Point and Interval Estimation of Weibull Parameters Based on Joint Progressively Censored Data

Shuvashree Mondal *and Debasis Kundu †

Abstract

Analysis of progressively censored data has received considerable attention in the last few years. In this paper we consider the joint progressive censoring scheme for two populations. It is assumed that the lifetime distribution of the items from the two populations follow Weibull distribution with the same shape but different scale parameters. Based on the joint progressive censoring scheme first we consider the maximum likelihood estimators of the unknown parameters whenever they exist. We provide the Bayesian inferences of the unknown parameters under a fairly general priors on the shape and scale parameters. The Bayes estimators and the associated credible intervals cannot be obtained in closed form, and we propose to use the importance sampling technique to compute the same. Further, we consider the problem when it is known apriori that the expected lifetime of one population is smaller than the other. We provide the order restricted classical and Bayesian inferences of the unknown parameters. Monte Carlo simulations are performed to observe the performances of the different estimators and the associated confidence and credible intervals. One real data set has been analyzed for illustrative purpose.

Key Words and Phrases: Joint progressive censoring scheme; Weibull distribution; Beta-Gamma distribution; log-concave density function; posterior analysis.

AMS Subject Classifications: 62N01, 62N02, 62F10.

*Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India.
†Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India.
Corresponding Author: E-mail: kundu@iitk.ac.in, Phone no. 91-512-2597141, Fax no. 91-512-2597500.

1 Introduction

In a life testing experiment it is common practice to terminate the experiment before all specimens under observation fail. In a type-I censoring scheme the test is terminated at a prefixed time point whereas in a type-II censoring scheme the experiment continues until a certain number of failures occurs. In a practical scenario it might be necessary to remove some of the experimental units during the experiment. Different progressive censoring schemes allow removal of experimental units during the experiment. A progressive type-II censoring scheme can be briefly described as follows. It is assumed that n items are put on a test. Suppose k, $R_1, \ldots, R_k$ are non-negative integers such that $n - k = R_1 + \ldots + R_k$. At the time of the first failure say t_1, R_1 units are chosen at random from the remaining $(n-1)$ items and they are removed from the experiment. Then at the time of the second failure, say t_2, R_2 units are chosen at random from the remaining $n - 2 - R_1$ units and they are removed. The process continues, finally at the time of the k-th failure all the remaining R_k items are removed from the experiment, and the experiment stops.

During the second world war, due to the demand of the highly reliable military equipment, engineers started laboratory investigation instead of long awaited field experiments. Due to tremendous pressure on cost and time, various schemes were introduced to reduce the cost of testing and multi-stage censoring was one of them. As it has been mentioned in Herd (1956) that in many laboratory evaluations, due to limited allocation of fund there are attempts to study the factors contributing to either the reliability or the unreliability of the components or the whole systems under study, as well as to estimate the reliability of the items. In some investigations in order to measure these auxiliary variables, some system needed to be disassembled or subject to measurement which are destructive in nature or might change the expected lifetime. In certain evaluations, systems are removed from the main experimental setup in order to measure certain electrical characteristics. Most of the cases systems are subject to interaction of human operator and to asses the effect of the human operator on the efficiency of the active system, sometimes it requires to withdraw few systems from the main experiment set up. During reliability study of production prototype systems, the addition of engineering modifications also require analyses on multistage censoring.

In Montanari and Cacciari(1988) experimenters conducted testing to evaluate endurance of XLPE-insulated cable to electric and combined thermal stress along with aging mechanism. Some specimens were removed from the test at selected times or at a time of breakdowns for the measurements of electrical, mechanical,chemical properties along with micro structural analyses in order to evaluate aging process. These measurements are destructive in nature. In this experiment the data obtained consist of failure time and censored time of specimens. Ng et al. (2009) presented a clinical study where multistage censored data arise quite naturally. In a study with plasma cell Myeloma at National Cancer Institute on 112 patients, few patients were dropping out at the end of certain intervals whose survival was ensured at that time but no further follow up was available. In his thesis, Herd (1956) discussed estimation under multistage censoring scheme, and refer the scheme as "multi-censored samples". Later on different multistage censoring schemes were referred as progressive censoring schemes. Cohen (1963) studied the importance of progressive censoring scheme in reliability experiment and Cohen (1966) discussed several cases where censored data occurred quite naturally.

To be more precise, in favor of progressive censoring scheme the following points can be put forward. In practical scenario experimental units might get damaged due to some unrelated reasons than the normal failure mechanism. To incorporate these information into inference study,we can rely on progressive censoring schemes. Sometimes the multistage censoring is also intentionally done to use censored units from one experiment to another related experiment due to budget constraints.

The progressive censoring scheme has received considerable attention in the literature. Mann (1971) and Lemon (1975) studied on estimation for Weibull parameters under progressive censoring scheme. Viveros and Balakrishnan (1994) provided interval estimation on progressively censored data. Ng et al. (2004) studied on optimal progressive censoring plan when the underlined distribution is Weibull distribution whereas Kundu (2008) provided Bayesian inference of Weibull population under progressive censoring scheme. Wang et al. (2010) studied the inference of certain life time distributions under progressive type-II right censored scheme. Recently published book by Balakrishnan and Cramer (2014) provided an excellent overview of the different developments on different aspects of the progressive censoring scheme, which have taken place on this topic over the last 20 years. Although, type-II progressive censoring scheme is the most popular one, several other progressive censoring schemes have also been introduced in the literature. In this article we have restricted our attention to the type-II progressive censoring scheme, although most of our results can be extended for other progressive censoring schemes also.

Although, extensive work has been done on the progressive censoring scheme for one group, not much work has been done when two or more groups are present. Rasouli and Balakrishnan (2010) introduced a joint progressive censoring (JPC) scheme, which can be used to compare the lifetime distributions of two products manufactured in different units under the same environmental conditions. The JPC scheme proposed by Rasouli and Balakrishnan (2010) can be briefly described as follows. Suppose m units of Product A (Group 1) and n units of product B (Group 2) are put on a test simultaneously at time zero. It is assumed that $R_1, \ldots, R_k$ are k non-negative integers such that $R_1 + \ldots + R_k = n + m - k$. At the time of the first failure, which may be either from Group 1 or Group 2, R_1 items from the remaining $n + m - 1$ remaining items have been selected at random, and they have been removed from the experiment. These R_1 censored units consist of S_1 units from Group 1 and $R_1 - S_1$ units from Group 2. Here S_1 is random. The time and group of the first failed item are recorded. Similarly, at the time of the second failure, R_2 items from the remaining $n + m - R_1 - 2$ items have been chosen at random and they have been removed from the experiment. Among censored R_2 units random number of S_2 units come from Group 1. The time and group of the second failed items are recorded. The process continues till the k-th failure takes place, when all the remaining items are removed and the experiment ends.

Based on the assumptions that the lifetime distributions of the experimental units of the two populations follow exponential distribution with different scale parameters, Rasouli and Balakrishnan (2010) provided the exact distributions of the maximum likelihood estimators (MLEs) of the unknown parameters and suggested several confidence intervals. Some of the related work on JPC scheme are by Shafay et al. (2014), Parsi and Bairamov (2009) and Doostparast et al. (2013). In most of these cases it has been assumed that the lifetime distributions of the items in two groups follow exponential distribution.

The exponential distribution has the constant hazard rate, which may not be very reasonable in a practical scenario. Again when two similar kind of products are tested, it is quite expected to have some common parameters from the underlined distributions. To justify these scenario, in this paper it is assumed that the lifetime distributions of the individual items of the two different groups follow Weibull distribution with the same shape parameter, but different scale parameters.

Our aim is to compare the lifetime distributions of the two populations. The problem is a typical two-sample problem. It can appear in a accelerated life testing problem, or in estimating the stress-strength parameter of a system.

