).
\end{aligned}$$

Observe that using Lemma 5, it immediately follows that

$$(a)\ \lim_{n\to\infty} \frac{1}{n} f_1(\boldsymbol{\theta}^0) = (\sigma_\alpha^2 + \mu_\alpha^2) \quad \text{and} \quad (b)\ \lim_{n\to\infty} \frac{1}{n} f_2(\boldsymbol{\theta}^0) = 0 \ \text{ a.s.} \tag{17}$$

Therefore, for large n, $\dfrac{\partial Q(\boldsymbol{\theta})}{\partial \theta_k} = \dfrac{2}{n} f_1(\boldsymbol{\theta}) g_1(k;\boldsymbol{\theta})$ ignoring the second term in (16) which involves $f_2(\boldsymbol{\theta})$. The second order derivatives with respect to θ_k for $k = 1, 2$ are

$$
\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_k^2} &= \frac{2}{n}\left\{-\sum_{t=1}^{n} z_R(t) 2t^k \sin(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t) 2t^k \cos(2\theta_1 t + 2\theta_2 t^2)\right\}^2 \\
&+ \frac{2}{n}\left\{\sum_{t=1}^{n} z_R(t) \cos(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t) \sin(2\theta_1 t + 2\theta_2 t^2)\right\} \times \\
&\left\{-\sum_{t=1}^{n} z_R(t) 4t^{2k} \cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_I(t) 4t^{2k} \sin(2\theta_1 t + 2\theta_2 t^2)\right\} \\
&+ \frac{2}{n}\left\{-\sum_{t=1}^{n} z_R(t) 2t^k \cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_I(t) 2t^k \sin(2\theta_1 t + 2\theta_2 t^2)\right\}^2 \\
&+ \frac{2}{n}\left\{-\sum_{t=1}^{n} z_R(t) \sin(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t) \cos(2\theta_1 t + 2\theta_2 t^2)\right\} \times \\
&\left\{\sum_{t=1}^{n} z_R(t) 4t^{2k} \sin(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_I(t) 4t^{2k} \cos(2\theta_1 t + 2\theta_2 t^2)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_1 \partial \theta_2} &= \frac{2}{n}\left\{\sum_{t=1}^{n} z_R(t) \cos(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t) \sin(2\theta_1 t + 2\theta_2 t^2)\right\} \times \\
&\left\{-\sum_{t=1}^{n} z_R(t) 4t^3 \cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_I(t) 4t^3 \sin(2\theta_1 t + 2\theta_2 t^2)\right\} \\
&+ \frac{2}{n}\left\{-\sum_{t=1}^{n} z_R(t) 2t^2 \sin(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t) 2t^2 \cos(2\theta_1 t + 2\theta_2 t^2)\right\} \times \\
&\left\{-\sum_{t=1}^{n} z_R(t) 2t \sin(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t) 2t \cos(2\theta_1 t + 2\theta_2 t^2)\right\} \\
&+ \frac{2}{n}\left\{-\sum_{t=1}^{n} z_R(t) \sin(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I(t) \cos(2\theta_1 t + 2\theta_2 t^2)\right\} \times \\
&\left\{\sum_{t=1}^{n} z_R(t) 4t^3 \sin(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_I(t) 4t^3 \cos(2\theta_1 t + 2\theta_2 t^2)\right\} \\
&+ \frac{2}{n}\left\{-\sum_{t=1}^{n} z_R(t) 2t^2 \cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_I(t) 2t^2 \sin(2\theta_1 t + 2\theta_2 t^2)\right\} \times
\end{aligned}
$$

$$\left\{-\sum_{t=1}^{n} z_R(t)2t\cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_I(t)2t\sin(2\theta_1 t + 2\theta_2 t^2)\right\}.$$

Now, we can show the following using Lemma 5, for $m = 0, \ldots, 4$,

$$\lim_{n\to\infty} \frac{1}{n^3}\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_1^2}\bigg|_{\boldsymbol{\theta}^0} = -\frac{2}{3}(\sigma_\alpha^2 + \mu_\alpha^2)^2, \tag{18}$$

$$\lim_{n\to\infty} \frac{1}{n^5}\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_2^2}\bigg|_{\boldsymbol{\theta}^0} = -\frac{32}{45}(\sigma_\alpha^2 + \mu_\alpha^2)^2, \tag{19}$$

$$\lim_{n\to\infty} \frac{1}{n^4}\frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_1 \partial \theta_2}\bigg|_{\boldsymbol{\theta}^0} = -\frac{2}{3}(\sigma_\alpha^2 + \mu_\alpha^2)^2. \tag{20}$$

