A New Two Sample Type-II Progressive Censoring Scheme

Shuvashree Mondal[*], Debasis Kundu [†]

Abstract

In this paper we introduce a new type-II progressive censoring scheme for two samples. It is observed that the proposed censoring scheme is analytically more tractable than the existing joint progressive type-II censoring scheme proposed by Rasouli and Balakrishnan (2010). The maximum likelihood estimators of the unknown parameters are obtained and their exact distributions are derived. Based on the exact distributions of the maximum likelihood estimators exact confidence intervals are also constructed. For comparison purposes we have used bootstrap confidence intervals also. One data analysis has been performed for illustrative purposes. Finally we propose some open problems.

Key Words and Phrases: Type-I censoring scheme; type-II censoring scheme; progressive censoring scheme; joint progressive censoring scheme; maximum likelihood estimator; confidence interval; bootstrap confidence interval.

AMS Subject Classifications: 62N01, 62N02, 62F10.

[*]Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India.
[†]Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India. Corresponding author. E-mail: kundu@iitk.ac.in, Phone no. 91-512-2597141, Fax no. 91-512-2597500.

1 Introduction

Different censoring schemes are extensively used in practice to make a life testing experiment to be more time and cost effective. In a type-I censoring scheme, the experiment is terminated at a prefixed time point. But it may happen that, no failure is observed during that time and it will lead to a very poor statistical analysis of the associated model parameters. To ensure a certain number of failures, type-II censoring scheme has been introduced in the literature. But in none of these censoring schemes any experimental unit can be removed during the experiment. The progressive censoring scheme allows to withdraw some experimental units during the experiment also. Different progressive censoring schemes have been introduced in the literature. The most popular one is known as the progressive type-II censoring scheme and it can be briefly described as follows. Suppose n identical units are put on a life testing experiment. The integer $k < n$ is prefixed, and $R_1, \ldots, R_k$ are k prefixed non-negative integers such that $\sum_{i=1}^{k} R_i + k = n$. At the time of the first failure, R_1 units are chosen randomly from the remaining $n - 1$ units and they are removed from the experiment. Similarly at the time of the second failure, R_2 units are chosen randomly from the remaining $n - R_1 - 2$ units and they are removed, and so on. Finally, at the time of k-th failure remaining R_k units are removed, and the experiment stops. Extensive work has been done during the last ten years on various aspects of different progressive censoring schemes. Interested readers may refer to the recent book by Balakrishnan and Cramer (2014) for a detailed account on different progressive censoring schemes and the related issues. See also Balakrishnan (2007), Pradhan and Kundu (2009) and Kundu (2008), in this respect.

Although extensive work has been done on different aspects of the progressive censoring schemes for one sample, not much work has been done related to two sample problems. Recently, Rasouli and Balakrishnan (2010) introduced the joint progressive type-II censoring for two samples. The joint progressive censoring scheme is quite useful to compare the lifetime distribution of products from different units which are being manufactured by two different lines in the same facility. The joint progressive censoring (JPC) scheme introduced by Rasouli and Balakrishnan (2010) can be briefly stated as follows. It is assumed that two samples of products of sizes m and n, respectively, are selected from these two lines of operation (say Line 1 and Line 2), and they are placed on a life testing experiment simultaneously. A type-II progressive censoring scheme is implemented on the combined sample of size $N = m + n$ as follows. Let $k < N$, and $R_1, \ldots, R_k$ are pre-fixed non-negative integers such that $\sum_{i=1}^{k} R_i + k = N$. At the time of the first failure, it may be from Line 1 or Line 2, R_1 units are chosen at random from the remaining combined $N - 1$ units which consists of S_1 units from Line 1 and T_1 units from Line 2, and they are removed from the experiment. Similarly at the the time of the second failure from the combined $N - 2 - R_1$ remaining units R_2 items are chosen at random, which consists of S_2 and T_2 units from Line 1 and Line 2, respectively, are removed, and so on. Finally at the k-th failure remaining $R_k = S_k + T_k$ units are removed from the experiment, and the experiment stops. Note that in a JPC, although R_j's are pre-fixed, S_j's and T_j's are random quantities, and that makes the analysis more difficult. Rasouli and Balakrishnan (2010) provided the exact likelihood inference for two exponential populations under the proposed JPC scheme. See also Parsi and Bairamov (2009), Ashour and Abo-Kasem (2014), Balakrishnan and Su (2015) for some problems related to the JPC scheme.

In this paper we introduce a new joint progressive type-II censoring (NJPC) scheme. It is observed that the proposed NJPC scheme is easier to handle analytically, therefore the properties of the proposed estimators can be derived quite conveniently. It has some other advantages also. In this paper we provide the exact inference for two exponential populations under the NJPC scheme, although the results can be extended for other lifetime distributions also. We obtain the maximum likelihood estimators (MLEs) of the unknown parameters when it exist, and provide the exact distributions of the MLEs. The generation of samples from the NJPC are quite simple, hence the simulation experiments can be performed quite conveniently. It is observed that the MLEs obtained from the NJPC scheme satisfy the stochastic monotonicity properties stated by Balakrishnan and Iliopoulos (2009), hence the exact distribution of the MLEs can be used to construct the confidence intervals of the unknown parameters. For comparison purposes we proposed to use bootstrap confidence intervals also. Some simulation experiments are performed to compare the performances of the estimators based on JPC and NJPC. It is observed that the estimators based on NJPC behave better than the corresponding estimators based on JPC for certain censoring schemes. One data analysis has been performed for illustrative purposes.

The rest of the paper is organized as follows. In Section 2 we introduce the model and provide the necessary assumptions. The MLEs are obtained and their exact distributions are provided in Section 3. In Section 4 we provide a simple algorithm to simulate data from a NJPC scheme and obtain the expected time of the experiment. The construction of confidence intervals are provided in Section 5. Simulation results and the analysis of one data set are provided in Section 6. Finally in Section 7 we propose some open problems and conclude the paper.

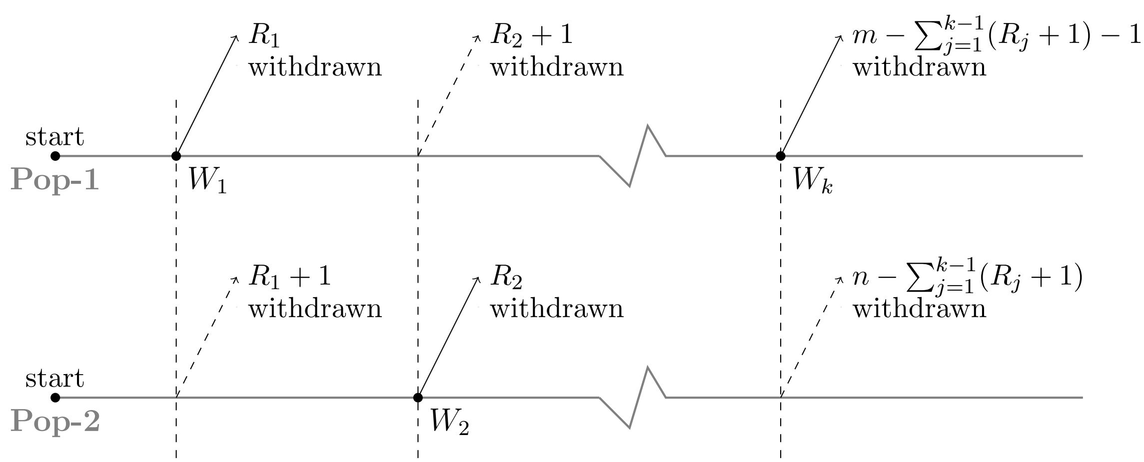
2 Model Description and Model Assumption

Suppose we have products from two different populations. We draw a random sample of size m from population one (Pop-1) and a random sample of size n from population two (Pop-2). We place two independent samples simultaneously on a life testing experiment. The proposed NJPC can be described as follows. Let $k < \min\{m, n\}$ be the total number of failures to be observed and $R_1, \ldots, R_{k-1}$ are such that $\sum_{i=1}^{k-1}(R_i + 1) < \min\{m, n\}$. Suppose the first failure takes place at the time point W_1 and it comes from Pop-1, then R_1 units are randomly chosen from the remaining $m - 1$ surviving units of Pop-1 and they are removed.