First we consider the MLEs of the unknown parameters based on the data obtained from a JPC scheme as proposed by Rasouli and Balakrishnan (2010). It has been shown that the MLEs exist under a very general condition and they are unique. The MLE of the common shape parameter can be obtained by solving one non-linear equation, and given the MLE of the common shape parameter, the MLEs of the scale parameters can be obtained in explicit forms. Although, the performances of the MLEs are quite satisfactory, the associated confidence intervals are not very easy to obtain. Hence, it seems the Bayesian inference is a natural choice in this case. It may be treated as an extension of the work of Kundu (2008), where the Bayesian inference of the unknown Weibull parameters for one sample problem was considered, and in this case the results have been generalized to two sample problem. Clearly, the generalization is a non-trivial generalization. Although, the whole development in this paper is for two groups, but the results can be easily generalized to more than two groups also.

For the Bayesian inference we need to assume some priors on the unknown parameters. If the common shape parameter is known, the most convenient but a fairly general conjugate prior on the scale parameters can be the Beta-Gamma prior, as it was suggested by Pena and Gupta (1990). In this case the explicit form of the Bayes estimates of the scale parameters can be obtained. When the shape parameter is unknown, the conjugate prior does not exist. In this case following the approach of Berger and Sun (1993) or Kundu (2008), it is assumed that the prior on the shape parameter has a support on $(0, \infty)$, and it has a log-concave density function. It may be mentioned that many well known distribution functions for example log-normal, Weibull, gamma etc. may have log-concave probability density function. Based on the prior distributions, the joint posterior density function can be obtained. As expected the explicit expressions of the Bayes estimates cannot be obtained in explicit forms. We propose to use importance sampling technique to compute the Bayes estimate of any function of the unknown parameters, and also to construct the associated highest posterior density (HPD) credible interval. Monte Carlo simulations are performed to see the performances of the proposed method, and one data analysis has been performed for illustrative purposes.

In many practical situations it is known apriori that one population is better than the other in terms of the expected lifetime of the experimental units. In accelerated life testing if units of one sample are put on higher stress than the units of other sample, it is quite expected to get shorter life time of the specimens under higher stress. We incorporate this information by considering the order restricted classical and Bayesian inference of the unknown parameters based on joint progressively censored samples. We obtain the MLEs of the unknown parameters based on the order restriction. The construction of the confidence intervals of the unknown parameters can be obtained using bootstrap method. For Bayesian inference, we propose a new order restricted Beta-Gamma prior. In this case the explicit expressions of the Bayes estimates cannot be obtained and we propose to use importance sampling technique to compute the Bayes estimates and the associated HPD credible intervals. We re-analyze the data set based on the order restriction.

Rest of the paper is organized as follows. In Section 2, we present the notations, preliminaries and the priors. Maximum likelihood estimators are presented in Section 3. Posterior analysis are provided in Section 4. In Section 5, we present the order restricted inferences of the unknown parameters. In Section 6, we present the simulation results and data analysis. Finally we conclude the paper in Section 7.

2 Notations, Model Assumptions and Priors

2.1 Notations and Model Assumptions

PDF :	Probability density function
HPD :	Highest posterior density
CDF :	Cumulative distribution function
MLE :	Maximum likelihood estimator
i.i.d. :	independent and identically distributed
F_1 :	CDF of the lifetime distribution of the items of Group 1
F_2 :	CDF of the lifetime distribution of the items of Group 2
T_1 :	Random variable with CDF F_1
T_2 :	Random variable with CDF F_2
k_1 :	Number of failures from Group 1
k_2 :	Number of failures from Group 2
k :	Total number of failures: $k = k_1 + k_2$
GA(α, λ) :	Gamma random variable with PDF: $(\lambda^\alpha/\Gamma(\alpha))x^{\alpha-1}e^{-\lambda x}; \quad x > 0.$
WE(α, λ) :	Weibull random variable with PDF: $\alpha\lambda x^{\alpha-1}e^{-\lambda x^\alpha}; \quad x > 0.$
Bin(n, p) :	Binomial random variable with probability mass function: $\binom{n}{i}p^i(1-p)^{n-i}$
Beta(a, b) :	Beta random variable with PDF: $(\Gamma(a+b)/\Gamma(a)\Gamma(b))p^{a-1}(1-p)^{b-1}; \quad 0 < p < 1.$

Suppose m and n independent units are placed on a test with the corresponding lifetimes being identically distributed with PDF $f_1(\cdot)$ and $f_2(\cdot)$, and CDF $F_1(\cdot)$ and $F_2(\cdot)$, respectively. It is assumed that the lifetime distribution of the items from Group 1 and Group 2, follow WE(α, λ_1) and WE(α, λ_2), respectively. For a given $(R_1, \ldots, R_k)$, as described before, in a JPC scheme, the observations are as follows;

$$\{(t_1, \delta_1, s_1), \ldots, (t_k, \delta_k, s_k)\}. \tag{1}$$

Here for $j = 1, \ldots, k$, $\delta_j = 1$, if the failure at t_j occurs from Group 1, otherwise $\delta_j = 0$, and s_j denotes the number of items from Group 1, which have been removed at the time t_j. Therefore, the likelihood function can be written as

$$L(data|\alpha, \lambda_1, \lambda_2) \propto \alpha^k \lambda_1^{k_1} \lambda_2^{k_2} \prod_{j=1}^{k} t_j^{\alpha-1} e^{-\lambda_1 U(\alpha)} e^{-\lambda_2 V(\alpha)}, \tag{2}$$

here $k_1 = \sum_{j=1}^{k} \delta_j$ and $k_2 = k - k_1$,

$$C_1 = \{j; \delta_j = 1\}, \quad C_2 = \{j; \delta_j = 0\},$$

$$U(\alpha) = \sum_{j=1}^{k} s_j t_j^{\alpha} + \sum_{C_1} t_j^{\alpha}, \quad V(\alpha) = \sum_{j=1}^{k} w_j t_j^{\alpha} + \sum_{C_2} t_j^{\alpha},$$

where $w_j = R_j - s_j$.

2.2 Prior Assumptions: Without Order Restriction

The following prior assumptions are made on the common shape parameter α and on the scale parameters λ_1 and λ_2, when there is no order restriction on λ_1 and λ_2. If we denote $\lambda = \lambda_1 + \lambda_2$, then similarly as in Pena and Gupta (1990), it is assumed that $\lambda \sim \text{GA}(a_0, b_0)$, with $a_0 > 0$, $b_0 > 0$ and $p = \lambda_1/(\lambda_1 + \lambda_2) \sim \text{Beta}(a_1, a_2)$, with $a_1 > 0$, $a_2 > 0$, and they are independently distributed. The joint PDF of λ_1 and λ_2 can be obtained as follows

$$\begin{aligned} \pi_1(\lambda_1, \lambda_2 | a_0, b_0, a_1, a_2) \quad &= \quad \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)} (b_0 \lambda)^{a_0 - a_1 - a_2} \times \frac{b_0^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1 - 1} e^{-b_0 \lambda_1} \\ &\quad \times \frac{b_0^{a_2}}{\Gamma(a_2)} \lambda_2^{a_2 - 1} e^{-b_0 \lambda_2}; \quad 0 < \lambda_1, \ \lambda_2 < \infty. \end{aligned} \tag{3}$$

It is known as the Beta-Gamma PDF, and it will be denoted by $\text{BG}(a_0, b_0, a_1, a_2)$.

The above Beta-Gamma prior is a very flexible prior on the scale parameters. Depending on the values of the hyper-parameters, the joint prior on λ_1 and λ_2 can take variety of shapes. Moreover, for a given α, it is a conjugate prior on (λ_1, λ_2). The correlation between λ_1 and λ_2 can be both positive and negative depending on the values of a_0, a_1 and a_2. If $a_0 = a_1 + a_2$, λ_1 and λ_2 are independent. If $a_0 > a_1 + a_2$, then they are positively correlated, and for $a_0 < a_1 + a_2$, they are negatively correlated. The following results will be useful for further development, and they can be established very easily.