Write $Q'(\boldsymbol{\theta}) = \left(\frac{\partial Q(\boldsymbol{\theta})}{\partial \theta_1}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \theta_2}\right)$ and $Q''(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_1^2} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 Q(\boldsymbol{\theta})}{\partial \theta_2^2} \end{pmatrix}$. Define a diagonal matrix $\mathbf{D} = \text{diag}\left\{\frac{1}{n^{\frac{3}{2}}}, \frac{1}{n^{\frac{5}{2}}}\right\}$. Expand $Q'(\widehat{\boldsymbol{\theta}})$ using bivariate Taylor series expansion around $\boldsymbol{\theta}^0$,

$$Q'(\widehat{\boldsymbol{\theta}}) - Q'(\boldsymbol{\theta}^0) = (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)Q''(\bar{\boldsymbol{\theta}}),$$

where $\bar{\boldsymbol{\theta}}$ is a point on the line joining $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$. As $\widehat{\boldsymbol{\theta}}$ maximizes $Q(\boldsymbol{\theta})$, $Q'(\widehat{\boldsymbol{\theta}}) = 0$, the above equation can be written as

$$\begin{aligned} -[Q'(\boldsymbol{\theta}^0)\mathbf{D}] &= (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1}\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D} \\ \Rightarrow\quad (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} &= -[Q'(\boldsymbol{\theta}^0)\mathbf{D}][\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}]^{-1}, \end{aligned}$$

provided $[\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}]$ is an invertible matrix a.s. Because $\widehat{\boldsymbol{\theta}} \to \boldsymbol{\theta}^0$ a.s. and $Q''(\boldsymbol{\theta})$ is a continuous function of $\boldsymbol{\theta}$, using continuous mapping theorem, we have

$$\lim_{n\to\infty}[\mathbf{D}Q''(\bar{\boldsymbol{\theta}})\mathbf{D}] = \lim_{n\to\infty}[\mathbf{D}Q''(\boldsymbol{\theta}^0)\mathbf{D}] = -\boldsymbol{\Sigma},$$

where $\boldsymbol{\Sigma}$ can be obtained using (18)-(20) as $\boldsymbol{\Sigma} = \frac{2(\sigma_\alpha^2 + \mu_\alpha^2)^2}{3}\begin{pmatrix} 1 & 1 \\ 1 & \frac{16}{15} \end{pmatrix}$. Using (17), the elements of $Q'(\boldsymbol{\theta}^0)\mathbf{D}$ are

$$\frac{1}{n^{\frac{3}{2}}}\frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \theta_1} = 2\frac{1}{n}f_1(\boldsymbol{\theta}^0)\frac{1}{n^{\frac{3}{2}}}g_1(1;\boldsymbol{\theta}^0) \text{ and} \frac{1}{n^{\frac{5}{2}}}\frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \theta_2} = 2\frac{1}{n}f_1(\boldsymbol{\theta}^0)\frac{1}{n^{\frac{5}{2}}}g_1(2;\boldsymbol{\theta}^0).$$

for large n. Therefore, to find the asymptotic distribution of $Q'(\boldsymbol{\theta}^0)\mathbf{D}$, we need to study the large sample distribution of $\frac{1}{n^{\frac{3}{2}}}g_1(1;\boldsymbol{\theta}^0)$ and $\frac{1}{n^{\frac{5}{2}}}g_1(2;\boldsymbol{\theta}^0)$. Replacing $z_R(t)$ and $z_I(t)$ in

$g_1(k;\boldsymbol{\theta}^0)$, $k=1,2$, we have

$$
\begin{aligned}
&\frac{1}{n^{\frac{2k+1}{2}}} g_1(k;\boldsymbol{\theta}^0) \\
= \;& \frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k z_I(t)\cos(2\theta_1^0 t + 2\theta_2^0 t^2) - \frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k z_R(t)\sin(2\theta_1^0 t + 2\theta_2^0 t^2) \\
= \;& \frac{4}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k e_R(t)e_I(t)\cos(2\theta_1^0 t + 2\theta_2^0 t^2) + \frac{4}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha(t)e_I(t)\cos(\theta_1^0 t + \theta_2^0 t^2) \\
& - \frac{4}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha(t)e_R(t)\sin(\theta_1^0 t + \theta_2^0 t^2) - \frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k (e_R^2(t) - e_I^2(t))\sin(2\theta_1^0 t + 2\theta_2^0 t^2).
\end{aligned}
$$

The random variables $e_R(t)e_I(t)$, $\alpha(t)e_R(t)$, $\alpha(t)e_I(t)$ and $(e_R^2(t) - e_I^2(t))$ are all mean zero finite variance random variables. Therefore, $E[\frac{1}{n^{\frac{3}{2}}} g_1(1;\boldsymbol{\theta}^0)] = 0$ and $E[\frac{1}{n^{\frac{5}{2}}} g_1(2;\boldsymbol{\theta}^0)] = 0$ for large n and all the terms above satisfy the Lindeberg-Feller's condition. So, $\frac{1}{n^{\frac{3}{2}}} g_1(1;\boldsymbol{\theta}^0)$ and $\frac{1}{n^{\frac{5}{2}}} g_1(2;\boldsymbol{\theta}^0)$ converge to normal distributions with zero mean and finite variances. In order to find the large sample covariance matrix of $Q'(\boldsymbol{\theta}^0)\mathbf{D}$, we first find the variances of $\frac{1}{n^{\frac{3}{2}}} g_1(1;\boldsymbol{\theta}^0)$ and $\frac{1}{n^{\frac{5}{2}}} g_1(2;\boldsymbol{\theta}^0)$ and their covariance.