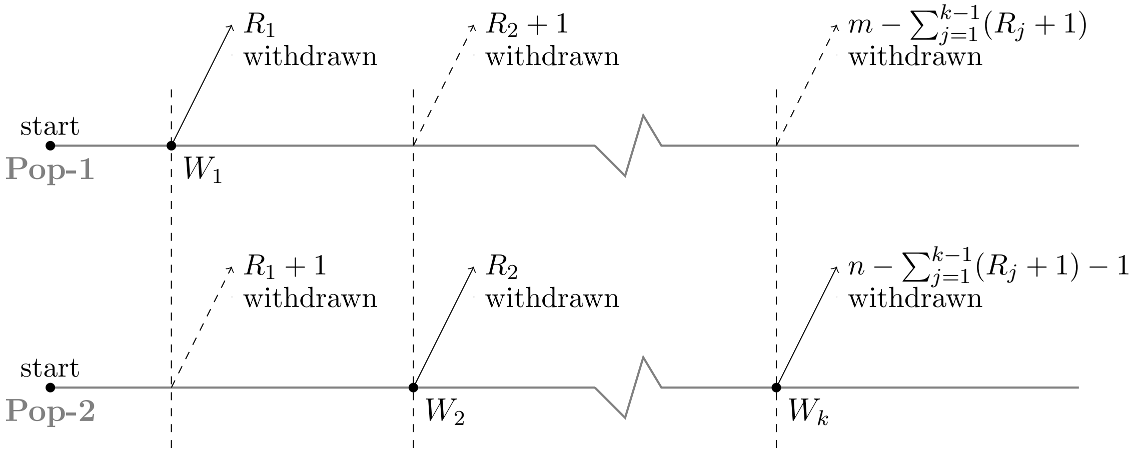
At the same time $(R_1 + 1)$ units are randomly chosen from n surviving units of Pop-2 and they are removed. Suppose the next failure takes place at the time point W_2 and it comes from Pop-2, then R_2+1 units are chosen at random from the remaining $m-1-R_1$ surviving units of Pop-1, and they are removed. At the same time R_2 units are chosen at random from the remaining $n-2-R_1$ surviving units of Pop-2, and they are removed, and so on. Finally, at the time of the k-th failure, it may be either from Pop-1 or from Pop-2, all the remaining items from both the populations are removed and the experiment stops.

We further define a new set of random variables $Z_1, \ldots, Z_k$, where $Z_j = 1$ if the j-th failure takes place from Pop-1 and $Z_j = 0$, otherwise. Hence for a NJPC scheme, the data will be of the form $(\mathbf{W}, \mathbf{Z})$, where $W = (W_1, \ldots, W_k)$, $W_1 \leq \ldots \leq W_k$ and $Z = (Z_1, \ldots, Z_k)$. Schematically, NJPC can be described as follows.

Case-I: k-th failure comes from Pop-1



Case-II: k-th failure comes from Pop-2



Suppose $X_1, \ldots, X_m$ denote the lifetimes of m units of Pop-1, and it is assumed that they are independent and identically distributed (i.i.d.) exponential random variables with mean θ_1 (Exp(θ_1)). Similarly, it is assumed that $Y_1, \ldots, Y_n$ denote the lifetimes of n units of Pop-2, and they are i.i.d exponential random variables with mean θ_2.

3 Maximum likelihood estimators And Their Exact Distributions

3.1 Maximum Likelihood Estimators

For a given sampling scheme m, n, k and $R_1, \ldots, R_{k-1}$ based on the observation $(\mathbf{W}, \mathbf{Z})$ the likelihood function can be written as

$$L(\theta_1, \theta_2 | \mathbf{w}, \mathbf{z}) = C \frac{1}{\theta_1^{m_k}} \frac{1}{\theta_2^{n_k}} e^{-(\frac{A_1}{\theta_1} + \frac{A_2}{\theta_2})}; \tag{1}$$

where the normalizing constant $C = \prod_{i=1}^{k}[(m - \sum_{j=1}^{i-1}(R_j + 1))z_i + (n - \sum_{j=1}^{i-1}(R_j + 1))(1 - z_i)]$,

$A_1 = \sum_{i=1}^{k-1}(R_i + 1)w_i + (m - \sum_{i=1}^{k-1}(R_i + 1))w_k$, $A_2 = \sum_{i=1}^{k-1}(R_i + 1)w_i + (n - \sum_{i=1}^{k-1}(R_i + 1))w_k$,

$m_k = \sum_{i=1}^{k} z_i$, $n_k = \sum_{i=1}^{k}(1-z_i) = k - m_k$. From (1) it follows that (m_k, n_k, A_1, A_2) is the joint complete sufficient statistics of the unknown parameters (θ_1, θ_1). It is immediate that the MLEs of both θ_1 and θ_2 exist when $1 \le m_k \le k-1$, and they are as follows:

$$\widehat{\theta}_1 = \frac{A_1}{m_k} \quad \text{and} \quad \widehat{\theta}_2 = \frac{A_2}{n_k}.$$

Hence $(\widehat{\theta}_1, \widehat{\theta}_2)$ is the conditional MLE of (θ_1, θ_2), conditioning on $1 \le m_k \le k-1$.

3.2 Joint and Marginal Distributions

In this section we provide the joint and marginal distribution function of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ based on the joint and marginal moment generating function (MGF) approach. Lemma 1 is needed for further development.

Lemma 1:

$$P(m_k = r) = \sum_{\mathbf{z} \in Q_r} \{\prod_{i=1}^{k} \frac{(m - \sum_{j=1}^{i-1}(R_j+1))z_i + (n - \sum_{j=1}^{i-1}(R_j+1))(1-z_i)}{(m - \sum_{j=1}^{i-1}(R_j+1))\theta_2 + (n - \sum_{j=1}^{i-1}(R_j+1))\theta_1}\}\theta_1^{k-r}\theta_2^{r},$$

where $Q_r = \left\{ \mathbf{z} = (z_1, \ldots, z_k) : \sum_{i=1}^{k} z_i = r \right\}; r = 0, \ldots, k.$

Proof: See in the Appendix. ∎

Note that when $m = n$, then

$$P(m_k = r) = \binom{k}{r} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{r} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{k-r}; \quad r = 0, \ldots, k. \tag{2}$$

Now we provide the joint moment generating function (MGF) of $(\widehat{\theta}_1, \widehat{\theta}_2)$ conditioning on $1 \le m_k \le k-1$.

Theorem 1: The joint MGF of $(\widehat{\theta}_1, \widehat{\theta}_2)$ conditioning on $1 \le m_k \le k-1$ is given by

$$M_{\widehat{\theta}_1, \widehat{\theta}_2}(t_1, t_2) = \frac{\sum_{r=1}^{k-1} P(m_k = r) \prod_{s=1}^{k}(1 - \alpha_{sr}t_1 - \beta_{sr}t_2)^{-1}}{P(1 \le m_k \le k-1)}, \tag{3}$$

where

$$\alpha_{sr} = \frac{(m - \sum_{i=1}^{s-1}(R_i+1))\theta_1\theta_2}{r\{(m - \sum_{i=1}^{s-1}(R_i+1))\theta_2 + (n - \sum_{i=1}^{s-1}(R_i+1))\theta_1\}}$$

$$\beta_{sr} = \frac{(n - \sum_{i=1}^{s-1}(R_i+1))\theta_1\theta_2}{(k-r)\{(m - \sum_{i=1}^{s-1}(R_i+1))\theta_2 + (n - \sum_{i=1}^{s-1}(R_i+1))\theta_1\}}.$$

PROOF: See in the Appendix. ∎

Using Theorem 1, we immediately get the following corollary.