RESULT 1: If $(\lambda_1, \lambda_2) \sim \text{BG}(a_0, b_0, a_1, a_2)$, then for $i = 1, 2$,

$$E(\lambda_i) = \frac{a_0 a_i}{b_0(a_1 + a_2)} \quad \text{and} \quad V(\lambda_i) = \frac{a_0 a_i}{b_0(a_1 + a_2)} \times \left\{ \frac{(a_i + 1)(a_0 + 1)}{a_1 + a_2 + 1} - \frac{a_0 a_i}{a_1 + a_2} \right\}. \qquad (4)$$

Moreover, the generation from a Beta-Gamma distribution is quite simple using the property that $(\lambda_1, \lambda_2) \sim \text{BG}(a_0, b_0, a_1, a_2)$, if and only, $\lambda_1 + \lambda_2$ has a gamma distribution and $\dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ has a beta distribution and they are independently distributed, see for example Kundu and Pradhan (2011).

No specific form of prior has been assumed here on the common shape parameter α. Following the idea of Berger and Sun (1993), it is assumed that $\pi(\alpha)$, the prior on α, has a support on the positive real line, and it has a log-concave PDF. Moreover, $\pi(\alpha)$ and $\pi(\lambda_1, \lambda_1 | b_0, a_), a_1, a_2)$ are independently distributed. It may be mentioned that many well known distribution has log-concave PDF. For example, log-normal, gamma, Weibull have log-concave PDFs. $\pi(\alpha)$ also has its hyper parameters. We do not make it explicit here, whenever it is needed, we will make it explicit.

2.3 Prior Assumptions: Order Restricted

If we have a order restriction on the scale parameters as $\lambda_1 < \lambda_2$, then we make the following prior assumption on λ_1 and λ_2.

$$\begin{aligned} \pi_2(\lambda_1, \lambda_2 | a_0, b_0, a_1, a_2) \quad = \quad & \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} b_0^{a_0} \lambda^{a_0 - a_1 - a_2} e^{-b_0(\lambda_1 + \lambda_2)} \times \\ & \left(\lambda_1^{a_1 - 1}\lambda_2^{a_2 - 1} + \lambda_1^{a_2 - 1}\lambda_2^{a_1 - 1}\right); \quad 0 < \lambda_1 < \lambda_2 < \infty. \qquad (5) \end{aligned}$$

We will call it as the ordered Beta-Gamma PDF, and it will be denoted by $\text{OBG}(a_0, b_0, a_1, a_2)$. Note that (5) is the PDF of the ordered random variable $(\lambda_{(1)}, \lambda_{(2)})$, where $(\lambda_{(1)}, \lambda_{(2)}) = (\lambda_1, \lambda_2)$ if $\lambda_1 < \lambda_2$, $(\lambda_{(1)}, \lambda_{(2)}) = (\lambda_2, \lambda_1)$ if $\lambda_2 < \lambda_1$ and (λ_1, λ_2) follows $(\sim)$ $\text{BG}(a_0, b_0, a_1, a_2)$. It may be noted that the generation from a ordered Beta-Gamma distribution is also quite straight forward. First we generate sample from a Beta-Gamma distribution and then by ordering them, we obtain a random sample from a ordered Beta-Gamma distribution. We assume the same prior $\pi(\alpha)$ on α as before and α and (λ_1, λ_2) are assumed to be independently distributed.

3 Maximum Likelihood Estimators

Based on the observations described in (1), the log-likelihood function without the additive constant becomes:

$$l(data|\alpha, \lambda_1, \lambda_2) = k \ln \alpha + k_1 \ln \lambda_1 + k_2 \ln \lambda_2 + (\alpha - 1) \sum_{j=1}^{k} \ln t_j - \lambda_1 U(\alpha) - \lambda_2 V(\alpha), \quad (6)$$

here k_1, k_2, $U(\alpha)$ and $V(\alpha)$ are same as defined before. The following result provides the uniqueness of the MLEs of λ_1 and λ_2 for a given α.

Theorem 1: If $k_1 > 0$ and $k_2 > 0$, then for a fixed $\alpha > 0$, $g_1(\lambda_1, \lambda_2) = l(data|\alpha, \lambda_1, \lambda_2)$ is a unimodal function of (λ_1, λ_2).

Proof: Note that $g_1(\lambda_1, \lambda_2)$ is a concave function as the Hessian matrix of $g_1(\lambda_1, \lambda_2)$ is a negative definite matrix. Now the result follows because for fixed $\lambda_1(\lambda_2)$, $g_1(\lambda_1, \lambda_2)$ tends to $-\infty$, as $\lambda_2(\lambda_1)$ tends to 0, or ∞. ∎

For known α, the MLEs of λ_1 and λ_2, say $\widehat{\lambda}_1(\alpha)$ and $\widehat{\lambda}_2(\alpha)$, respectively, can be obtained as follows:

$$\widehat{\lambda}_1(\alpha) = \frac{k_1}{U(\alpha)} \quad \text{and} \quad \widehat{\lambda}_2(\alpha) = \frac{k_2}{V(\alpha)}. \quad (7)$$

When α is unknown, first the MLE of α can be obtained by maximizing the profile log-likelihood function of α without the additive constant, and that is

$$p_1(\alpha) = l(data|\alpha, \widehat{\lambda}_1(\alpha), \widehat{\lambda}_2(\alpha)) = k \ln \alpha - k_1 \ln U(\alpha) - k_2 \ln V(\alpha) + (\alpha - 1) \sum_{j=1}^{k} \ln t_j. \quad (8)$$

The following result will provide the existence and uniqueness of the MLE of α.

Theorem 2: If $k_1 > 0$ and $k_2 > 0$, $p_1(\alpha)$ is a unimodal function of α.

Proof: See in the appendix A.

Therefore, combining Theorem 1 and Theorem 2, it is immediately obtained that for $k_1 > 0$ and $k_2 > 0$, the MLEs of α, λ_1 and λ_2 are unique. It is quite simple to compute the MLE of α as $p_1(\alpha)$ is a unimodal function. Use the bisection or Newton-Raphson method to compute the MLEs of α, and once the MLE of α is obtained the MLEs of λ_1 and λ_2 can be obtained from (7). Although, the MLEs can be obtained quite efficiently in this case, the exact distribution of the MLEs is not possible to obtain. Hence, the construction of confidence intervals of the unknown parameters may not be very simple. The Fisher information matrix may be used to construct the asymptotic confidence intervals of the unknown parameters and it is provided in Appendix B. Alternatively, we propose to use the bootstrap method to construct the confidence intervals of the unknown parameters in this case. Since the exact confidence intervals cannot be obtained, the Bayesian inference seems to be a natural choice in this case.