$$
\begin{aligned}
&\mathrm{Var}\Big[\frac{1}{n^{\frac{3}{2}}} g_1(1;\boldsymbol{\theta}^0)\Big] \\
= \;& \mathrm{Var}\Big[\frac{4}{n^{\frac{3}{2}}}\sum_{t=1}^{n} t e_R(t)e_I(t)\cos(2\theta_1^0 t + 2\theta_2^0 t^2) + \frac{4}{n^{\frac{3}{2}}}\sum_{t=1}^{n} t\alpha(t)e_I(t)\cos(\theta_1^0 t + \theta_2^0 t^2) \\
& - \frac{4}{n^{\frac{3}{2}}}\sum_{t=1}^{n} t\alpha(t)e_R(t)\sin(\theta_1^0 t + \theta_2^0 t^2) - \frac{2}{n^{\frac{3}{2}}}\sum_{t=1}^{n} t(e_R^2(t) - e_I^2(t))\sin(2\theta_1^0 t + 2\theta_2^0 t^2)\Big] \\
= \;& E\Big[\frac{16}{n^3}\sum_{t=1}^{n} t^2 e_R^2(t)e_I^2(t)\cos^2(2\theta_1^0 t + 2\theta_2^0 t^2) + \frac{16}{n^3}\sum_{t=1}^{n} t^2\alpha^2(t)e_I^2(t)\cos^2(\theta_1^0 t + \theta_2^0 t^2) \\
& + \frac{16}{n^3}\sum_{t=1}^{n} t^2\alpha^2(t)e_R^2(t)\sin^2(\theta_1^0 t + \theta_2^0 t^2) + \frac{4}{n^3}\sum_{t=1}^{n} t^2(e_R^2(t) - e_I^2(t))^2\sin^2(2\theta_1^0 t + 2\theta_2^0 t^2)\Big] \\
& \Big[\text{The cross-product terms vanish due to Lemma 1 and independence of} \\
& \alpha(t),\ e_R(t) \text{ and } e_I(t).\Big] \\
\to \;& 16.\frac{\sigma^2}{2}.\frac{\sigma^2}{2}.\frac{1}{6} + 16.\frac{\sigma^2}{2}.(\sigma_\alpha^2 + \mu_\alpha^2).\frac{1}{6} + 16.\frac{\sigma^2}{2}.(\sigma_\alpha^2 + \mu_\alpha^2).\frac{1}{6} + 4.(2\gamma - \frac{\sigma^4}{2})\frac{1}{6} \\
= \;& \frac{8}{3}\big[(\sigma_\alpha^2 + \mu_\alpha^2)\sigma^2 + \frac{1}{2}\gamma + \frac{1}{8}\sigma^4\big].
\end{aligned}
$$

Similarly, we can show that for large n

$$\mathrm{Var}\Big[\frac{1}{n^{\frac{5}{2}}}g_1(2;\boldsymbol{\theta}^0)\Big] \to \frac{8}{5}\big[(\sigma_\alpha^2+\mu_\alpha^2)\sigma^2+\frac{1}{2}\gamma+\frac{1}{8}\sigma^4\big],$$
$$\mathrm{Cov}\Big[\frac{1}{n^{\frac{3}{2}}}g_1(1;\boldsymbol{\theta}^0), \frac{1}{n^{\frac{5}{2}}}g_1(2;\boldsymbol{\theta}^0)\Big] \to 2\big[(\sigma_\alpha^2+\mu_\alpha^2)\sigma^2+\frac{1}{2}\gamma+\frac{1}{8}\sigma^4\big].$$

Now, note that $Q'(\boldsymbol{\theta}^0)\mathbf{D}$ can be written as

$$Q'(\boldsymbol{\theta}^0)\mathbf{D} = \frac{2}{n}f_1(\boldsymbol{\theta}^0)\Big[\frac{1}{n^{\frac{3}{2}}}g_1(1;\boldsymbol{\theta}^0), \frac{1}{n^{\frac{5}{2}}}g_1(2;\boldsymbol{\theta}^0)\Big]. \tag{21}$$

Then, as $n\to\infty$, $\frac{2}{n}f_1(\boldsymbol{\theta}^0) \xrightarrow{a.s.} 2(\sigma_\alpha^2+\mu_\alpha^2)$ and

$$\Big[\frac{1}{n^{\frac{3}{2}}}g_1(1;\boldsymbol{\theta}^0), \frac{1}{n^{\frac{5}{2}}}g_1(2;\boldsymbol{\theta}^0)\Big] \xrightarrow{d} \mathcal{N}_2(\mathbf{0},\boldsymbol{\Gamma}),$$

where

$$\boldsymbol{\Gamma} = 8\big[(\sigma_\alpha^2+\mu_\alpha^2)\sigma^2+\frac{1}{2}\gamma+\frac{1}{8}\sigma^4\big]\begin{bmatrix}\frac{1}{3} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{5}\end{bmatrix}.$$