COROLLARY 1: Conditioning on $1 \le m_k \le k-1$, the marginal MGF of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are given by

$$M_{\widehat{\theta}_1}(t) = \frac{\sum_{r=1}^{k-1} P(m_k = r)\prod_{s=1}^{k}(1-\alpha_{sr}t)^{-1}}{P(1 \le m_k \le k-1)} \quad \text{and}$$

$$M_{\widehat{\theta}_2}(t) = \frac{\sum_{r=1}^{k-1} P(m_k = r)\prod_{s=1}^{k}(1-\beta_{sr}t)^{-1}}{P(1 \le m_k \le k-1)},$$

respectively.

Hence we have the PDFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ as follows.

THEOREM 2: Conditioning on $1 \le m_k \le k-1$, the PDF of $\widehat{\theta}_1$ is given by

$$f_{\widehat{\theta}_1}(t) = \frac{\sum_{r=1}^{k-1} P(m_k = r) g_{X_r}(t)}{P(1 \le m_k \le k-1)}. \tag{4}$$

Here $X_r \stackrel{d}{=} \sum_{s=1}^{k} U_{sr}$, where $U_{sr} \sim \text{Exp}(\alpha_{sr})$ and they are independently distributed. Also, $g_{X_r}(t)$ is the PDF of X_r, and when $m \neq n$,

$$g_{X_r}(t) = \prod_{s=1}^{k}\frac{1}{\alpha_{sr}} \times \sum_{s=1}^{k}\frac{e^{-\frac{t}{\alpha_{sr}}}}{\prod_{j=1, j\neq s}^{k}(\frac{1}{\alpha_{jr}} - \frac{1}{\alpha_{sr}})}; \quad t > 0,$$

and 0, otherwise. When $m = n$,

$$g_{X_r}(t) = \frac{1}{\Gamma(k)\alpha_r^k} t^{k-1} e^{-\frac{t}{\alpha_r}}; \quad t > 0,$$

and 0, otherwise. Here $\alpha_r = \dfrac{\theta_1\theta_2}{r(\theta_1+\theta_2)}$.

The PDF of $\widehat{\theta}_2$ is given by

$$f_{\widehat{\theta}_2}(t) = \frac{\sum_{r=1}^{k-1} P(m_k = r) g_{Y_r}(t)}{P(1 \le m_k \le k-1)}. \tag{5}$$

Here $Y_r \overset{d}{=} \sum_{s=1}^{k} V_{sr}$, where $V_{sr} \sim \text{Exp}(\beta_{sr})$ and they are independently distributed. Also, $g_{Y_r}(t)$ is the PDF of Y_r, and when $m \neq n$,

$$g_{Y_r}(t) = \prod_{s=1}^{k} \frac{1}{\beta_{sr}} \times \sum_{s=1}^{k} \frac{e^{-\frac{t}{\beta_{sr}}}}{\prod_{j=1, j\neq s}^{k} (\frac{1}{\beta_{jr}} - \frac{1}{\beta_{sr}})}; \quad t > 0,$$

and 0, otherwise. When $m = n$,

$$g_{Y_r}(t) = \frac{1}{\Gamma(k)\beta_r^k} t^{k-1} e^{-\frac{t}{\beta_r}}; \quad t > 0,$$

and 0, otherwise. Here $\beta_r = \dfrac{\theta_1\theta_2}{(k-r)(\theta_1+\theta_2)}$.

Proof: It immediately follows from Corollary 1. ∎

Remark: The distribution of the MLE is a mixture of $k-1$ components, where each component is a sum of k independent exponentially distributed random variables. When $m = n$, it is a weighted mixture of gamma distributions.

We can easily obtain the moments of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. When $m \neq n$, the first two moments are

$$\begin{aligned}
E(\widehat{\theta}_1) &= \frac{\sum_{r=1}^{k-1} P(m_k = r) \sum_{s=1}^{k} \alpha_{sr}}{P(1 \le m_k \le k-1)} \\
E(\widehat{\theta}_1^2) &= \frac{\sum_{r=1}^{k-1} P(m_k = r)(2\sum_{s=1}^{k} \alpha_{sr}^2 + \sum_{i\neq j} \alpha_{ir}\alpha_{jr})}{P(1 \le m_k \le k-1)} \\
E(\widehat{\theta}_2) &= \frac{\sum_{r=1}^{k-1} P(m_k = r) \sum_{s=1}^{k} \beta_{sr}}{P(1 \le m_k \le k-1)} \\
E(\widehat{\theta}_2^2) &= \frac{\sum_{r=1}^{k-1} P(m_k = r)(2\sum_{s=1}^{k} \beta_{sr}^2 + \sum_{i\neq j} \beta_{ir}\beta_{jr})}{P(1 \le m_k \le k-1)}.
\end{aligned}$$

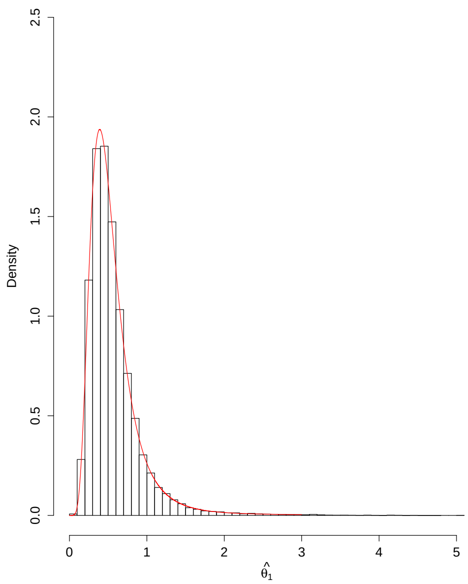
When $m = n$,

$$E(\widehat{\theta}_1) = \frac{\sum_{r=1}^{k-1} P(m_k = r) k\alpha_r}{P(1 \le m_k \le k-1)} \quad \text{and} \quad E(\widehat{\theta}_1^2) = \frac{\sum_{r=1}^{k-1} P(m_k = r) k(k+1){\alpha_r}^2}{P(1 \le m_k \le k-1)}$$

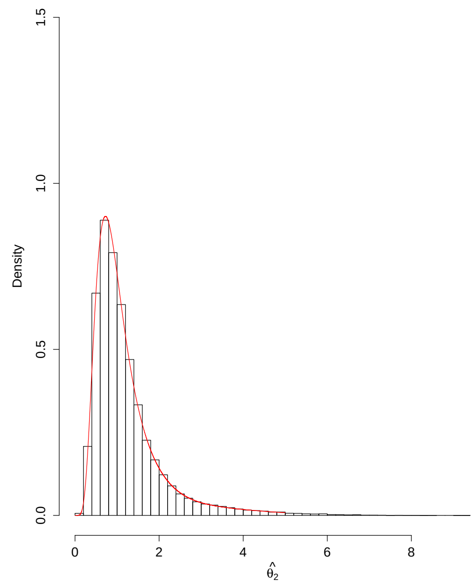
$$E(\widehat{\theta}_2) = \frac{\sum_{r=1}^{k-1} P(m_k = r) k \beta_r}{P(1 \le m_k \le k-1)} \quad \text{and} \quad E(\widehat{\theta}_2^2) = \frac{\sum_{r=1}^{k-1} P(m_k = r) k(k+1) {\beta_r}^2}{P(1 \le m_k \le k-1)}.$$

Here α_r and β_r are same as defined before, and $P(m_k = r)$ is given by (2).