4 Bayes Estimates and Credible Intervals

In this section we provide the Bayes estimates of the unknown parameters, and the corresponding credible sets based on JPC scheme as described before. We mainly assume the squared error loss function, although any other loss function can be easily incorporated. Now to compute the Bayes estimates of the unknown parameters, we need to assume some specific form of the prior distribution of α, and it is assumed that $\pi(\alpha)$ has a Gamma(a, b) distribution. Hence, the joint posterior density function of λ_1, λ_2 and α can be written as

$$\begin{aligned} \pi(\lambda_1, \lambda_2, \alpha | Data) \quad \propto \quad & (\lambda_1 + \lambda_2)^{a_0 - a_1 - a_2} \lambda_1^{a_1 + k_1 - 1} \lambda_2^{a_2 + k_2 - 1} e^{-\lambda_1 (b_0 + U(\alpha))} e^{-\lambda_2 (b_0 + V(\alpha))} \\ & \alpha^{k + a - 1} e^{-b\alpha} \prod_{i=1}^{k} t_i^{\alpha}. \end{aligned} \tag{9}$$

We re-write (9) in the following manner

$$\begin{aligned}\pi(\lambda_1, \lambda_2, \alpha|Data) \;\; &\propto \;\; (\lambda_1+\lambda_2)^{a_0-a_1-a_2}\lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}e^{-(\lambda_1+\lambda_2)(b_0+W(\alpha))} \\ &\qquad \alpha^{k+a-1}e^{-\alpha(b-\sum_{i=1}^{k}\ln t_i)} \times e^{-\lambda_1(U(\alpha)-W(\alpha))}e^{-\lambda_2(V(\alpha)-W(\alpha)}. \qquad (10)\end{aligned}$$

Here $W(\alpha) = \min\{U(\alpha), V(\alpha)\}$. The posterior density function of α, λ_1 and λ_2 can be written as

$$\pi(\alpha, \lambda_1, \lambda_2|Data) \propto \pi_1^*(\lambda_1, \lambda_2|data, \alpha) \times \pi_2^*(\alpha|data) \times g(\alpha, \lambda_1, \lambda_2|data). \qquad (11)$$

Here $\pi_1^*(\lambda_1, \lambda_2|\alpha, Data)$ is the PDF of a BG$(a_0 + k_1 + k_2, b_0 + W(\alpha), a_1 + k_1, a_2 + k_2)$, and

$$\pi_2^*(\alpha) \propto \frac{\alpha^{k+a-1}e^{-\alpha((b-\sum_{i=1}^{k}\ln t_i))}}{(b_0+W(\alpha))^{a_0+k}}, \qquad g(\alpha, \lambda_1, \lambda_2|Data) = e^{-\lambda_1(U(\alpha)-W(\alpha))}e^{-\lambda_2(V(\alpha)-W(\alpha))} \quad (12)$$

Therefore, the Bayes estimate of $h(\alpha, \lambda_1, \lambda_2)$, any function of α, λ_1, λ_2 with respect to squared error loss function is

$$E_{\pi(\alpha,\lambda_1,\lambda_2|Data)}(h(\alpha, \lambda_1, \lambda_2)) = \int_0^\infty \int_0^\infty \int_0^\infty h(\alpha, \lambda_1, \lambda_2)\pi(\alpha, \lambda_1, \lambda_2|Data)d\alpha d\lambda_1 d\lambda_2 = \frac{K_1}{K_2}, \qquad (13)$$

provided it exists. Here

$$K_1 = \int_0^\infty \int_0^\infty \int_0^\infty h(\alpha, \lambda_1, \lambda_2)\times\pi_1^*(\lambda_1, \lambda_2|Data, \alpha)\times\pi_2^*(\alpha|data)\times g(\alpha, \lambda_1, \lambda_2|Data)d\alpha d\lambda_1 d\lambda_2. \qquad (14)$$

and

$$K_2 = \int_0^\infty \int_0^\infty \int_0^\infty \pi_1^*(\lambda_1, \lambda_2|Data, \alpha) \times \pi_2^*(\alpha|data) \times g(\alpha, \lambda_1, \lambda_2|Data)d\alpha d\lambda_1 d\lambda_2. \qquad (15)$$

It is clear that (13) cannot be obtained in closed form. We may use Lindley's approximation to compute (13), but we may not be able to compute the associated credible interval using that. We propose to use importance sampling technique to compute simulation consistent Bayes estimate and the associated credible interval. The details will be explained later. We use the following result for that purpose.

Theorem 3: The density function $\pi_2^*(\alpha|Data)$ is log-concave.

Proof: It can be obtained along the same line as in Theorem 1, the details are avoided. ∎

Therefore, it is quite simple to generate samples from $l(\alpha|Data)$ using the method of Devroye (1984) or Kundu (2008), and for a given α, random samples from $l(\lambda_1, \lambda_2|Data, \alpha)$ can be easily generated using the method of Kundu and Pradhan (2011). The following algorithm can be used to compute the Bayes estimate and also the associated HPD credible interval of $h(\alpha, \lambda_1, \lambda_2)$.

Algorithm

- Step 1: Generate α from $\pi_2^*(\alpha|data)$.
- Step 2: For a given α, generate λ_1 and λ_2 from $\pi_1^*(\lambda_1, \lambda_2|data, \alpha)$.
- Step 3: Repeat the procedure N times to generate $(\alpha_1, \lambda_{11}, \lambda_{21}), \ldots, (\alpha_N, \lambda_{1N}, \lambda_{2N})$.
- Step 4: To obtain Bayes estimate of $h(\alpha, \lambda_1, \lambda_2)$, compute $(h_1, \ldots, h_N)$, where $h_i = h(\alpha_i, \lambda_{1i}, \lambda_{2i})$ as well as compute $(g_1, \ldots, g_N)$, where $g_i = g(\alpha_i, \lambda_{1i}, \lambda_{2i})$.
- Step 5: Bayes estimate of $h(\beta, \lambda_1, \lambda_2)$ can be approximated as $\frac{\sum_{i=1}^N g_i h_i}{\sum_{j=1}^N g_j} = \sum_{i=1}^N v_i h_i$ where $v_i = \frac{g_i}{\sum_{j=1}^N g_j}$.
- Step 6: To compute $100(1-\beta)\%$ CRI of $h(\alpha, \lambda_1, \lambda_2)$, arrange h_i in ascending order to obtain $(h_{(1)}, \ldots h_{(N)})$ and record the corresponding v_i as $(v_{(1)}, \ldots, v_{(N)})$. A $100(1-\gamma)\%$ CRI can be obtained as $(h_{(j_1)}, h_{(j_2)})$ where j_1, j_2 such that

$$j_1 < j_2, \quad j_1, j_2 \in \{1, \ldots, N\} \quad \text{and} \quad \sum_{i=j_1}^{j_2} v_i \leq 1 - \beta < \sum_{i=j_1}^{j_2+1} v_i \tag{16}$$

The $100(1-\beta)\%$ highest posterior density (HPD) CRI can be obtained as $(h_{(j_1^*)}, h_{(j_2^*)})$, such that $h_{(j_2^*)} - h_{(j_1^*)} \leq h_{(j_2)} - h_{(j_1)}$ and j_1^*, j_2^* satisfying (16) for all j_1, j_2 satisfying (16).

5 Order Restricted Inference

In this section we consider the inference on the unknown parameters under the restriction $\lambda_1 < \lambda_2$. In many practical situations experimenter may have the information that one population has a smaller expected lifetime than the other. In our case, this leads to the above restriction. Therefore, our problem can be stated as follows. Based on the same set of assumptions as in Section 2.2 and with $\lambda_1 < \lambda_2$, the problem is to estimate the unknown parameters α, λ_1 and λ_2 using the data (1).

5.1 Maximum Likelihood Estimators

In this case also we proceed along the same line as before. For a given α, the log-likelihood function (6) is concave as a function of λ_1 and λ_2 and it has a unique maximum. The maximum value of the function (6) is obtained at the point $(\widehat{\lambda}_1(\alpha), \widehat{\lambda}_2(\alpha))$. Clearly if $\widehat{\lambda}_1(\alpha) < \widehat{\lambda}_1(\alpha)$, then the order restricted MLEs of λ_1 and λ_2, say $\widetilde{\lambda}_1(\alpha) = \widehat{\lambda}_1(\alpha)$ and $\widetilde{\lambda}_2(\alpha) = \widetilde{\lambda}_2(\alpha)$, respectively. On the other hand if $\widehat{\lambda}_1(\alpha) \geq \widehat{\lambda}_1(\alpha)$, then the maximum value of $g_1(\lambda_1, \lambda_2)$ will be on the line $\lambda_1 = \lambda_2$ under the order restriction $\lambda_1 < \lambda_2$. Therefore, in this case

$$\widetilde{\lambda}_1(\alpha) = \widetilde{\lambda}_2(\alpha) = \text{arg max } g_1(\lambda, \lambda).$$

Hence for a given α, the order restricted MLEs of λ_1 and λ_2 become

$$(\widetilde{\lambda}_1(\alpha), \widetilde{\lambda}_2(\alpha)) = \begin{cases} (\widehat{\lambda}_1(\alpha), \widehat{\lambda}_2(\alpha)) & \text{if} \quad \widehat{\lambda}_1(\alpha) < \widehat{\lambda}_2(\alpha) \\ \left(\frac{k}{\sum_{j=1}^{k}(R_j+1)t_j^\alpha}, \frac{k}{\sum_{j=1}^{k}(R_j+1)t_j^\alpha} \right) & \text{if} \quad \widehat{\lambda}_1(\alpha) \geq \widehat{\lambda}_2(\alpha). \end{cases} \tag{17}$$

The MLE of α, say $\widetilde{\alpha}$ can be obtained by maximizing $p_2(\alpha) = l(data|\alpha, \widetilde{\lambda}_1(\alpha), \widetilde{\lambda}_2(\alpha))$ with respect to α. The following result provides the existence and uniqueness of the MLE of α.