Therefore, using (17) and (21), Slutsky's theorem can be applied. Hence, as $n\to\infty$,

$$Q'(\boldsymbol{\theta}^0)\mathbf{D} \xrightarrow{d} \mathcal{N}_2(\mathbf{0}, 4(\sigma_\alpha^2+\mu_\alpha^2)^2\boldsymbol{\Gamma}),$$

and hence

$$(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0)\mathbf{D}^{-1} \xrightarrow{d} \mathcal{N}_2(\mathbf{0}, 4(\sigma_\alpha^2+\mu_\alpha^2)^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Sigma}^{-1}).$$

That proves the theorem. ∎

Appendix C

As defined in section 3, $z_R^p(t)$ and $z_I^p(t)$ are real and imaginary parts of $y^2(t)$ in case of multicomponent model (5) and are given as follows:

$$\begin{aligned} z_R^p(t) &= \sum_{k=1}^{p}\alpha_k^2(t)\cos(2(\theta_{1k}^0 t+\theta_{2k}^0 t^2)) + 2\sum_{k\neq j}\alpha_k(t)\alpha_j(t)\cos(\theta_{1k}^0 t+\theta_{1j}^0 t+\theta_{2k}^0 t^2+\theta_{2j}^0 t^2) \\ &\quad +(e_R^2(t)-e_I^2(t)) + 2\sum_{k=1}^{p}\alpha_k(t)\Big\{e_R(t)\cos(\theta_{1k}^0 t+\theta_{2k}^0 t^2) - e_I(t)\sin(\theta_{1k}^0 t+\theta_{2k}^0 t^2),\Big\} \\ z_I^p(t) &= \sum_{k=1}^{p}\alpha_k^2(t)\sin(2(\theta_{1k}^0 t+\theta_{2k}^0 t^2)) + 2\sum_{k\neq j}\alpha_k(t)\alpha_j(t)\sin(\theta_{1k}^0 t+\theta_{1j}^0 t+\theta_{2k}^0 t^2+\theta_{2j}^0 t^2) \\ &\quad +2e_R(t)e_I(t) + 2\sum_{k=1}^{p}\alpha_k(t)\Big\{e_R(t)\sin(\theta_{1k}^0 t+\theta_{2k}^0 t^2) + e_I(t)\cos(\theta_{1k}^0 t+\theta_{2k}^0 t^2)\Big\}. \end{aligned}$$

Now a lemma similar to Lemma 5 is required for the multicomponent model (5).

Lemma 6. *Under assumptions 2, 5 and 6, the following results are true for model* (5).

$$\begin{aligned}
\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_R^p(t)\cos(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2) &= \frac{1}{2(m+1)}(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2),\\
\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_I^p(t)\cos(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2) &= 0,\\
\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_R^p(t)\sin(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2) &= 0,\\
\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_I^p(t)\sin(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2) &= \frac{1}{2(m+1)}(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2),
\end{aligned}$$

$k = 1, 2, \ldots, p$ *and* $m = 0, 1, \ldots, 4$.

Proof. of Lemma 6: Along the same line, as in the proof of Lemma 5, it can be shown that

$$\begin{aligned}
&\{e_R(t)e_I(t)\} \overset{i.i.d.}{\sim} \left(0, \frac{\sigma^4}{2}\right), \quad \{e_R^2(t) - e_I^2(t)\} \overset{i.i.d.}{\sim} \left(0, 2\gamma - \frac{\sigma^4}{2}\right),\\
&\{\alpha_k(t)e_R(t)\} \overset{i.i.d.}{\sim} \left(0, (\sigma_{k\alpha}^2 + \mu_{k\alpha}^2)\frac{\sigma^2}{2}\right), \quad \{\alpha_k(t)e_I(t)\} \overset{i.i.d.}{\sim} \left(0, (\sigma_{k\alpha}^2 + \mu_{k\alpha}^2)\frac{\sigma^2}{2}\right). \qquad (22)
\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m z_R^p(t)\cos(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2)\\
= \; &\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m \left\{\sum_{j=1}^{p} \alpha_j^2(t)\cos^2(2\theta_{1j}^0 t + 2\theta_{2j}^0 t^2)\right\}\cos^2(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2)\\
&+\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m (e_R^2(t) - e_I^2(t))\cos(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2)\\
&+\frac{2}{n^{m+1}}\sum_{t=1}^{n} t^m \sum_{j=1}^{p}\alpha_j(t)\Big\{e_R(t)\cos(\theta_{1j}^0 t + \theta_{2j}^0 t^2) - e_I(t)\sin(\theta_{1j}^0 t + \theta_{2j}^0 t^2)\Big\}\cos(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2)\\
&+\frac{2}{n^{m+1}}\sum_{t=1}^{n} t^m \sum_{j\neq k}\alpha_j(t)\alpha_k(t)\cos(\theta_{1k}^0 t + \theta_{1j}^0 t + \theta_{2k}^0 t^2 + \theta_{2j}^0 t^2)\cos(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2).
\end{aligned}$$