Now to get an idea about the shape of the PDFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$, for different censoring schemes, we have plotted in Figures 1 to 4 the PDFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ along with the histograms of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ based on 10,000 replications.

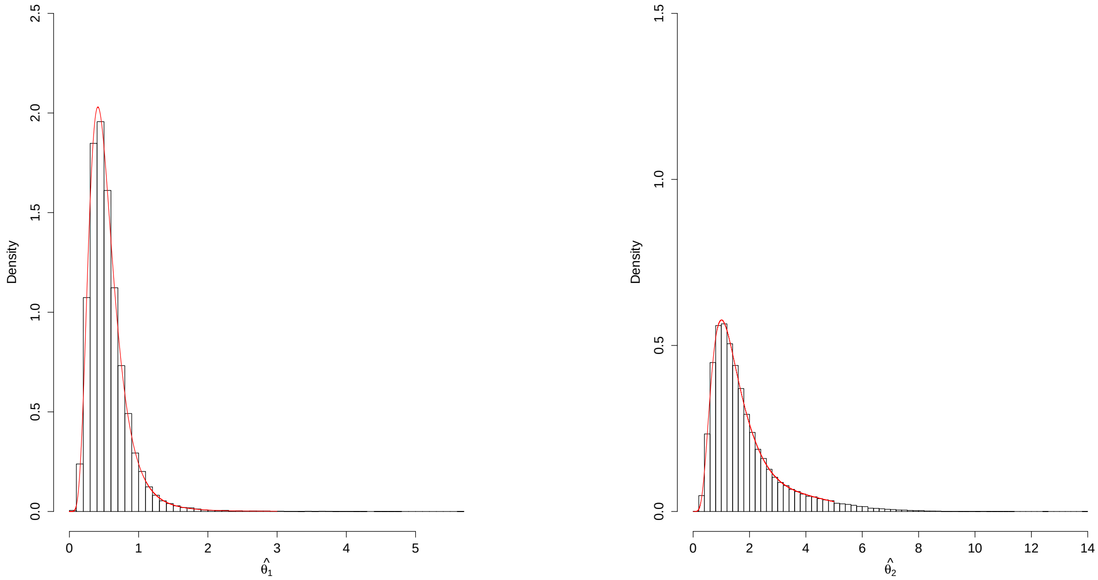


(a) Histogram of $\widehat{\theta}_1$ along with its PDF



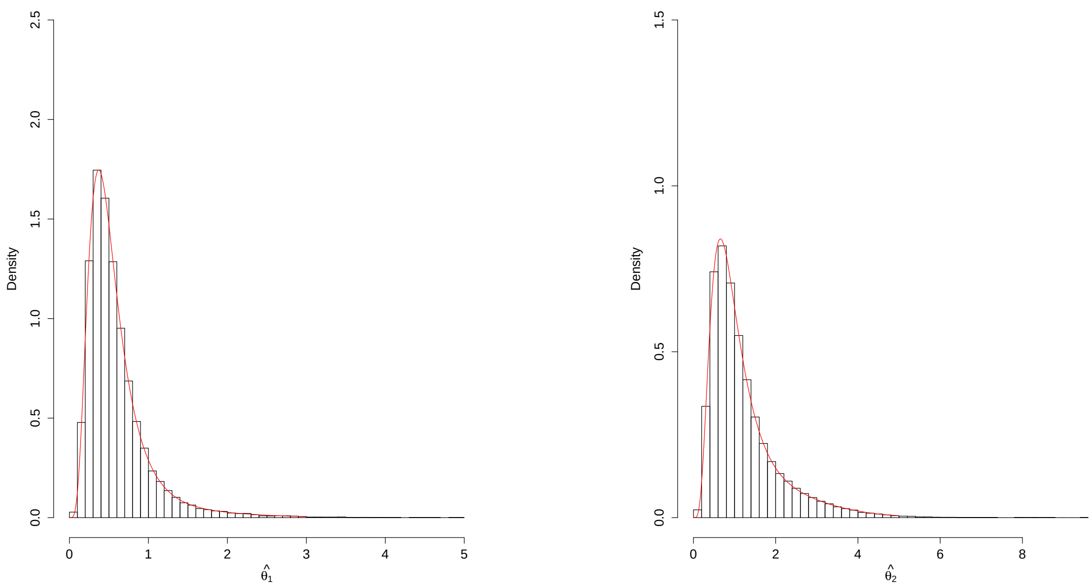
(b) Histogram of $\widehat{\theta}_2$ along with its PDF

Figure 1: Histogram of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ along with its PDF,taking $\theta_1 = .5$, $\theta_2 = 1$, $m = 20$, $n = 25$, $k = 8$,$R = (7, 0_{(6)})$



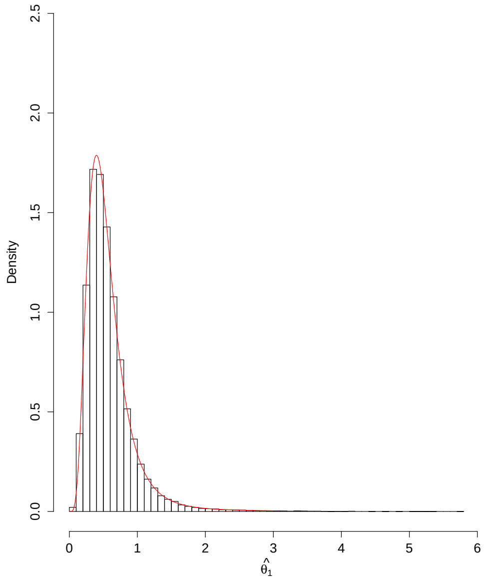
(a) Histogram of $\widehat{\theta}_1$ along with its PDF (b) Histogram of $\widehat{\theta}_2$ along with its PDF

Figure 2: Histogram of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ along with its PDF,taking $\theta_1 = .5$, $\theta_2 = 1.5$, $m = 20$, $n = 25$, $k = 8$,$R = (7, 0_{(6)})$

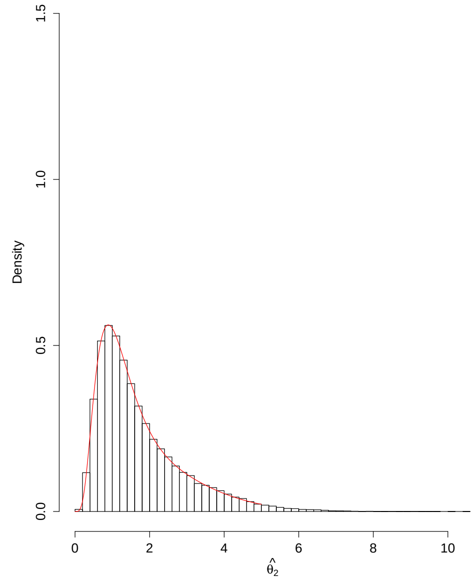


(a) Histogram of $\widehat{\theta}_1$ along with its PDF (b) Histogram of $\widehat{\theta}_2$ along with its PDF

Figure 3: Histogram of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ along with its PDF,taking $\theta_1 = .5$, $\theta_2 = 1$, $m = 20$, $n = 25$, $k = 6$, $R = (2_{(5)})$



(a) Histogram of $\widehat{\theta}_1$ along with its PDF



(b) Histogram of $\widehat{\theta}_2$ along with its PDF

Figure 4: Histogram of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ along with its PDF,taking $\theta_1 = .5$, $\theta_2 = 1.5$, $m = 20$, $n = 25$, $k = 6$,$R = (2_{(5)})$

Some of the points are quite clear from the PDFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. The PDFs of both $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are unimodal and are right skewed for different parameter values and for different sample sizes. Moreover, in all the cases it is observed that the modes of the PDFs are very close to the corresponding true parameter values, as expected.