THEOREM 4: If $k_1 > 0$ and $k_2 > 0$, $p_2(\alpha)$ is a unimodal function of α.

PROOF: The result follows along the same line as in Theorem 2 by observing the fact that $p_2(\alpha)$ is log-concave in both the region, and $p_2(\alpha)$ is a continuous function of α. ∎

Once the MLE of α is obtained, the MLEs of λ_1 and λ_2 can be obtained from (17) explicitly. We propose to use bootstrap method to construct the confidence intervals of the unknown parameters.

5.2 Bayes Estimates and Credible Intervals

In this section we will provide the order restricted Bayesian inference of the unknown parameters based on the prior assumptions as provided in Section 2.4. Similarly as before, for specific implementation we assume that $\pi(\alpha)$ has a Gamma(a, b) distribution. The joint posterior density function of α, λ_1 and λ_2 for $\alpha > 0, 0 < \lambda_1 < \lambda_2$, can be written as

$$\begin{aligned}\pi(\lambda_1, \lambda_2, \alpha|Data) \;\; \propto \;\; & (\lambda_1+\lambda_2)^{a_0-a_1-a_2}\left(\lambda_1^{a_1-1}\lambda_2^{a_2-1} + \lambda_1^{a_2-1}\lambda_2^{a_1-1}\right)\lambda_1^{k_1}\lambda_2^{k_2} \\ & e^{-\lambda_1(b_0+U(\alpha))}e^{-\lambda_2(b_0+V(\alpha))}\alpha^{k+a-1}e^{-b\alpha}\prod_{i=1}^{k} t_i^{\alpha}. \end{aligned} \qquad (18)$$

We re-write (18) as follows

$$\begin{aligned}\pi(\lambda_1, \lambda_2, \alpha|Data) \;\; \propto \;\; & (\lambda_1+\lambda_2)^{a_0-a_1-a_2}\left(\lambda_1^{a_1+J-1}\lambda_2^{a_2+J-1} + \lambda_1^{a_2+J-1}\lambda_2^{a_1+J-1}\right)e^{-(\lambda_1+\lambda_2)(b_0+W(\alpha))} \\ & \lambda_1^{k_1-J}\lambda_2^{k_2-J}e^{-\lambda_1(U(\alpha)-W(\alpha))}e^{-\lambda_2(V(\alpha)-W(\alpha))}\alpha^{k+a-1}e^{-\alpha(b-\sum_{i=1}^{k}\ln t_i)}, \end{aligned} \qquad (19)$$

here $J = \min\{k_1, k_2\}$ and $W(\alpha)$ is same as defined before.

The posterior density function of α, λ_1 and λ_2 in this case can be written as

$$\pi(\alpha, \lambda_1, \lambda_2|Data) \propto \pi_1^*(\lambda_1, \lambda_2|data, \alpha) \times \pi_2^*(\alpha|data) \times g(\alpha, \lambda_1, \lambda_2|data). \qquad (20)$$

Here $\pi_1^*(\lambda_1, \lambda_2|\alpha, Data)$ is the PDF of a OBG$(a_0 + 2J, b_0 + W(\alpha), a_1 + J, a_2 + J)$, and

$$\pi_2^*(\alpha) \propto \frac{\alpha^{k+a-1} e^{-\alpha((b-\sum_{i=1}^{k} \ln t_i))}}{(b_0 + W(\alpha))^{a_0+2J}} \tag{21}$$

and

$$g(\alpha, \lambda_1, \lambda_2|Data) = \lambda_1^{k_1-J} \lambda_2^{k_2-J} e^{-\lambda_1(U(\alpha)-W(\alpha))} e^{-\lambda_2(V(\alpha)-W(\alpha))}. \tag{22}$$

Since $\pi_2^*(\alpha|Data)$ is a log-concave function, and the generation from a $\pi_1^*(\lambda_1, \lambda_2|\alpha)$ can be performed quite conveniently, we can use the same importance sampling technique as in Section 4, to compute the Bayes estimate and the associated HPD credible interval of any function of α, λ_1 and λ_2.

6 Numerical Experiments and Data Analysis

6.1 Numerical Experiments

In this section we have performed some simulation experiments to see the effectiveness of the proposed methods and also to observe how the ordered restricted inference behaves in practice. We have considered $m = 20$, $n = 25$, $\lambda_1 = 1.0$, $\lambda_2 = 0.5$. We have taken different effective sample sizes, $k = 20$, 25, different censoring schemes, and different α values, $\alpha = 1$ and 2. In each case we obtain the MLEs of the unknown parameters. We compute the average estimates (AE) and the mean squared errors (MSE) based on 10,000 replications. The results are reported in Table 1 and Table 2. In each case we also compute 90% symmetric percentile bootstrap confidence interval based on 500 bootstrap samples. We repeat the process 1000 times, and obtain the average lengths (AL) of the confidence intervals and their coverage percentages (CP). The results are reported in Table 3 and Table 4.

We further compute the Bayes estimates and the associated 90% credible intervals based on both informative priors (IP) and non-informative priors (NIP). In case of non-informative priors it is assumed that $b_0 = a_0 = a_1 = a_2 = a = b = 0$. For informative priors, when $\alpha = 1$, then $b_0 = 1$, $a_0 = \frac{3}{2}b_0$, $a_1 = 2$, $a_2 = 4$, $a = 2$, $b = 2$, and when $\alpha = 2$, $b_0 = 1$, $a_0 = \frac{3}{2}b_0$, $a_1 = 2$, $a_2 = 4$, $a = 4$, $b = 2$. We compute the Bayes estimates and the associated credible intervals based on 1000 samples. We report the average Bayes estimates and the corresponding MSEs in Table 1 and Table 2. The average lengths of the credible intervals and the associated coverage percentages are reported in Table 3 and Table 4. We use the following notation for a particular progressive censoring scheme. For example $k = 6$ and $R = (4, 0_{(5)})$ means $R_1 = 4$, $R_2 = R_3 = R_4 = R_5 = R_6 = 0$.