The second, third and fourth terms converge to zero a.s. using Corollary of Lemma 3 and results given in (22). Using the same trick as used in the proof of Lemma 5, we can show that the first term

$$\frac{1}{n^{m+1}}\sum_{t=1}^{n} t^m \left\{\sum_{j=1}^{p} \alpha_j^2(t)\cos^2(2\theta_{1j}^0 t + 2\theta_{2j}^0 t^2)\right\}\cos^2(2\theta_{1k}^0 t + 2\theta_{2k}^0 t^2) \xrightarrow{a.s.} \frac{1}{2(m+1)}(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2),$$

using Lemma 1 for $m = 0, 1, \ldots, 4$. Similarly, the other three identities can be proved. ∎

Lemma 7. *Let $\widehat{\boldsymbol{\theta}}_k = (\widehat{\theta}_{1k}, \widehat{\theta}_{2k})$ be an estimate of $\boldsymbol{\theta}_k^0 = (\theta_{1k}^0, \theta_{2k}^0)$ that maximizes $Q_p(\boldsymbol{\theta})$ locally and for any $\epsilon > 0$, let $S_{k\epsilon} = \big\{\boldsymbol{\theta}_k : |\boldsymbol{\theta}_k - \boldsymbol{\theta}_k^0| > \epsilon\big\}$ for some fixed $\boldsymbol{\theta}_k^0 \in (0,\pi)\times(0,\pi)$. If for any $\epsilon > 0$,*

$$\overline{\lim}_{n\to\infty} \sup_{S_{k\epsilon}} \frac{1}{n}\big[Q_p(\boldsymbol{\theta}_k) - Q_p(\boldsymbol{\theta}_k^0)\big] < 0, \quad a.s. \tag{23}$$

then as $n \to \infty$, $\widehat{\boldsymbol{\theta}}_k \xrightarrow{a.s.} \boldsymbol{\theta}_k^0$, that is, $\widehat{\theta}_{1k} \xrightarrow{a.s.} \theta_{1k}^0$ and $\widehat{\theta}_{2k} \xrightarrow{a.s.} \theta_{2k}^0$, $k = 1, \ldots, p$.

The proof of Lemma 7 is similar to the proof of Lemma 2, so is omitted.

Proof of Theorem 3.1: We first consider $\frac{1}{n}Q_p(\boldsymbol{\theta}_k^0)$. Using Lemma 6, it can be shown similarly, as in the proof of Theorem 2.1 that

$$\frac{1}{n}Q_p(\boldsymbol{\theta}_k^0) \xrightarrow{a.s.} (\sigma_{k\alpha}^2 + \mu_{k\alpha}^2). \tag{24}$$

Now, write $Q_p(\boldsymbol{\theta}_k) = R_1 + R_2$, where

$$\begin{aligned} R_1 &= \Big[\frac{1}{n}\sum_{t=1}^{n}\Big\{z_R^p(t)\cos(2\theta_{1k}t + 2\theta_{k2}t^2) + z_I^p(t)\sin(2\theta_{1k}t + 2\theta_{2k}t^2)\Big\}\Big]^2, \\ R_2 &= \Big[\frac{1}{n}\sum_{t=1}^{n}\Big\{-z_R^p(t)\sin(2\theta_{1k}t + 2\theta_{2k}t^2) + z_I^p(t)\cos(2\theta_{1k}t + 2\theta_{2k}t^2)\Big\}\Big]^2. \end{aligned}$$

Consider

$$\begin{aligned} &\overline{\lim}_{n\to\infty} \sup_{S_{k\epsilon}} R_1 \\ = &\overline{\lim}_{n\to\infty} \sup_{S_{k\epsilon}} \Big[\frac{1}{n}\sum_{t=1}^{n}\Big\{z_R^p(t)\cos(2\theta_{1k}t + 2\theta_{k2}t^2) + z_I^p(t)\sin(2\theta_{1k}t + 2\theta_{2k}t^2)\Big\}\Big]^2 \\ = &\overline{\lim}_{n\to\infty} \sup_{S_{k\epsilon}} \Big[\frac{1}{n}\sum_{t=1}^{n}\Big(\sum_{j=1}^{p}\Big\{\alpha_j^2(t)\cos[2(\theta_{1j}^0 - \theta_{1k})t + 2(\theta_{2j}^0 - \theta_{2k})t^2]\Big\} \\ &+2e_R^2(t)e_I^2(t)\sin(2\theta_{1k}t + 2\theta_{2k}t^2) + (e_R^2(t) - e_I^2(t))\cos(2\theta_{1k}t + 2\theta_{2k}t^2) \\ &+2\sum_{j=1}^{p}\alpha_j(t)\Big\{e_R(t)\cos[(2\theta_{1j}^0 - \theta_{1k})t + (2\theta_{2j}^0 - \theta_{2k})t^2] \\ &\qquad\qquad + e_I(t)\sin[(2\theta_{1j}^0 - \theta_{1k})t + (2\theta_{2j}^0 - \theta_{2k})t^2]\Big\} \\ &+2\sum_{j\neq l}\alpha_j(t)\alpha_l(t)\cos(\theta_{1k}^0 t + \theta_{1l}^0 t + \theta_{2k}^0 t^2 + \theta_{2l}^0 t^2)\cos(2\theta_{1k}t + 2\theta_{2k}t^2) \\ &+2\sum_{j\neq l}\alpha_j(t)\alpha_l(t)\cos(\theta_{1k}^0 t + \theta_{1l}^0 t + \theta_{2k}^0 t^2 + \theta_{2l}^0 t^2)\cos(2\theta_{1k}t + 2\theta_{2k}t^2)\Big)\Big]^2 \end{aligned}$$