4 Generation of the Data and the Expected Experimental Time

It is observed that for the proposed NJPC scheme, it is quite simple to generate samples for a given censoring scheme, hence simulation experiments can be performed quite efficiently. In this section we provide an algorithm to generate sample from a given NJPC scheme. This algorithm is based on the following lemma.

LEMMA 2: If $W_1 \leq \ldots \leq W_k$ are the ordered lifetime from a NJPC, then

$$W_i \overset{d}{=} \sum_{s=1}^{i} V_s,$$

where V_s's are independent random variables such that

$$V_s \sim \text{Exp}\left(\frac{1}{E_s}\right), \quad E_s = \frac{(m - \sum_{j=1}^{s-1}(R_j+1))}{\theta_1} + \frac{(n - \sum_{j=1}^{s-1}(R_j+1))}{\theta_2}.$$

PROOF: See in the Appendix. ∎

Now we can use the following algorithm to generate $(\mathbf{W}, \mathbf{Z})$ for a given $n, m, k, R_1, \ldots, R_{k-1}$.

ALGORITHM:

- Step 1: Compute E_s, for $s = 1, \ldots, k$.
- Step 2: Generate $V_s \sim \text{Exp}\left(\frac{1}{E_s}\right)$, $s = 1, \ldots, k$.
- Step 3: Compute $W_i = \sum_{s=1}^{i} V_s, i = 1, \ldots, k$.
- Step 4: Generate $Z_i \sim \text{Bin}(1, p_i), i = 1, \ldots, k$, where

$$p_i = \frac{(m - \sum_{j=1}^{i-1}(R_j+1))\theta_2}{(m - \sum_{j=1}^{i-1}(R_j+1))\theta_2 + (n - \sum_{j=1}^{i-1}(R_j+1))\theta_1}.$$

∎

Using Lemma 2, we can easily obtain the expected experimental time as

$$E(W_k) = \sum_{s=1}^{k} E(V_s) = \sum_{s=1}^{k} \frac{1}{E_s}.$$

5 Construction of Confidence Interval

5.1 Exact Confidence Interval

Based on the assumptions that $P_{\theta_1}(\widehat{\theta}_1 > t)$ is a strictly increasing function of θ_1 for any point $t > 0$ when θ_2 is fixed, a $100(1-\alpha)\%$ exact confidence interval of θ_1 can be constructed. Similarly, based on the assumption that $P_{\theta_2}(\widehat{\theta}_2 > t)$ is a strictly increasing function of θ_2 for any point t when θ_1 is fixed, a $100(1-\alpha)\%$ exact confidence interval of θ_2 can be constructed as follows, see for example Lehmann and Romano (2005).

Conditioning on $1 \le m_k \le k-1$, a $100(1-\alpha)\%$ exact confidence interval for θ_1 as $(\theta_{1L}, \theta_{1U})$ can be obtained by solving the following two nonlinear equations keeping θ_2 fixed.

$$\begin{cases} P_{\theta_{1L}}(\widehat{\theta}_1 > \widehat{\theta}_{1obs} | 1 \le m_k \le k-1) = \frac{\alpha}{2}, \\ P_{\theta_{1U}}(\widehat{\theta}_1 > \widehat{\theta}_{1obs} | 1 \le m_k \le k-1) = 1 - \frac{\alpha}{2}. \end{cases} \tag{6}$$

Similarly, conditioning on $1 \le m_k \le k-1$, a $100(1-\alpha)\%$ exact confidence interval for θ_2 as $(\theta_{2L}, \theta_{2U})$ can be obtained by solving the following nonlinear equations keeping θ_1 fixed.

$$\begin{cases} P_{\theta_{2L}}(\widehat{\theta}_2 > \widehat{\theta}_{2obs} | 1 \le m_k \le k-1) = \frac{\alpha}{2}, \\ P_{\theta_{2U}}(\widehat{\theta}_2 > \widehat{\theta}_{2obs} | 1 \le m_k \le k-1) = 1 - \frac{\alpha}{2}. \end{cases} \tag{7}$$

In practice to compute $(\theta_{1L}, \theta_{1U})$, we replace θ_2 by its MLE $\widehat{\theta}_2$, similarly, to compute $(\theta_{2L}, \theta_{2U})$, we replace θ_1 by its MLE $\widehat{\theta}_1$. One can use the standard bisection method or Newton-Raphson method to solve these two (6) and (7) non-linear equations.

The following result provides the necessary monotonicity properties of $P_{\theta_1}(\widehat{\theta}_1 > t)$ and $P_{\theta_2}(\widehat{\theta}_2 > t)$. It also justifies using (6) and (7) to construct the exact confidence intervals of θ_1 and θ_2, respectively.

Lemma 3:

(i) $P_{\theta_1}(\widehat{\theta}_1 > t | 1 \le m_k \le k-1)$ is a strictly increasing function of θ_1 for any point t when θ_2 is kept fixed.

(ii) $P_{\theta_2}(\widehat{\theta}_2 > t | 1 \leq m_k \leq k-1)$ is a strictly increasing function of θ_2 for any point t when θ_1 is kept fixed.

Proof: See in appendix. ∎

5.2 Bootstrap Confidence Interval

Since the exact confidence intervals can be obtained by solving two non-linear equations we propose to use parametric bootstrap confidence intervals also as an alternative. The following steps can be followed to construct parametric bootstrap confidence intervals.

Step 1: Given the original data, compute $\widehat{\theta}_1$, $\widehat{\theta}_2$.

Step 2: Generate a bootstrap sample $\{(W_1{}^*, Z_1{}^*) \ldots, (W_k{}^*, Z_k{}^*)\}$ using the algorithm provided in Section 4 for a given m, n, k, $(R_1, \ldots R_{k-1})$, $\widehat{\theta}_1$, $\widehat{\theta}_2$,.

Step 3: Compute $\widehat{\theta}_1^*$, $\widehat{\theta}_2^*$ based on the bootstrap sample.

Step 4: Repeat Step 1-Step 3 say B times and obtain $\{\widehat{\theta}_{11}^*, \ldots, \widehat{\theta}_{1B}^*\}$ and $\{\widehat{\theta}_{21}^*, \ldots, \widehat{\theta}_{2B}^*\}$. Sort $\widehat{\theta}_{1j}^*$ in ascending order to get $(\widehat{\theta}_{1(1)}^*, \ldots, \widehat{\theta}_{1(B)}^*)$. Similarly sort $\widehat{\theta}_{2j}^*$ in ascending order to get $(\widehat{\theta}_{2(1)}^*, \ldots, \widehat{\theta}_{2(B)}^*)$.

Step 5: Construct a $100(1-\alpha)\%$ confidence interval for θ_1 as $\big(\widehat{\theta}_{1([\frac{\alpha}{2}B])}^*, \widehat{\theta}_{1([(1-\frac{\alpha}{2})B])}^*\big)$ and a $100(1-\alpha)\%$ confidence interval for θ_2 as $\big(\widehat{\theta}_{2([\frac{\alpha}{2}B])}^*, \widehat{\theta}_{2([(1-\frac{\alpha}{2}B)])}^*\big)$. Here $[x]$ denotes the largest integer less than or equal to x.

6 Simulation Results And Data Analysis

6.1 Simulation Results

We perform some simulation experiments to compare the performances of the estimators based on NJPC and JPC schemes. We have taken different m, n, k, different (θ_1, θ_2) and different $R_1, \ldots, R_{k-1}$ values. For a given set of parameters and the sample sizes, we generate sample based on the algorithm provided in Section 4. In each case we compute the MLEs based on the observed sample, and report their average estimates (AE) and mean squared errors (MSEs) based on 10,000 replications. In each case for the NJPC scheme we construct the exact confidence intervals of θ_1 and θ_2, and we report the average lengths (AL) and the coverage percentages (CP) based on 1000 replications. For each sample we compute the bootstrap confidence intervals based on 1000 replications and we report the average lengths and the coverage percentages based on 1000 replications. All the results are reported in Tables 1 - 4. We use the following notation to denote a particular progressive censoring scheme. For example when $m = 15$, $n = 12$, $k = 6$ and $R = (4, 0_{(4)})$ means $R_1 = 4$, $R_2 = R_3 = R_4 = R_5 = 0$.