Table 1: $m = 20, n = 22, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE		Bayes IP		Bayes NIP	
		AE	MSE	AE	MSE	AE	MSE
k=20,R=(7,$0_{(18)}$,15)	α	1.097	0.063	1.075	0.050	1.095	0.064
	λ_1	0.554	0.057	0.536	0.035	0.543	0.047
	λ_2	1.102	0.147	1.066	0.086	1.086	0.123
k=20,R=($0_{(9)}$,7,$0_{(9)}$,15)	α	1.101	0.067	1.062	0.048	1.066	0.055
	λ_1	0.562	0.062	0.535	0.031	0.557	0.057
	λ_2	1.123	0.185	1.064	0.077	1.081	0.127
k=20,R=($0_{(18)}$,7,15)	α	1.105	0.072	1.068	0.052	1.064	0.060
	λ_1	0.569	0.074	0.548	0.035	0.551	0.052
	λ_2	1.126	0.191	1.077	0.092	1.085	0.117
k=25,R=(7,$0_{(23)}$,10)	α	1.077	0.044	1.063	0.043	1.064	0.040
	λ_1	0.532	0.036	0.523	0.025	0.532	0.036
	λ_2	1.062	0.095	1.042	0.068	1.044	0.085
k=25,R=($0_{(11)}$,7,$0_{(12)}$,10)	α	1.078	0.044	1.062	0.036	1.075	0.044
	λ_1	0.537	0.038	0.524	0.025	0.530	0.034
	λ_2	1.071	0.099	1.053	0.069	1.064	0.101
k=25,R=($0_{(23)}$,7,10)	α	1.080	0.049	1.073	0.043	1.079	0.047
	λ_1	0.540	0.040	0.529	0.026	0.535	0.035
	λ_2	1.071	0.099	1.059	0.075	1.069	0.098

From Table 1 and Table 2, it is clear that as the effective sample size increases in all the cases the average biases and the MSEs decrease. The performances of the Bayes estimators with respect to the NIPs are slightly better than MLEs both in terms of biases and MSEs in most of the cases investigated here. Also the Bayes estimators with respect to IPS are performing better than the Bayes estimators with respect to NIPs in terms of biases and

Table 2: $m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE		Bayes IP		Bayes NIP	
		AE	MSE	AE	MSE	AE	MSE
k=20,R=(7,$0_{(18)}$,15)	α	2.191	0.252	2.127	0.157	2.185	0.243
	λ_1	0.555	0.057	0.545	0.035	0.550	0.049
	λ_2	1.097	0.143	1.055	0.078	1.104	0.131
k=20,R=($0_{(9)}$,7,$0_{(9)}$,15)	α	2.209	0.264	2.108	0.145	2.147	0.228
	λ_1	0.563	0.063	0.536	0.034	0.556	0.049
	λ_2	1.118	0.165	1.097	0.099	1.081	0.126
k=20,R=($0_{(18)}$,7,15)	α	2.207	0.273	2.120	0.157	2.160	0.251
	λ_1	0.561	0.067	0.548	0.035	0.541	.046
	λ_2	1.122	0.183	1.064	0.082	1.112	0.148
k=25,R=(7,$0_{(23)}$,10)	α	2.149	0.169	2.133	0.151	2.139	0.171
	λ_1	0.534	0.037	0.523	0.025	0.524	0.034
	λ_2	1.065	0.101	1.056	0.073	1.061	0.094
k=25,R=($0_{(11)}$,7,$0_{(12)}$,10)	α	2.150	0.176	2.127	0.135	2.146	0.192
	λ_1	0.5380	0.036	0.523	0.023	0.541	0.035
	λ_2	1.066	0.102	1.045	0.068	1.047	0.088
k=25,R=(R=($0_{(23)}$,7,10)	α	2.156	0.193	2.122	0.144	2.132	0.191
	λ_1	0.537	0.038	0.527	0.025	0.545	0.039
	λ_2	1.071	0.101	1.048	0.062	1.049	0.093

MSEs, as expected. Now from Table 3 and Table 4 we study the performances of the percentile bootstrap confidence intervals and the Bayes credible intervals. In most of the cases the coverage percentages are very close to the nominal values. The average credible lengths based on the non-informative priors are larger than the informative priors, but they are slightly lower than the average bootstrap confidence intervals.

Further, we perform the order restricted inference of the unknown parameters. In this case we have taken the same set of parameter values, the sample sizes and the censoring schemes. For the Bayesian inference we have considered the same set of hyper parameters also. The average estimates and the associated MSEs are reported in Table 5 and Table 6. The confidence and credible intervals are reported in Table 7 and Table 8.

From Tables 5 and 6, it is clear that the performances of the Bayes estimators based on non-informative priors behave slightly better than the MLEs in terms of MSEs particularly for the scale parameters. In this case also the performances of the Bayes estimators with

Table 3: $m = 20, n = 22, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	90% HPD CRI IP		90% HPD CRI NIP		90% Bootstrap CI	
		AL	CP	AL	CP	AL	CP
k=20,R=(7,$0_{(18)}$,15)	α	0.621	86.6%	0.663	82.8%	0.804	82.2%
	λ_1	0.560	90.0%	0.627	86.8%	0.814	87.0%
	λ_2	0.894	89.6%	1.018	87.6%	1.358	83.8%
k=20,R=($0_{(9)}$,7,$0_{(9)}$,15)	α	0.603	84.6%	0.624	85.0%	0.818	83.0%
	λ_1	0.565	87.6%	0.635	87.2%	0.907	85.6%
	λ_2	0.900	91.6%	0.977	89.2%	1.454	82.8%
k=20,R=($0_{(18)}$,7,15)	α	0.615	87.0%	0.624	82.0%	0.854	83.6%
	λ_1	0.582	92.0%	0.649	87.2%	0.900	87.2%
	λ_2	0.903	91.4%	0.981	89.4%	1.463	86.0%

Table 4: $m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	90% HPD CRI IP		90% HPD CRI NIP		90% Bootstrap CI	
		AL	CP	AL	CP	AL	CP
k=20,R=(7,$0_{(18)}$,15)	α	1.198	90.8%	1.309	87.4%	1.612	80.2%
	λ_1	0.562	88.2%	0.639	85.0%	0.827	87.2%
	λ_2	0.897	90.8%	0.993	89.6%	1.293	87.8%
k=20,R=($0_{(9)}$,7,$0_{(9)}$,15)	α	1.166	89.4%	1.209	82.2%	1.690	77.2%
	λ_1	0.569	91.4%	0.614	85.0%	0.903	88.4%
	λ_2	0.910	90.4%	1.000	88.4%	1.492	85.6%
k=20,R=($0_{(18)}$,7,15)	α	1.171	89.0%	1.258	81.8%	1.758	81.0%
	λ_1	0.572	90.6%	0.652	85.0%	0.902	87.8%
	λ_2	0.907	91.8%	0.988	89.0%	1.468	84.4%

respect to the informative priors perform much than the non-informative priors, as expected. From Tables 7 and 8, it is observed that coverage percentages of the bootstrap confidence intervals are very similar to the Bayes estimators based on the non-informative priors. The coverage percentages of the bootstrap confidence intervals are slightly closer to the nominal value than the Bayes estimator with non-informative priors in most of the cases considered here.

Now comparing the results between the ordered inference and the unordered inference, it is observed that the estimators obtained using ordered information are slightly better than the unordered ones both the in terms of biases and MSEs. Moreover, the confidence

Table 5: $m = 20, n = 22, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$
(With Order Restriction)

Censoring scheme	Parameter	MLE		Bayes inf-prior		Bayes noninf-prior	
		AE	MSE	AE	MSE	AE	MSE
k=20,R=$(7,0_{(18)},15)$	α	1.093	0.063	1.080	0.051	1.078	0.056
	λ_1	0.551	0.052	0.526	0.024	0.511	0.031
	λ_2	1.113	0.155	1.041	0.067	1.103	0.110
k=20,R=$(0_{(9)}\ ,7,0_{(9)},15)$	α	1.102	0.064	1.057	0.055	1.077	0.057
	λ_1	0.553	0.058	0.527	0.026	0.506	0.034
	λ_2	1.124	0.178	1.051	0.076	1.095	0.109
k=20,R=$(0_{(18)},7,15)$	α	1.102	0.072	1.078	0.054	1.081	0.066
	λ_1	0.560	0.062	0.535	0.028	0.504	0.028
	λ_2	1.129	0.184	1.057	0.080	1.085	0.102
k=25,R=$(7,0_{(23)},10)$	α	1.075	0.043	1.081	0.052	1.066	0.045
	λ_1	0.529	0.033	0.523	0.020	0.508	0.022
	λ_2	1.068	0.986	1.032	0.062	1.056	0.081
k=25,R=$(0_{(11)},7,0_{(12)},10)$	α	1.079	0.045	1.053	0.035	1.079	0.048
	λ_1	0.535	0.034	0.529	0.022	0.501	0.022
	λ_2	1.072	0.099	1.034	0.058	1.073	0.080
k=25,R=$(0_{(23)},7,10)$	α	1.082	0.050	1.061	0.042	1.070	0.046
	λ_1	0.536	0.036	0.526	0.021	0.508	0.023
	λ_2	1.075	0.101	1.029	0.058	1.065	0.075

intervals and the credible intervals based on the ordered inference are slightly smaller than the unordered ones. Hence, if we have some knowledge about the ordering on the scale parameters, it is better to use that information.