$$
\begin{aligned}
= \quad & \overline{\lim}_{n\to\infty} \sup_{|\boldsymbol{\theta}_k - \boldsymbol{\theta}_k^0| > \epsilon} \Big[\frac{1}{n}\sum_{t=1}^{n}\Big(\sum_{j=1}^{p}\Big\{\Big(\alpha_j^2(t) - (\sigma_{j\alpha}^2 + \mu_{j\alpha}^2)\Big)\cos[2(\theta_{1j}^0 - \theta_{1k})t + 2(\theta_{2j}^0 - \theta_{2k})t^2] \\
& + (\sigma_{j\alpha}^2 + \mu_{j\alpha}^2)\Big)\cos[2(\theta_{1j}^0 - \theta_{1k})t + 2(\theta_{2j}^0 - \theta_{2k})t^2]\Big\} \\
& + 2\sum_{j=1}^{p}\alpha_j(t)\Big\{e_R(t)\cos[(2\theta_{1j}^0 - \theta_{1k})t + (2\theta_{2j}^0 - \theta_{2k})t^2] \\
& \qquad\qquad + e_I(t)\sin[(2\theta_{1j}^0 - \theta_{1k})t + (2\theta_{2j}^0 - \theta_{2k})t^2]\Big\} \\
& + 2\sum_{j\neq l}\alpha_j(t)\alpha_l(t)\cos(\theta_{1k}^0 t + \theta_{1l}^0 t + \theta_{2k}^0 t^2 + \theta_{2l}^0 t^2)\cos(2\theta_{1k}t + 2\theta_{2k}t^2) \\
& + 2\sum_{j\neq l}\alpha_j(t)\alpha_l(t)\cos(\theta_{1k}^0 t + \theta_{1l}^0 t + \theta_{2k}^0 t^2 + \theta_{2l}^0 t^2)\cos(2\theta_{1k}t + 2\theta_{2k}t^2)\Big)\Big]^2
\end{aligned}
$$

[The second and third terms inside the squared bracket vanish using Lemma 3.]

$$\xrightarrow{a.s.} \quad 0,$$

using Lemmas 1, 3 and 6. Similarly, we can show that $\overline{\lim}_{n\to\infty}\sup_{S_{k\epsilon}} R_2 \xrightarrow{a.s.} 0$. Therefore,

$$\overline{\lim}_{n\to\infty}\sup_{S_\epsilon}\frac{1}{n}\big[Q_p(\boldsymbol{\theta}_k) - Q_p(\boldsymbol{\theta}_k^0)\big] = \overline{\lim}_{n\to\infty}\sup_{S_{k\epsilon}}\Big[R_1 + R_2 - \frac{1}{n}Q_p(\boldsymbol{\theta}_k^0)\Big] \to -(\sigma_{k\alpha}^2 + \mu_{k\alpha}^2)^2 < 0 \ \text{ a.s.}$$

Then, using Lemma 7, $\widehat{\theta}_{1k}$ and $\widehat{\theta}_{2k}$ which maximizes $Q_p(\boldsymbol{\theta})$ locally, are strongly consistent estimators for θ_{1k}^0 and θ_{2k}^0, respectively. ∎

Appendix D

In this Appendix, we prove Theorem 3.3. The unknown parameters are estimated by maximizing $Q_p(\boldsymbol{\theta})$ locally. Let $Q_p'(\boldsymbol{\theta})$ be the $p \times 1$ vector of first order derivatives and $Q_p''(\boldsymbol{\theta})$ be the matrix of second derivatives. Then, the elements of $Q_p'(\boldsymbol{\theta})$ can be written as

$$\frac{\partial Q_p(\boldsymbol{\theta})}{\partial \theta_k} = \frac{2}{n} f_1^p(\boldsymbol{\theta}) g_1^p(k;\boldsymbol{\theta}) + \frac{2}{n} f_2^p(\boldsymbol{\theta}) g_2^p(k;\boldsymbol{\theta}), \quad k = 1,\ldots,p$$

$$
\begin{aligned}
f_1^p(\boldsymbol{\theta}) &= \sum_{t=1}^{n} z_R^p(t)\cos(2\theta_1 t + 2\theta_2 t^2) + \sum_{t=1}^{n} z_I^p(t)\sin(2\theta_1 t + 2\theta_2 t^2), \\
g_1^p(k;\boldsymbol{\theta}) &= \sum_{t=1}^{n} z_I^p(t) 2t^k \cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_R^p(t) 2t^k \sin(2\theta_1 t + 2\theta_2 t^2), \\
f_2^p(\boldsymbol{\theta}) &= \sum_{t=1}^{n} z_I^p(t)\cos(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_R^p(t)\sin(2\theta_1 t + 2\theta_2 t^2),
\end{aligned}
$$

$$g_2^p(k;\boldsymbol{\theta}) \quad = \quad -\sum_{t=1}^{n} z_I^p(t) 2t^k \sin(2\theta_1 t + 2\theta_2 t^2) - \sum_{t=1}^{n} z_R^p(t) 2t^k \cos(2\theta_1 t + 2\theta_2 t^2).$$