Table 1: AE and MSE of the MLE's taking $\theta_1 = .5, \theta_2 = 1, m = 15, n = 12$

Censoring scheme	MLE	NJPC		JPC	
		AE	MSE	AE	MSE
k=6,R=(4,0$_{(4)}$)	$\widehat{\theta}_1$	0.575	0.099	0.563	0.113
	$\widehat{\theta}_2$	0.995	0.377	1.125	0.607
k=6,R=(0,4,0$_{(3)}$)	$\widehat{\theta}_1$	0.577	0.106	0.565	0.114
	$\widehat{\theta}_2$	1.001	0.380	1.122	0.599
k=6,R=(0$_{(2)}$,4,0$_{(2)}$)	$\widehat{\theta}_1$	0.573	0.106	0.571	0.112
	$\hat{\theta}_2$	1.016	0.388	1.147	0.622
k=6,R=(0$_{(3)}$,4,0)	$\hat{\theta}_1$	.580	0.108	0.567	0.112
	$\hat{\theta}_2$	1.034	0.411	1.133	0.598
k=6,R=(0$_{(4)}$,4)	$\hat{\theta}_1$	0.571	0.103	0.569	0.106
	$\hat{\theta}_2$	1.044	0.421	1.124	0.585

Table 2: AE and MSE of the MLE's taking $\theta_1 = .5$, $\theta_2 = 1$, $m = 15$, $n = 12$

Censoring scheme	MLE	NJPC		JPC	
		AE	MSE	AE	MSE
k=8,R=$(3,0_{(6)})$	$\widehat{\theta}_1$	0.538	0.056	0.537	0.062
	$\widehat{\theta}_2$	1.121	0.504	1.238	0.838
k=8,R=$(0_{(2)},3,0_{(4)})$	$\widehat{\theta}_1$	0.541	0.059	0.534	0.063
	$\widehat{\theta}_2$	1.134	0.523	1.226	0.805
k=8,R=$(0_{(3)},3,0_{(3)})$	$\widehat{\theta}_1$	.539	0.056	0.534	0.061
	$\widehat{\theta}_2$	1.138	0.543	1.238	0.817
k=8,R=$(0_{(5)},7,0)$	$\widehat{\theta}_1$	0.540	0.059	0.537	0.061
	$\widehat{\theta}_2$	1.156	0.577	1.231	0.792
k=8,R=$(0_{(6)},7)$	$\widehat{\theta}_1$	0.543	0.063	0.538	0.066
	$\widehat{\theta}_2$	1.159	0.574	1.227	0.834

Table 3: AL and CP of CI's taking $\theta_1 = .5, \theta_2 = .6, m = 20, n = 25$

Censoring scheme	Parameter	Exact 90% CI		Bootstrap 90%CI	
		AL	CP	AL	CP
k=8,R=$(7,0_{(6)})$	θ_1	2.920	89.80%	1.279	91.80%
	θ_2	2.190	90.90%	1.384	89.00%
k=8,R=$(0_{(3)},7,0_{(3)})$	θ_1	2.912	89.40%	1.288	90.70%
	θ_2	2.101	91.70%	1.395	90.60%
k=8,R=$(0_{(5)},7,0)$	θ_1	2.799	88.80%	1.237	89.60%
	θ_2	2.214	91.40%	1.479	91.10%
k=8,R=$(0_{(6)},7)$	θ_1	2.871	89.30%	1.246	89.50%
	θ_2	2.399	90.50%	1.409	89.20%
k=8,R=$(0_{(7)})$	θ_1	2.476	90.40%	1.223	90.50%
	θ_2	2.455	91.40%	1.485	89.20%

Table 4: AL and CP of CI's taking $\theta_1 = .5, \theta_2 = .6, m = 20, n = 25$

Censoring scheme	Parameter	Exact 90% CI		Bootstrap 90%CI	
		AL	CP	AL	CP
k=6,R=(10,$0_{(4)}$)	θ_1	4.410	89.10%	1.213	92.90%
	θ_2	3.188	88.90%	1.531	91.40%
k=6,R=($0_{(2)}$,10,$0_{(2)}$)	θ_1	4.252	88.50%	1.241	92.30%
	θ_2	3.201	89.40%	1.578	90.80%
k=6,R=($0_{(4)}$,10)	θ_1	4.008	88.40%	1.293	91.70%
	θ_2	3.550	90.90%	1.543	92.60%
k=6,R=($0_{(5)}$)	θ_1	3.642	89.70%	1.253	90.90%
	θ_2	3.860	90.10%	1.511	89.20%

Some of the points are quite clear from the above Tables. It is clear that for both the censoring schemes the estimators are quite satisfactory. In most of the cases considered here it is observed that the MSEs of both the estimators are smaller in case of NJPC than the JPC. Regarding the confidence intervals it is observed that the confidence intervals obtained using the exact distribution and also using the bootstrap method provide satisfactory results. In all the cases the coverage percentages are very close to the nominal level. Regarding the length of the confidence intervals, the bootstrap confidence intervals perform slightly better than the exact confidence intervals. Moreover, the implementation of the bootstrap method is also quite simple in this case.

Now we would like to discuss some of the computational issues we have encountered during the simulation experiments mainly to calculate the exact confidence intervals of θ_1 and θ_2. It is observed that for $m \neq n$, and when k is large the computation of $P(X_r > t)$ and $P(Y_r > t)$ become quite difficult for large value of t. For small value of k, if θ_1 and θ_2 are quite different, then solving the two non-linear equations (6) and (7) become quite difficult. In this case $P_{\theta_{1U}}(\widehat{\theta}_1 > \widehat{\theta}_{1obs} | 1 \leq m_k \leq k-1)$ and $P_{\theta_{2U}}(\widehat{\theta}_2 > \widehat{\theta}_{2obs} | 1 \leq m_k \leq k-1)$ become very flat for large values of θ_{1U} and θ_{2U}, respectively. Hence the confidence intervals become very wide. On the other hand the construction of confidence intervals based on bootstrapping does not have any numerical issues.

Considering all these points we propose to use bootstrap method for constructing the confidence intervals in this case.

6.2 Data Analysis

In this section we provide the analysis of a data set mainly for illustrative purposes. These data sets were used by Rasouli and Balakrishnan (2010) also and they were originally taken from Proschan (1963). The data represent the intervals between failures (in hours) of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes. It is observed by Proschan (1963) that the failure time distribution of the air conditioning system for each of the planes can be well approximated by exponential distributions. We have considered the planes "7913" and "7914" for our illustrative purposes. The data are presented below:

Plane 7914: 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210.

Plane 7913: 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.

In this case $m = 24$ and $n = 27$. We have considered two different NJPC with $k = 8$, and different R_i values.

Censoring Scheme 1: $k = 8$ and $R = (0_{(7)})$

Based on the above censoring scheme we generate $\mathbf{W}$ and $\mathbf{Z}$, and they are as follows. $w = (1, 3, 4, 5, 5, 11, 13, 15)$ $z = (0, 1, 0, 1, 1, 0, 1, 1)$. We compute the MLEs of the unknown parameters and 90% exact and bootstrap confidence intervals in both the cases. The results are reported in Table 5.

