Noise Space Decomposition Method for two dimensional sinusoidal model

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Abstract

The estimation of the parameters of the two dimensional sinusoidal signal model has been addressed. The proposed method is the two dimensional extension of the one dimensional noise space decomposition method. It provides consistent estimators of the unknown parameters and they are non-iterative in nature. Two pairing algorithms, which help in identifying the frequency pairs have been proposed. It is observed that the mean squares errors of the proposed estimators are quite close to the asymptotic variance of the least squares estimators. For illustrative purposes two data sets have been analyzed, and it is observed that the proposed model and the method work quite well for analyzing real symmetric textures.

Key Words and Phrases: Sinusoidal Model, Prony's Algorithm, Monte Carlo Simulation; Strong Consistency, Symmetric Texture.

1 Introduction

We consider the following two-dimensional (2-D) sinusoidal model

$$y(s,t) = \sum_{k=1}^{p} \Big(A_k^0 \cos(s\lambda_k^0 + t\mu_k^0) + B_k^0 \sin(s\lambda_k^0 + t\mu_k^0) \Big) + e(s,t) \qquad (1)$$
$$s = 1,\ldots,M; \quad t = 1,\ldots,N,$$

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where A_k^0s and B_k^0s are the unknown amplitudes, λ_k^0s and μ_k^0s are the unknown frequencies and $\lambda_k^0, \mu_k^0 \in (0, \pi)$. The additive component $\{e(s,t)\}$ is from an independent and identically distributed (*i.i.d.*) random field, and the number of components, p is assumed to be known. Given a sample $\{y(s,t), s = 1, \ldots, M;\ t = 1, \ldots, N\}$, the problem is to estimate A_k^0, B_k^0, λ_k^0 and μ_k^0, $k = 1, \ldots, p$. Note that the model (1) is a natural 2-D extension of the one dimensional model, see for example Mitra, Kundu and Agarwal [21], Kundu [13] and see the references cited therein.

The first term on the right hand side of (1) is known as the signal component, and the second term as the noise component. The 2-D sinusoidal model as defined in (1) has received considerable attention in the signal processing literature because of its applicability in texture analysis. It is first observed by Francos, Meiri and Porat [8], see also Zhang and Mandrekar [34] and in Yuan and Subba Rao [32], that model (1) can be used quite effectively to model 2-D symmetric gray-scale texture images. Francos, Meiri and Porat [8] estimated the unknown frequencies by selecting the sharpest peaks at the Fourier frequencies of the 2-D periodogram function of the observed signals.

Figure 1 represents the 2-D image plot of a simulated $y(s,t)$ whose gray level at (s,t) is proportional to the value of $y(s,t)$. Our problem is to extract the regular texture from the contaminated one. Figure 2 represents the image plot of a real texture, and it will be shown in this paper that the data given by Figure 2 can be analyzed very effectively using model (1).

The problem has a special interest in spectrography, and it is studied by Malliavan [19, 20] using some group theoretic methods. This problem can be interpreted as 'signal detection' and it has different applications in Multidimensional Signal Processing. This model has been used quite extensively in antenna array processing, geophysical perception, biomedical spectral analysis etc, see, for example, the work of Barbieri and Barone [4], Cabrera and Bose [6], Chun and Bose [7], Hua [10] and Lang and McClellan [18]

The most natural estimators will be the least squares estimators (LSEs), which can be obtained by minimizing

$$\sum_{s=1}^{M}\sum_{t=1}^{N}\left[y(s,t) - \sum_{k=1}^{p}\left(A_k\cos(s\lambda_k + t\mu_k) + B_k\sin(s\lambda_k + t\mu_k)\right)\right]^2 \qquad (2)$$

with respect to the unknown parameters. It is well known, that the model does not satisfy the sufficient conditions of Jennrich [11] or Wu [31] for the LSEs to be consistent. Therefore, the consistency and asymptotic normality

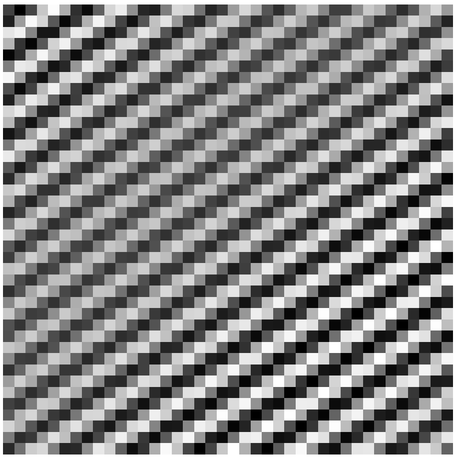

Figure 1: The image plot of a simulated dataset.