Using Lemma 6, similarly as (17), we have

$$(a)\ \lim_{n\to\infty} \frac{1}{n} f_1^p(\boldsymbol{\theta}_k^0) = (\sigma_{k\alpha}^2 + \mu_{k\alpha}^2) \quad \text{and} \quad (b)\ \lim_{n\to\infty} \frac{1}{n} f_2^p(\boldsymbol{\theta}_k^0) = 0 \ \text{ a.s.} \tag{25}$$

for $k = 1, \ldots, p$. Now, as in the proof of Theorem 2.2, we can write

$$(\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k^0)\mathbf{D}^{-1} = -[Q_p'(\boldsymbol{\theta}_k^0)\mathbf{D}][\mathbf{D}Q_p''(\bar{\boldsymbol{\theta}}_k)\mathbf{D}]^{-1}, \ \ k = 1, \ldots, p$$

where $\bar{\boldsymbol{\theta}}_k$ is a point on the line joining $\widehat{\boldsymbol{\theta}}_k$ and $\boldsymbol{\theta}_k^0$, $k = 1, \ldots, p$. Therefore, to prove the asymptotic independence of $(\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k^0)\mathbf{D}^{-1}$ and $(\widehat{\boldsymbol{\theta}}_j - \boldsymbol{\theta}_j^0)\mathbf{D}^{-1}$, $k \neq j$ we need to show that

$$\boldsymbol{\Sigma}^p = ((\Sigma_{ij}^p)) = \operatorname{Cov}\Big[Q_p'(\boldsymbol{\theta}_k^0)\mathbf{D}, Q_p'(\boldsymbol{\theta}_j^0)\mathbf{D}\Big] = \mathbf{0}.$$

Assume $p = 2$. Now because of (25)(b), to prove

$$\Sigma_{kj}^2 = \operatorname{Cov}\left[\frac{1}{n^{\frac{2k+1}{2}}} \frac{\partial Q_2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}_1^0}, \frac{1}{n^{\frac{2j+1}{2}}} \frac{\partial Q_2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}_2^0}\right] = 0, \quad k, j = 1, 2,$$

it is enough to show that for $k, j = 1, 2$

$$\operatorname{Cov}\left[\frac{1}{n^{\frac{2k+1}{2}}} g_1^2(k, \boldsymbol{\theta}_1^0), \frac{1}{n^{\frac{2j+1}{2}}} g_1^2(j, \boldsymbol{\theta}_2^0),\right] = 0.$$

Here Σ_{kj}^2 denote the (k,j)-th element of $\boldsymbol{\Sigma}^p$ with $p = 2$. Expanding $\frac{1}{n^{\frac{2k+1}{2}}} g_1^2(k; \boldsymbol{\theta}_1^0)$, using $z_R^p(t)$ and $z_I^p(t)$, we have

$$\begin{aligned}
&\frac{1}{n^{\frac{2k+1}{2}}} g_1^2(k; \boldsymbol{\theta}_1^0) \\
= \ & \frac{2}{n^{\frac{2k+1}{2}}} \left[\sum_{t=1}^{n} z_I^p(t) t^k \cos(2\theta_{11}^0 t + 2\theta_{21}^0 t^2) - \sum_{t=1}^{n} z_R^p(t) t^k \sin(2\theta_{11}^0 t + 2\theta_{21}^0 t^2)\right] \\
= \ & \frac{2}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k \alpha_2^2(t) \sin(2\theta_{12}^0 t - 2\theta_{11}^0 t + 2\theta_{22}^0 t^2 - 2\theta_{21}^0 t^2) + \frac{2}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k e_R(t) e_I(t) \cos(2\theta_{11}^0 t + 2\theta_{21}^0 t^2) \\
& + \frac{2}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k \alpha_1(t)\alpha_2(t) \sin(\theta_{11}^0 t - \theta_{12}^0 t + \theta_{21}^0 t^2 - \theta_{22}^0 t^2) - \frac{1}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k (e_R^2(t) - e_I^2(t)) \sin(2\theta_{11}^0 t + 2\theta_{21}^0 t^2) \\
& - \frac{2}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k \alpha_1(t) e_R(t) \sin(\theta_{11}^0 t + \theta_{21}^0 t^2) - \frac{2}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k \alpha_2(t) e_R(t) \sin(2\theta_{11}^0 t - \theta_{12}^0 t + 2\theta_{21}^0 t^2 - \theta_{22}^0 t^2) \\
& + \frac{2}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k \alpha_1(t) e_I(t) \cos(\theta_{11}^0 t + \theta_{21}^0 t^2) + \frac{2}{n^{\frac{2k+1}{2}}} \sum_{t=1}^{n} t^k \alpha_2(t) e_I(t) \cos(2\theta_{11}^0 t - \theta_{12}^0 t + 2\theta_{21}^0 t^2 - \theta_{22}^0 t^2)
\end{aligned}$$