Table 5: Results related to Censoring Scheme 1.

parameter	MLE	Bootstrap 90% CI	Exact 90% CI
θ_1	59.4	(27.862,132.911)	(30.027,141.049)
θ_2	114.0	(49.146,345.655)	(49.183,422.490)

Censoring Scheme 2: $k = 8$ and $R = (2_{(7)})$

For the Censoring Scheme 2, the generated **W** and **Z** are $w = (1, 3, 4, 5, 5, 14, 15, 16)$ and $z = (0, 1, 0, 1, 1, 1, 1, 0)$. In this case the MLEs and the associate confidence intervals are reported in Table 6

Table 6: Results related to Censoring Scheme 2.

parameter	MLE	Bootstrap 90% CI	Exact 90% CI
θ_1	37.8	(17.239,82.119)	(19.318,93.453)
θ_2	79.0	(31.003,249.636)	(34.588,283.294)

It is clear that the MLEs of the unknown parameters depend quite significantly on the censoring schemes, as expected. The length of the confidence intervals based on bootstrapping are smaller than the exact confidence intervals.

7 Conclusion

In this paper we introduce a new joint progressive censoring scheme for two samples. Based on the assumptions that the lifetime distributions of the two populations follow exponential distributions we obtain the MLE's of the unknown parameters, and derive their exact distributions. It is observed that analytically the proposed model is easier to handle than the existing joint progressive censoring scheme of Rasouli and Balakrishnan (2010). We perform some simulation experiments and it is observed that in certain cases the MLEs of the unknown parameters based on the proposed model behave better than the existing model. Moreover, performing the simulation experiments based on the proposed model is easier compared to the existing model. Therefore, the proposed model can be used for two sample problem quite conveniently in practice.

In this paper we have assumed that the lifetimes of the items follow exponential distribution. In practice it may not be the case always because exponential distribution has a constant hazard rate. It is well known that because of the flexibility, the Weibull distribution or the generalized exponential distribution are more useful in practice. Therefore, it is important to develop the proper inferential procedures for other lifetime distributions for a two sample problem. More work is needed along these directions.

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Appendix

Proof of Lemma 1: Note that

$$\begin{aligned}
P(m_k = r) &= \sum_{\mathbf{z} \in Q_r} P(Z_1 = z_1, \ldots, Z_k = z_k) \\
&= \sum_{\mathbf{z} \in Q_r} P(Z_1 = z_1) P(Z_2 = z_2 | Z_1 = z_1) \cdots P(Z_k = z_k | Z_{k-1} = z_{k-1}, \ldots, Z_1 = z_1).
\end{aligned}$$

Now

$$P(Z_i = z_i | Z_{i-1}, \ldots, Z_1 = z_1) = \frac{(m - \sum_{j=1}^{i-1}(R_j+1))z_i + (n - \sum_{j=1}^{i-1}(R_j+1))(1-z_i)}{(m - \sum_{j=1}^{i-1}(R_j+1))p + (n - \sum_{j=1}^{i-1}(R_j+1))q} p^{z_i} q^{1-z_i},$$

where $p = P(X < Y) = \dfrac{\theta_2}{\theta_1 + \theta_2}$, $q = 1 - p$. Hence $Z_i's$ are independent, therefore

$$\begin{aligned}
P(m_k = r) &= \sum_{\mathbf{z} \in Q_r} \prod_{i=1}^{k} \{ \frac{(m - \sum_{j=1}^{i-1}(R_j+1))z_i + (n - \sum_{j=1}^{i-1}(R_j+1))(1-z_i)}{(m - \sum_{j=1}^{i-1}(R_j+1))p + (n - \sum_{j=1}^{i-1}(R_j+1))q} p^{z_i} q^{1-z_i} \} \\
&= \sum_{\mathbf{z} \in Q_r} \prod_{i=1}^{k} \frac{(m - \sum_{j=1}^{i-1}(R_j+1))z_i + (n - \sum_{j=1}^{i-1}(R_j+1))(1-z_i)}{(m - \sum_{j=1}^{i-1}(R_j+1))p + (n - \sum_{j=1}^{i-1}(R_j+1))q} p^{m_k} q^{n_k} \\
&= \sum_{\mathbf{z} \in Q_r} \prod_{i=1}^{k} \frac{(m - \sum_{j=1}^{i-1}(R_j+1))z_i + (n - \sum_{j=1}^{i-1}(R_j+1))(1-z_i)}{(m - \sum_{j=1}^{i-1}(R_j+1))\theta_2 + (n - \sum_{j=1}^{i-1}(R_j+1))\theta_1} {\theta_1}^{k-r} {\theta_2}^{r}.
\end{aligned}$$

Proof of Theorem 1: Conditioning on $1 \le m_k \le k-1$,

$$\begin{aligned}
M_{\widehat{\theta}_1, \widehat{\theta}_2}(t_1, t_2) &= E(e^{t_1 \widehat{\theta}_1 + t_2 \widehat{\theta}_2} | 1 \le m_k \le k-1) \\
&= \sum_{r=1}^{k-1} E(e^{t_1 \widehat{\theta}_1 + t_2 \widehat{\theta}_2} | m_k = r) P(m_k = r | 1 \le m_k \le k-1) \\
&= \sum_{r=1}^{k-1} \sum_{\mathbf{z} \in Q_r} E(e^{t_1 \widehat{\theta}_1 + t_2 \widehat{\theta}_2} | m_k = r, \mathbf{Z} = \mathbf{z}) P(\mathbf{Z} = \mathbf{z} | m_k = r) P(m_k = r | 1 \le m_k \le k-1) \\
&= \frac{C}{P(1 \le m_k \le k-1)} \sum_{r=1}^{k-1} \sum_{\mathbf{z} \in Q_r} \frac{1}{{\theta_1}^r} \frac{1}{{\theta_2}^{k-r}} \times \\
&\quad \int_0^\infty \int_{w_1}^\infty \ldots \int_{w_{k-1}}^\infty e^{\frac{t_1\{\sum_{i=1}^{k-1}(R_i+1)w_i + (m - \sum_{i=1}^{k-1}(R_i+1))w_k\}}{r}} \times e^{\frac{t_2\{\sum_{i=1}^{k-1}(R_i+1)w_i + (n - \sum_{i=1}^{k-1}(R_i+1))w_k\}}{k-r}} \\
&\quad \times e^{-\frac{1}{\theta_1}\{\sum_{i=1}^{k-1}(R_i+1)w_i + (m - \sum_{i=1}^{k-1}(R_i+1))w_k\}} \\
&\quad \times e^{-\frac{1}{\theta_2}\{\sum_{i=1}^{k-1}(R_i+1)w_i + (n - \sum_{i=1}^{k-1}(R_i+1))w_k\}} dw_k \ldots dw_2 dw_1 \\
&= \frac{C}{P(1 \le m_k \le k-1)} \sum_{r=1}^{k-1} \sum_{\mathbf{z} \in Q_r} \frac{1}{{\theta_1}^r} \frac{1}{{\theta_2}^{k-r}} \\
&\quad \{\prod_{j=1}^{k} \frac{(m - \sum_{i=1}^{j-1}(R_i+1))}{\theta_1} + \frac{(n - \sum_{i=1}^{j-1}(R_i+1))}{\theta_2}\}^{-1} \times \prod_{s=1}^{k} (1 - \alpha_{sr} t_1 - \beta_{sr} t_2)^{-1} \\
&= \frac{1}{P(1 \le m_k \le k-1)} \sum_{r=1}^{k-1} P(m_k = r) \prod_{s=1}^{k} (1 - \alpha_{sr} t_1 - \beta_{sr} t_2)^{-1}.
\end{aligned}$$