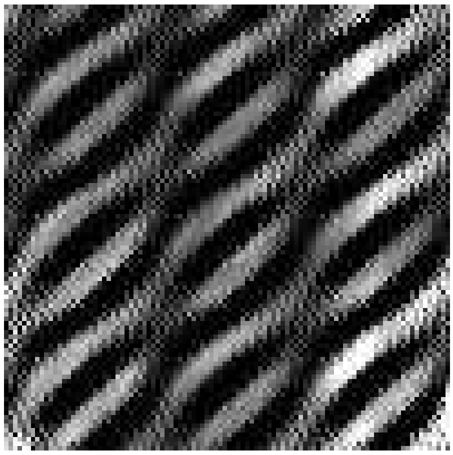

Figure 2: The image Plot of a real dataset.

properties are not immediate in this case. Rao, Zhao and Zhou [25] first proved the consistency and asymptotic normality of the LSEs of the unknown parameters of an equivalent model, under the assumptions that the errors are *i.i.d.* Gaussian random variables. Kundu and Mitra [15] extended the results when the errors are *i.i.d.* random variables with mean zero and finite variance. Kundu and Nandi [17] further extended their results when the errors are from a stationary sequence of random variables. Different variations of model (1) can be found in the literature, see for example, Bansal, Hamedani and Zhang [3] and Zhang [33].

Unfortunately, it is well known that even though the LSEs are the most efficient estimators, finding the LSEs of the frequencies is a numerically challenging problem and the procedure tends to be computationally quite intensive. The function, required to be optimized, is highly non-linear in its parameters even in case of one dimensional model and one needs to use an iterative procedure. Due to the presence of several local minima, convergence might be a tricky problem, see for example Rice and Rosenblatt [29]. In higher dimension the problem becomes more severe. The standard Newton-Raphson algorithm or its variants do not work. It often converges to a local minima rather than the global minimum.

Recently, Nandi, Prasad and Kundu [23] proposed an efficient algorithm to estimate the unknown parameters of (1) which provides estimators asymptotically equivalent to the LSEs. In this paper, we develop a non iterative procedure to estimate the unknown frequencies of model (1) extending the one-dimensional (1-D) noise space decomposition (NSD) method proposed by Kundu and Mitra [16] to 2-D model (1). The proposed 2-D NSD method provides consistent estimators of the unknown frequencies and it does not require any initial guesses unlike the method proposed by Nandi, Prasad and Kundu [23]. We have performed some experiments based on simulation to compare the effectiveness of the proposed method with the LSEs and approximate least squares estimators (ALSEs) obtained by maximizing the 2-D periodogram function. It is observed that the performance of the proposed method is much better than that of ALSEs, and comparable reasonably well with the LSEs. We have analyzed two data sets using the proposed method and the performances are quite satisfactory.

The organization of the rest of the paper is as follows. Some preliminary ideas are given in section 2. Here we provide a different formulation of model (1) and briefly discuss the Prony's estimators and their extension in 2-D. The 2-D NSD method for model (1) is discussed in section 3. Two pairing algorithms are proposed in section 4. The consistency results of the proposed estimators are provided in section 5. Numerical results are provided in section 6, two data analysis are performed in 7. Finally we conclude the paper in section 8.

2 Preliminaries

In this section, we first provide an equivalent formulation of the signal component of model (1) using complex exponentials. Then, we briefly discuss the Prony's method, which was proposed in 1795 [28], to find the non-linear parameters of a similar 1-D model in noiseless situation.

We write model (1) as $y(s,t) = m(s,t) + e(s,t)$, where $m(s,t)$ is the signal component. Then we observe that using complex exponentials, $m(s,t)$ can be written as

$$m(s,t) = \sum_{k=1}^{2p} C_k^0 e^{i(s\gamma_k^0 + t\delta_k^0)} \tag{3}$$

with

$$i = \sqrt{-1}, \qquad C_{2k}^0 = \frac{A_k^0 + iB_k^0}{2}, \qquad C_{2k-1}^0 = \frac{A_k^0 - iB_k^0}{2},$$
$$\gamma_{2k}^0 = -\lambda_k^0, \qquad \gamma_{2k-1}^0 = \lambda_k^0, \qquad \delta_{2k}^0 = -\mu_k^0, \qquad \delta_{2k-1}^0 = \mu_k^0, \qquad k = 1, \ldots, p.$$

Now $\gamma_k^0, \delta_k^0 \in (0, \pi)$ and C_k^0, $k = 1, \ldots, 2p$ are complex-valued. This form is quite useful in tackling the technical details.