$$
\begin{aligned}
= \;& \frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k\Big(\alpha_2^2(t)-(\sigma_{2\alpha}^2+\mu_{2\alpha}^2\Big)\sin(2\theta_{12}^0 t-2\theta_{11}^0 t+2\theta_{22}^0 t^2-2\theta_{21}^0 t^2)\\
&+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k e_R(t)e_I(t)\cos(2\theta_{11}^0 t+2\theta_{21}^0 t^2)\\
&+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_1(t)\alpha_2(t)\sin(\theta_{11}^0 t-\theta_{12}^0 t+\theta_{21}^0 t^2-\theta_{22}^0 t^2)-\frac{1}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k(e_R^2(t)-e_I^2(t))\sin(2\theta_{11}^0 t+2\theta_{21}^0 t^2)\\
&-\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_1(t)e_R(t)\sin(\theta_{11}^0 t+\theta_{21}^0 t^2)-\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_2(t)e_R(t)\sin(2\theta_{11}^0 t-\theta_{12}^0 t+2\theta_{21}^0 t^2-\theta_{22}^0 t^2)\\
&+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_1(t)e_I(t)\cos(\theta_{11}^0 t+\theta_{21}^0 t^2)+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_2(t)e_I(t)\cos(2\theta_{11}^0 t-\theta_{12}^0 t+2\theta_{21}^0 t^2-\theta_{22}^0 t^2),
\end{aligned}
\tag{26}
$$

using Lemma 4. Similarly, $\frac{1}{n^{\frac{2k+1}{2}}}g_1^2(k;\boldsymbol{\theta}_2^0)$ can be expressed as

$$
\begin{aligned}
&\frac{1}{n^{\frac{2k+1}{2}}}g_1^2(k;\boldsymbol{\theta}_2^0)\\
= \;& \frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k\Big(\alpha_1^2(t)-(\sigma_{1\alpha}^2+\mu_{1\alpha}^2\Big)\sin(2\theta_{11}^0 t-2\theta_{12}^0 t+2\theta_{21}^0 t^2-2\theta_{22}^0 t^2)\\
&+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k e_R(t)e_I(t)\cos(2\theta_{12}^0 t+2\theta_{22}^0 t^2)\\
&+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_1(t)\alpha_2(t)\sin(\theta_{12}^0 t-\theta_{11}^0 t+\theta_{22}^0 t^2-\theta_{21}^0 t^2)-\frac{1}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k(e_R^2(t)-e_I^2(t))\sin(2\theta_{12}^0 t+2\theta_{22}^0 t^2)\\
&-\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_2(t)e_R(t)\sin(\theta_{12}^0 t+\theta_{22}^0 t^2)-\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_1(t)e_R(t)\sin(2\theta_{12}^0 t-\theta_{11}^0 t+2\theta_{22}^0 t^2-\theta_{21}^0 t^2)\\
&+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_2(t)e_I(t)\cos(\theta_{12}^0 t+\theta_{22}^0 t^2)+\frac{2}{n^{\frac{2k+1}{2}}}\sum_{t=1}^{n} t^k \alpha_1(t)e_I(t)\cos(2\theta_{12}^0 t-\theta_{11}^0 t+2\theta_{22}^0 t^2-\theta_{21}^0 t^2).
\end{aligned}
\tag{27}
$$

Under the assumption of independence of $\alpha_1(t)$ and $\alpha_2(t)$ and using Lemmas 1, 3 and 4 on expressions (26) and (27), we can show that

$$
\mathrm{Cov}\left[\frac{1}{n^{\frac{2k+1}{2}}}g_1^2(k,\boldsymbol{\theta}_1^0),\frac{1}{n^{\frac{2j+1}{2}}}g_1^2(j,\boldsymbol{\theta}_2^0),\right]=0, \quad \text{for } \; k\neq j=1,2.
$$

Therefore, $\boldsymbol{\Sigma}^2$ is a zero matrix and $(\widehat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_1^0)\mathbf{D}^{-1}$ and $(\widehat{\boldsymbol{\theta}}_2-\boldsymbol{\theta}_2^0)\mathbf{D}^{-1}$ are asymptotically independently distributed when $p=2$. Similarly, for $p>2$, it can be proved that $(\widehat{\boldsymbol{\theta}}_k-\boldsymbol{\theta}_k^0)\mathbf{D}^{-1}$ and $(\widehat{\boldsymbol{\theta}}_j-\boldsymbol{\theta}_j^0)\mathbf{D}^{-1}$ for $k\neq j$ are asymptotically independently distributed. ∎

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