Proof of Lemma 2:

$$
\begin{aligned}
E(e^{tW_j}) &= \sum_{r=0}^{k}\sum_{\mathbf{z}\in Q_r} E(e^{tW_j}|m_k=r,\mathbf{Z}=\mathbf{z})P(\mathbf{Z}=\mathbf{z}|m_k=r)P(m_k=r)\\
&= C\sum_{r=1}^{k-1}\sum_{\mathbf{z}\in Q_r}\int_0^\infty\int_{w_1}^\infty\cdots\int_{w_{k-1}}^\infty \frac{1}{{\theta_1}^r}\frac{1}{{\theta_2}^{k-r}}e^{tw_j}\times e^{-\frac{1}{\theta_1}\{\sum_{i=1}^{k-1}(R_i+1)w_i+(m-\sum_{i=1}^{k-1}(R_i+1))w_k\}}\\
&\quad\times e^{-\frac{1}{\theta_2}\{\sum_{i=1}^{k-1}(R_i+1)w_i+(n-\sum_{i=1}^{k-1}(R_i+1))w_k\}}dw_k\ldots dw_2dw_1\\
&= C\sum_{r=0}^{k}\sum_{\mathbf{z}\in Q_r}\frac{1}{{\theta_1}^r}\frac{1}{{\theta_2}^{k-r}}\\
&\quad\times\{a_k(a_k+a_{k-1})\cdots(a_k+a_{k-1}+\cdots+a_{j+1})(a_k+a_{k-1}+\cdots+a_{j+1}+a_j'+a_{j-1})\}^{-1}\cdots\\
&\quad\times(a_k+a_{k-1}+\cdots+a_{j+1}+a_j'+a_{j-1}+\cdots+a_1)^{-1}\\
&= C\sum_{r=0}^{k}\sum_{\mathbf{z}\in Q_r}\frac{1}{{\theta_1}^r}\frac{1}{{\theta_2}^{k-r}}\\
&\quad\times\{a_k(a_k+a_{k-1})\cdots(a_k+a_{k-1}+\cdots+a_j)\cdots(a_k+a_{k-1}+\cdots+a_j+a_{j-1}+\cdots a_1)\}^{-1}\\
&\quad\times\frac{(a_k+a_{k-1}+\cdots+a_j)\cdots(a_k+a_{k-1}+\cdots+a_j+a_{j-1}+\cdots+a_1)}{(a_k+a_{k-1}+\cdots+a_j')\cdots(a_k+a_{k-1}+\cdots+a_j'+aj-1+\cdots+a_1)}\\
&= \sum_{r=0}^{k}P(m_k=r)\left\{\frac{(a_k+a_{k-1}+\cdots+a_j)\cdots(a_k+a_{k-1}+\cdots+a_j+a_{j-1}+\cdots+a_1)}{(a_k+a_{k-1}+\cdots+a_j')\cdots(a_k+a_{k-1}+\cdots+a_j'+a_{j-1}+\cdots+a_1)}\right\}\\
&= \left\{\frac{(a_k+a_{k-1}+\cdots+a_j)\cdots(a_k+a_{k-1}+\cdots+a_j+a_{j-1}+\cdots+a_1)}{(a_k+a_{k-1}+\cdots+a_j')\cdots(a_k+a_{k-1}+\cdots+a_j'+a_{j-1}+\cdots+a_1)}\right\}\sum_{r=0}^{k}P(m_k=r)\\
&= \prod_{s=1}^{j}\left(1-\frac{t}{E_s}\right)^{-1}.
\end{aligned}
$$

Here

$$
\begin{aligned}
a_j &= \frac{(R_j+1)}{\theta_1}+\frac{(R_j+1)}{\theta_2},\quad j=1,\ldots,k-1;\\
a_k &= \frac{(m-\sum_{j=1}^{k-1}(R_j+1))}{\theta_1}+\frac{(n-\sum_{j=1}^{k-1}(R_j+1))}{\theta_2};\quad a_j'=a_j-t;\\
E_s &= \frac{(m-\sum_{j=1}^{s-1}(R_j+1))}{\theta_1}+\frac{(n-\sum_{j=1}^{s-1}(R_j+1))}{\theta_2}.
\end{aligned}
$$

Proof of Lemma 3: To prove Lemma 3, we mainly use the "Three Monotonicity Lemmas" of Balakrishnan and Iliopoulos (2009). We briefly state the "Three Monotonicity Lemmas" for convenience, and we will show that both $\widehat{\theta}_1$ and $\widehat{\theta}_2$ satisfy the "Three Monotonicity Lemmas".

Suppose $\widehat{\theta}$ is an estimate of θ, and the survival function of $\widehat{\theta}$ can be written in the following form:

$$P_\theta(\widehat{\theta} > x) = \sum_{d \in \mathcal{D}} P_\theta(\widehat{\theta} > x | D = d) P_\theta(D = d),$$

where $\mathcal{D}$ is a finite set.

Lemma (Three Monotonicity Lemmas:) Assume that the following hold true:

(M1) $P_\theta(\widehat{\theta} > x | D = d)$ is increasing in θ for all x and $d \in \mathcal{D}$;

(M2) For all x and $\theta > 0$, $P_\theta(\widehat{\theta} > x | D = d)$ is decreasing in $d \in \mathcal{D}$;

(M3) D is stochastically decreasing in θ.

Then $P_\theta(\widehat{\theta} > x)$ is increasing in θ for any fixed x.

Now to prove (i), first observe that

$$P_{\theta_1}(\widehat{\theta}_1 > t | 1 \leq m_k \leq k-1) = \sum_{r=1}^{k-1} P_{\theta_1}(\widehat{\theta}_1 > t | m_k = r) P_{\theta_1}(m_k = r | 1 \leq m_k \leq k-1).$$

Hence, (i) can be proved if we can show that

(M1) $P_{\theta_1}(\widehat{\theta}_1 > t | m_k = r)$ is increasing in θ_1, $\forall t, r \in \{1, \ldots k-1\}$;

(M2) $P_{\theta_1}(\widehat{\theta}_1 > t | m_k = r)$ is decreasing in r, $\forall t, \theta_1 > 0$;

(M3) The conditional distribution of m_k is stochastically decreasing in θ_1.

From the moment generating function of $E(e^{t\widehat{\theta}_1} | m_k = r)$ it is easily observe that conditioning on $m_k = r$, $\widehat{\theta}_1 \overset{d}{=} \sum_{s=1}^{k} X_{sr}$, where $X_{sr} \sim \text{Exp}(\alpha_{sr})$ and they are independently distributed. Here α_{sr}'s are same as defined in Theorem 1. Since α_{sr} is increasing with θ_1, the distribution of X_{sr} is stochastically increasing with θ_1. Since X_{sr}'s are independently distributed, (M1) is satisfied.

Now to prove (M2), observe that

$$\widehat{\theta_1}|\{m_k = r\} \stackrel{d}{=} \frac{\sum_{i=1}^{k-1}(R_i+1)w_i + (m - \sum_{i=1}^{k-1}(R_i+1))w_k}{r}$$

$$\widehat{\theta_1}|\{m_k = r+1\} \stackrel{d}{=} \frac{\sum_{i=1}^{k-1}(R_i+1)w_i + (m - \sum_{i=1}^{k-1}(R_i+1))w_k}{r+1}.$$

Hence for all t and for $\theta_1 > 0$, $P_{\theta_1}(\widehat{\theta_1} > t|m_k = r) > P_{\theta_1}(\widehat{\theta_1} > t|m_k = r+1)$. This proves (M2).

To prove (M3) it is enough to show m_k has monotone likelihood ratio property with respect to θ_1. For $\theta_1 < {\theta_1}'$

$$\frac{P_{\theta_1}(m_k = r|1 \le m_k \le k-1)}{P_{{\theta_1}'}(m_k = r|1 \le m_k \le k-1)} \propto \frac{P_{\theta_1}(m_k = r)}{P_{{\theta_1}'}(m_k = r)} \propto \left(\frac{\theta_1}{{\theta_1}'}\right)^{k-r} \uparrow r.$$

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