2.1 Different Other Estimators

The LSEs of the unknown parameters are obtained by minimizing the following residual sum of squares with respect to unknown parameters A_k, B_k λ_k and μ_k, $k = 1, \ldots, p$.

$$\sum_{s=1}^{M}\sum_{t=1}^{N}\left(y(s,t) - \sum_{k=1}^{p}\left[A_k\cos(s\lambda_k + t\mu_k) + B_k\sin(s\lambda_k + t\mu_k)\right]\right)^2.$$

In section 6, we compare the LSE and ALSE with the proposed NSD estimators. The ALSE of λ_k and μ_k, $k = 1, \ldots, p$ are obtained by maximizing the 2-D periodogram function defined as follows;

$$I(\lambda, \mu) = \frac{1}{MN}\left|\sum_{s=1}^{M}\sum_{t=1}^{N} y(s,t)e^{-i(s\lambda + t\mu)}\right|^2.$$

The main idea of the 2-D periodogram estimators come from the 1-D periodogram estimators. Similarly, as the 1-D periodogram estimators, see for example Fuller (1976), as M, N tend to infinity, the peaks of the periodogram converge to the true frequencies.

For 2-D case, the maximization is done locally and sequentially under the constraints

$$|A_1^0|^2 + |B_1^0|^2 \geq |A_2^0|^2 + |B_2^0|^2 \geq \cdots \geq |A_p^0|^2 + |B_p^0|^2,$$

required to resolve the identifiability issues. Once the non-linear frequencies are obtained, corresponding linear parameters, A_ks and B_ks, are estimated as

$$\tilde{A}_k = \frac{2}{MN}\sum_{s=1}^{M}\sum_{t=1}^{N} y(s,t)\cos(s\tilde{\lambda}_k + t\tilde{\mu}_k),$$

$$\tilde{B}_k = \frac{2}{MN}\sum_{s=1}^{M}\sum_{t=1}^{N} y(s,t)\sin(s\tilde{\lambda}_k + t\tilde{\mu}_k),$$

where $\tilde{\lambda}_k$ and $\tilde{\mu}_k$ are the ALSEs of λ_k and μ_k, respectively. The ALSE's are asymptotically equivalent to the LSE's with the same rate of convergence (Kundu and Nandi [17]). Therefore, the ALSE's are also consistent and asymptotically normally distributed with the same asymptotic variance covariance matrix as the LSEs.

2.2 Prony's Method

Prony's [28] idea of fitting the sum of exponentials to the data has been extensively used in Signal Processing and Numerical Analysis. The method is described in several text books (Barrodale and Oleski [5], Kay [12]) in details. The proposed 2-D NSD method is based on 1-D NSD method and the later uses some concepts of Prony's method. So we briefly describe it here. Suppose $m_1(1), \ldots, m_1(n)$ are n data points from

$$m_1(t) = \sum_{k=1}^{q} \alpha_k e^{i\beta_k t}, \quad t = 1, \ldots, n \quad \text{where} \quad \beta_i \neq \beta_k, \quad i \neq k. \tag{4}$$

Here α_k's are complex numbers and β_k's are real numbers lying between $(0, 2\pi)$. For model (4), Prony observed that there exists $q+1$ constants, say, $g_1, \ldots, g_{q+1}$ such that they satisfy

$$\begin{array}{ccccccccc}
g_1 m_1(1) & + & g_2 m_1(2) & + & \cdots & + & g_{q+1} m_1(q+1) & = & 0 \\
g_1 m_1(2) & + & g_2 m_1(3) & + & \cdots & + & g_{q+1} m_1(q+2) & = & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
g_1 m_1(n-q) & + & g_2 m_1(n-q+1) & + & \cdots & + & g_{q+1} m_1(n) & = & 0
\end{array}$$

and there is a one - one correspondence between $\mathbf{g} = (g_1, \ldots, g_{q+1})$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)$ subject to the condition $\sum_{i=1}^{q+1} g_i^2 = 1$ and $g_1 > 0$. Then for $n \geq 2q+1$, the following q-degree polynomial

$$g_1 + g_2 x^2 + \cdots + g_{q+1} x^q = 0$$

has roots $e^{i\beta_1}, e^{i\beta_2}, \ldots, e^{i\beta_q}$. Thus $\beta_1, \ldots, \beta_q$ can be estimated once $g_1, \ldots, g_{q+1}$ are estimated. It is also observed that g_ks are independent of $\alpha_1, \ldots, \alpha_q$, $k = 1, \ldots, q$. Since Prony's algorithm is applicable to noiseless data, several problem specific adoptions have been considered and the method can be used to estimate the starting values of the nonlinear parameters of any iterative scheme.