The following conclusions can be made from the above results. As effective sample size increases all the point estimators are performing better in terms of average bias and mean squared errors. When order restriction is not imposed and we have information about the hyper parameters of prior densities we should use point estimation based on informative priors rather than MLEs or estimators based on non-informative priors. In case we have no information about hyper-parameters, we should use Bayes estimators based on non-informative priors than MLEs. For interval estimation, we should choose HPD credible intervals based on informative priors than bootstrap confidence intervals or non-informative prior based HPD credible intervals. When no information about hyper-parameters are available we must opt

Table 6: $m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$
(With Order Restriction)

Censoring scheme	Parameter	MLE		Bayes inf-prior		Bayes noninf-prior	
		AE	MSE	AE	MSE	AE	MSE
k=20,R=(7,$0_{(18)}$,15)	α	2.185	0.247	2.123	0.160	2.161	0.239
	λ_1	0.548	0.051	0.528	0.028	0.505	0.033
	λ_2	1.107	0.147	1.056	0.075	1.091	0.110
k=20,R=($0_{(9)}$,7,$0_{(9)}$,15)	α	2.197	0.266	2.108	0.153	2.146	0.219
	λ_1	0.555	0.057	0.524	0.025	0.508	0.032
	λ_2	1.119	0.167	1.046	0.074	1.092	0.113
k=20,R=($0_{(18)}$,7,15)	α	2.198	0.271	2.111	0.163	2.145	0.249
	λ_1	0.558	0.058	0.525	0.025	0.521	0.034
	λ_2	1.129	0.171	1.047	0.069	1.089	0.106
k=25,R=(7,$0_{(23)}$,10)	α	2.149	0.175	2.127	0.171	2.121	0.157
	λ_1	0.527	0.033	0.517	0.021	0.508	0.023
	λ_2	1.061	0.094	1.034	0.063	1.072	0.082
k=25,R=($0_{(11)}$,7,$0_{(12)}$,10)	α	2.149	0.176	2.115	0.126	2.102	0.150
	λ_1	0.536	0.033	0.514	0.020	0.502	0.020
	λ_2	1.073	0.101	1.024	0.063	1.058	0.073
k=25,R=(R=($0_{(23)}$,7,10)	α	2.169	0.201	2.126	0.144	2.154	0.180
	λ_1	0.540	0.039	0.530	0.022	0.502	0.023
	λ_2	1.079	0.102	1.039	0.060	1.052	0.072

for HPD credible intervals based on non-informative prior than bootstrap confidence intervals.

When order restriction is imposed on scale parameters, for point estimation we must prefer informative Bayes estimators than MLEs or non-informative prior based Bayes estimators. When information about the values of hyper-parameters are not available, we should choose Bayes estimators bases on non-informative prior than MLEs. For interval estimation, we should opt for HPD credible intervals than HPD credible intervals based on non-informative prior and bootstrap confidence intervals .

Moreover if information on order restriction is available we should incorporate this information in the inferential procedure.

Table 7: $m = 20, n = 22, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$
(With Order Restriction)

Censoring scheme	Parameter	90% HPD CRI inf-prior		90% HPD CRI noninf-prior		90% Bootstrap CI	
		AL	CP	AL	CP	AL	CP
k=20,R=$(7,0_{(18)},15)$	α	0.600	83.8%	0.624	82.2%	0.801	80.8%
	λ_1	0.502	88.6%	0.515	86.2%	0.740	89.2%
	λ_2	0.711	83.4%	0.722	81.4%	1.312	84.0%
k=20,R=$(0_{(9)},7,0_{(9)},15)$	α	0.587	83.6%	0.612	82.4%	0.813	81.4%
	λ_1	0.515	88.5%	0.537	87.2%	0.763	89.8%
	λ_2	0.685	80.0%	0.731	81.4%	1.368	86.6%
k=20,R=$(0_{(18)},7,15)$	α	0.591	81.4%	0.630	81.0%	0.865	83.0%
	λ_1	0.517	91.4%	0.534	82.4%	0.829	89.8%
	λ_2	0.683	82.6%	0.728	77.0%	1.433	85.8%

Table 8: $m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$
(With Order Restriction)

Censoring scheme	Parameter	90% HPD CRI inf-prior		90% HPD CRI noninf-prior		90% Bootstrap CI	
		AL	CP	AL	CP	AL	CP
k=20,R=$(7,0_{(18)},15)$	α	1.141	87.8%	1.280	84.6%	1.618	80.0%
	λ_1	0.499	89.2%	0.517	86.8%	0.743	88.8%
	λ_2	0.724	83.4%	0.746	79.8%	1.314	85.4%
k=20,R=$(0_{(9)},7,0_{(9)},15)$	α	1.135	88.4%	1.230	82.0%	1.639	82.4%
	λ_1	0.492	89.2%	0.526	84.8%	0.780	88.6%
	λ_2	0.703	82.8%	0.740	80.2%	1.378	88.0%
k=20,R=$(0_{(18)},7,15)$	α	1.157	86.6%	1.239	84.4%	1.702	82.4%
	λ_1	0.509	89.6%	0.527	85.2%	0.800	88.8%
	λ_2	0.694	84.2%	0.704	82.8%	1.416	86.8%

6.2 Data Analysis

In this section we present the analysis of real data sets to show how the different methods work in practice. The data represent the strength measured in GPA for single carbon fibers and can be obtained from Kundu and Gupta (2006). Single fibers were tested under tension at different gauge lengths. Data set 1 are measurements on single fiber of gauge length 20 mm and Data set 2 are obtained from single fiber of gauge length 10 mm. They are presented below for an easy reference:

Data set 1:

1.312,1.314,1.479,1.552,1.700,1.803,1.861,1.865,1.944,1.958,1.966,1.997,2.006,2.021, 2.027,2.055, 2.063,2.098,2.140,2.179,2.224,2.240,2.253,2.270,2.272,2.274,2.301,2.301,2.359,2.382,2.382,2.426, 2.434,2.435,2.478,2.490,2.511,2.514,2.535,2.554,2.566,2.570,2.586,2.629,2.633,2.642,2.648,2.684, 2.697,2.726,2.770,2.773,2.800,2.809,2.818,2.821,2.848,2.880,2.954,3.012,3.067,3.084,3.090,3.096, 3.128,3.233,3.433,3.585,3.585.

Data set 2:

1.901,2.132,2.203,2.228,2.257,2.350,2.361,2.396,2.397,2.445,2.454,2.474,2.518,2.522,2.525,2.532, 2.575,2.614,2.616,2.618,2.624,2.659,2.675,2.738,2.740,2.856,2.917,2.928,2.937,2.937,2.977,2.996, 3.030,3.125,3.139,3.145,3.220,3.223,3.235,3.243,3.264,3.272,3.294,3.332,3.346,3.377,3.408,3.435, 3.493,3.501,3.537,3.554,3.562,3.628,3.852,3.871,3.886,3.971,4.024,4.027,4.225,4.395, 5.020.

Kundu and Gupta (2006) subtract 0.75 from both the data sets and fit Weibull distributions for both the modified data sets separately. The MLEs of shape parameters and scale parameters and Kolmogorov-Smirnov (K-S) distances between empirical distribution functions and fitted distribution functions along with p values are provided in Table 9 for both the data sets. The K-S distances and p values clearly indicate both the data sets after subtracting 0.75 are well fitted by Weibull distributions.