The above idea can be extended to the 2-D case also. We concentrate on form (3) of $m(s,t)$. We write the signal component $\{m(s,t), s = 1, \ldots, M; t = 1, \ldots, N\}$ of model (1) in the following matrix form

$$\begin{bmatrix} m(1,1) & \cdots & m(1,N) \\ \vdots & \vdots & \cdots \\ m(M,1) & \cdots & m(M,N) \end{bmatrix} = \mathbf{M_S}, \quad \text{(say)}. \tag{5}$$

Note that there exists a vector $\mathbf{a} = (a_1, \ldots, a_{2p+1})$, with $|\mathbf{a}|^2 = 1$ and $a_1 > 0$, such that

$$\begin{bmatrix} m(1,1) & \cdots & m(1,N) \\ \vdots & \vdots & \cdots \\ m(M,1) & \cdots & m(M,N) \end{bmatrix} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & a_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{2p+1} & \vdots & \cdots & 0 \\ 0 & a_{2p+1} & \cdots & a_1 \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & a_{2p+1} \end{bmatrix} = \mathbf{0} \qquad (6)$$

$$= \mathbf{M_S A}, \quad \text{(say)}.$$

Similarly there exists a vector $\mathbf{b} = (b_1, \ldots, b_{2p+1})$, with $|\mathbf{b}|^2 = 1$ and $b_1 > 0$, such that

$$\begin{bmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & b_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ b_{2p+1} & \vdots & \cdots & 0 \\ 0 & b_{2p+1} & \cdots & b_1 \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & b_{2p+1} \end{bmatrix}^T \begin{bmatrix} m(1,1) & \cdots & m(1,N) \\ \vdots & \vdots & \cdots \\ m(M,1) & \cdots & m(M,N) \end{bmatrix} = \mathbf{0}$$

$$= \mathbf{B M_S}, \quad \text{(say)}.$$

Consider the two polynomial equations

$$B_1(z) = a_1 + a_2 z + \cdots + a_{2p+1} z^{2p} = 0, \qquad (7)$$

$$B_2(z) = b_1 + b_2 z + \cdots + b_{2p+1} z^{2p} = 0, \qquad (8)$$

then equation (7) has the roots $e^{i\gamma_1}, \ldots, e^{i\gamma_{2p}}$ and equation (8) has the roots $e^{i\delta_1}, \ldots, e^{i\delta_{2p}}$. Thus the roots are basically in the form $\exp(\pm i\lambda_k)$ and $\exp(\pm i\mu_k)$.

2.3 One-Dimensional NSD Method

Kundu and Mitra [16] proposed 1-D NSD decomposition method for the following model

$$y(t) = \sum_{k=1}^{q} \alpha_k e^{i\beta_k t} + \epsilon(t), \qquad (9)$$

here $\epsilon(t)$'s are complex valued random variables with mean zero and finite variance. The problem is same as before, *i.e.* to estimate the unknown parameters namely α_k's and β_k's based on a sample of size n, $\{y(1), \cdots, y(n)\}$. The 1-D NSD method can be described as follows. Let L be any integer within the interval $q < L < n-q$, and the $(n-q) \times (L+1)$ matrix

$$\mathbf{A} = \begin{bmatrix} y(1) & \cdots & y(L+1) \\ \vdots & \ddots & \vdots \\ y(n-L) & \cdots & y(n) \end{bmatrix}. \tag{10}$$