Table 9: MLEs of parameters, K-S and p values for the fitted Weibull models to modified Data set 1 and 2

data	MLE from complete sample		K-S distance	p value
	shape parameter	scale parameter		
Data set1	$\alpha = 3.843$	$\lambda = 0.088$	0.046	0.998
Data set 2	$\alpha = 3.909$	$\lambda = 0.025$	0.079	0.815

To test whether two data sets have common shape parameter, we perform conventional likelihood ratio test (L-R). The MLE of common shape parameter, two scale parameters along with p value of likelihood ratio test are recorded in Table 10. We continue further computation assuming modified Data set 1 follows $WE(\alpha, \lambda_1)$ and modified Data set 2 follows $WE(\alpha, \lambda_2)$.

Table 10: MLEs of parameters, p value of L-R test for for common shape parameter on fitted Weibull models to modified Data set 1 and 2

shape parameter	scale parameter	p value
$\alpha = 3.876$	$\lambda_1 = 0.0861$, $\lambda_2 = 0.026$	0.895

For illustrative purpose we have generated a joint progressive type-II censored sample with $k = 20$, $R_i = 4$ for $i = 1, \ldots, 19$ and $R_{20} = 36$. The data set is as follows: (1.312,1,2) , (1.314,1,2), (1.479,1,1) ,(1.552,1,1), (,1.700,1,3), (1.861,1,1) ,(1.865,1,1), (1.901,0,4), (1.944,1,2), (1.966,1,3), (1.997,1,1), (2.006,1,2), (2.027,1,1), (2.055,1,3), (2.098,1,2), (2.132 ,0,3),(2.140,1,2), (2.179 ,1,2), (2.203,0,3), (2.257,0,14).

First we obtain the MLEs of the unknown parameters without any order restriction and the associate 90% confidence intervals. The Bayes estimators (BEs) and the associated 90% confidence and credible intervals without any order restriction are obtained based on non-informative priors setting $a_0 = 0, b_0 = 0, a_1 = 0, a_2 = 0$ and $a = 0, b = 4$. We set non-informative prior of α as $GA(0, 4)$ so that the posterior density $\pi_2^*(\alpha|data)$ is integrable based on the given data. The results are presented in Table 11 and Table 12. The corresponding results based on order restriction are presented in Table 13 and Table 14.

Table 11: The MLEs and BEs of the unknown parameters for the real data set

Parameter	MLE	BE
α	4.495	3.896
λ_1	0.071	0.098
λ_2	0.016	0.028

Table 12: The CIs and CRIs of the unknown parameters for the real data set

Parameter	90% HPD CRI		90% Bootstrap CI	
	LL	UL	LL	UL
α	3.472	4.338	3.461	6.693
λ_1	0.070	0.167	0.030	0.117
λ_2	0.007	0.049	0.0004	0.037

Table 13: The MLEs and BEs of the unknown parameters for the real data set with order restriction

Parameter	MLE	BE
α	4.495	3.728
λ_1	0.071	0.088
λ_2	0.016	0.022

Based on the joint type-II progressively censored data, among non-informative Bayes estimates and MLEs, non-informative Bayes estimates are more close to the estimates based on complete data sets (see Table 10) Also the length of the HPD credible intervals are less than that of the Bootstrap CIs.

7 Conclusions

In this paper we consider the analysis of joint progressively censored data for two populations. It is assumed that the lifetime of the items of the two populations follow Weibull distributions with the same shape parameter but different scale parameters. We obtain the MLEs of the unknown parameters, and showed that they always exist and they are unique. We also obtain the Bayes estimate and the associated credible interval of a function of the unknown parameters based on a fairly general prior distribution both on the scale and shape

Table 14: The CIs and CRIs of the unknown parameters for the real data set with order restriction

Parameter	90% HPD CRI		90% Bootstrap CI	
	LL	UL	LL	UL
α	3.442	4.360	3.461	6.774
λ_1	0.078	0.112	0.028	0.117
λ_2	0.007	0.033	0.004	0.038

parameters. We further consider the order restricted inference on the unknown parameters. Based on an extensive simulation experiment it is observed that the Bayes estimators with respect to the informative priors perform significantly better than the corresponding Bayes estimators based on non-informative priors for point estimation in terms of average bias and mean squared errors. Also non-informative prior based estimators are slightly better than MLEs. For interval estimation credible intervals based on informative prior are performing better than other two methods in terms of average length of intervals and coverage percentage. Also information on order restriction makes the estimation process better. It may be mentioned that in this paper we have mainly considered two populations. It will be interesting to extend the results for more than two populations also. Su (2013) developed joint progressive censoring scheme for multiple populations. For multiple Weibull populations with common shape parameter, we can assume multivariate gamma-Dirichlet prior for scale parameters and a log-concave prior for common shape parameter as described in this article. More work is needed along that direction.

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Appendix A: Proof of Theorem 1

To prove Theorem 1, first we show the following result. Suppose $a_j \geq 0$, for $j = 1, \ldots, k$, $g(\alpha) = \sum_{j=1}^{k} a_j t_j^{\alpha}$, then -ln$(g(\alpha))$ is a concave function. Note that

$$g'(\alpha) = \sum_{j=1}^{k} a_j t_j^{\alpha} \ln t_j \quad \text{and} \quad g''(\alpha) = \sum_{j=1}^{k} a_j t_j^{\alpha} (\ln t_j)^2.$$

Moreover,

$$\left(\sum_{j=1}^{k} a_j t_j^{\alpha} (\ln t_j)^2\right) \left(\sum_{j=1}^{m} a_j t_j^{\alpha}\right) - \left(\sum_{j=1}^{m} a_j t_j^{\alpha} \ln t_j\right)^2 = \sum_{1 \leq i < j \leq k} a_i a_j (\ln t_i - \ln t_j)^2 \geq 0.$$

Therefore,

$$-\frac{d^2 \ln g(\alpha)}{d\alpha^2} = -\frac{g''(\alpha) g(\alpha) - (g'(\alpha))^2}{(g(\alpha))^2} \leq 0.$$

Now from the above observation, it immediately follows that $p(\alpha)$ is a concave function. The result follows by observing the fact that $p(\alpha) \to -\infty$, as $\alpha \to 0$ or $\alpha \to \infty$. ∎

Appendix B: Fisher Information Matrix

If we denote the matrix $\mathbf{A}$ as

$$\mathbf{A} = -\begin{bmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \lambda_1} & \frac{\partial^2 l}{\partial \alpha \partial \lambda_2} \\ \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} & \frac{\partial^2 l}{\partial \lambda_1^2} & \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} \\ \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} & \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 l}{\partial \lambda_2^2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then

$$\begin{aligned} a_{11} &= \frac{k}{\alpha^2} + \lambda_1 \left[\sum_{j=1}^{k} s_j t_j^{\alpha} (\ln t_j)^2 + \sum_{C_1} t_j^{\alpha} (\ln t_j)^2\right] + \lambda_2 \left[\sum_{j=1}^{k} w_j t_j^{\alpha} (\ln t_j)^2 + \sum_{C_2} t_j^{\alpha} (\ln t_j)^2\right] \\ a_{12} &= a_{21} = \left[\sum_{j=1}^{k} s_j t_j^{\alpha} \ln t_j + \sum_{C_1} t_j^{\alpha} \ln t_j\right] \\ a_{13} &= a_{31} = \left[\sum_{j=1}^{k} w_j t_j^{\alpha} \ln t_j + \sum_{C_2} t_j^{\alpha} \ln t_j\right] \\ a_{22} &= \frac{k_1}{\lambda_1^2}, \quad a_{33} = \frac{k_1}{\lambda_1^2}, \quad a_{23} = a_{32} = 0. \end{aligned}$$

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