Construct the $(L+1)\times(L+1)$ matrix, $\mathbf{T}$ as $\mathbf{T} = \dfrac{1}{n}\mathbf{A}^H\mathbf{A}$; $\mathbf{A}^H$ is the hermitian matrix of $\mathbf{A}$. Let the spectral decomposition of the matrix $\mathbf{T}$ be

$$\mathbf{T} = \sum_{i=1}^{L+1} \widehat{\sigma}^2 \mathbf{w}_i \mathbf{w}_i^H, \tag{11}$$

where $\widehat{\sigma}_1^2 > \cdots > \widehat{\sigma}_{L+1}^2$ are the ordered eigen values of the matrix $\mathbf{T}$ and $\mathbf{w}_i$'s are orthogonal eigenvectors corresponding to $\widehat{\sigma}_i^2$. Construct the $(L+1) \times (L+1-q)$ matrix $\mathbf{C}$ as

$$\mathbf{C} = \left[\mathbf{w}_{q+1} : \cdots : \mathbf{w}_{L+1}\right]. \tag{12}$$

Partition the matrix $\mathbf{C}$ as

$$\mathbf{C} = \left[\mathbf{C}_{1k}^H : \mathbf{C}_{2k}^H : \mathbf{C}_{3k}^H\right], \tag{13}$$

for $k = 0, \cdots, L-q$, where $\mathbf{C}_{1k}^H$, $\mathbf{C}_{2k}^H$ and $\mathbf{C}_{3k}^H$ are of the orders $(L+1-q)\times k$, $(L+1-q)\times(q+1)$ and $(L+1-q)\times(L-k-q)$ respectively. Find a vector $\mathbf{V}_k$ such that

$$\begin{bmatrix} \mathbf{C}_{1k}^H \\ \\ \mathbf{C}_{3k}^H \end{bmatrix} \mathbf{V}_k = \mathbf{0}. \tag{14}$$

Let us denote the vector $\mathbf{c}_k = \mathbf{C}_{2k}^H\mathbf{V}_k$ for $k = 0, \cdots, L-q$. Consider the average of the vector $\mathbf{c}_k$ as the vector $\mathbf{c}$, *i.e.*,

$$\mathbf{c} = \frac{1}{L-q+1}\sum_{k=0}^{L-q} \mathbf{c}_k = (c_1, \cdots, c_{q+1}), \tag{15}$$

with $c_{q+1} = 1$. Construct the polynomial equation

$$c_1 + c_2 x + \cdots + c_q x^{q-1} + x^q = 0, \tag{16}$$

and obtain q roots, and estimate the frequencies from there. It has been shown by Kundu and Mitra (1995) that the estimated frequencies are strongly consistent.

3 Two-Dimensional NSD Method

In this section we propose the 2-D NSD method to estimate the non-linear frequencies of the 2-D sinusoidal model (1). Like the Prony's method presented in section 2, we will use form (3) for developing the algorithm. The proposed method which is basically an extension of 1-D NSD method to two-dimension, is as follows.

From the s^{th} row of the data matrix

$$\mathbf{Y} = \begin{bmatrix} y(1,1) & \cdots & y(1,N) \\ \vdots & \ddots & \vdots \\ y(s,1) & \cdots & y(s,N) \\ \vdots & \ddots & \vdots \\ y(M,1) & \cdots & y(M,N) \end{bmatrix}, \tag{17}$$

construct the matrix $\mathbf{A_s}$ for any $N - 2p \geq L \geq 2p$ as follows,

$$\mathbf{A_s} = \begin{bmatrix} y(s,1) & \cdots & y(s,L+1) \\ \vdots & \vdots & \vdots \\ y(s,N-L) & \cdots & y(s,N) \end{bmatrix}.$$

Obtain the $(L+1) \times (L+1)$ matrix $\mathbf{B}$ as

$$\mathbf{B} = \frac{1}{(N-L)M} \sum_{s=1}^{M} \mathbf{A_s}^H \mathbf{A_s}.$$

Suppose the singular value decomposition of $\mathbf{B}$ is

$$\mathbf{B} = \sum_{i=1}^{L+1} \lambda_i \mathbf{u_i} \mathbf{u_i}^H,$$

where $\lambda_1 \geq \cdots \geq \lambda_{L+1}$ are the ordered eigen values of $\mathbf{B}$ and $\mathbf{u}_i$ is the normalized eigen vector corresponding to λ_i.

Now as the 1-D NSD method, construct the estimated signal subspace $\mathcal{S}$ and the estimated noise subspace $\mathcal{N}$ as follows:

$$\mathcal{S} = \{\mathbf{u_1} : \cdots : \mathbf{u_{2p}}\} \quad \text{and} \quad \mathcal{N} = \{\mathbf{u_{2p+1}} : \cdots : \mathbf{u_{L+1}}\}.$$

We use the estimated noise space $\mathcal{N}$ to estimate $\mathbf{a} = (a_1, \ldots, a_{2p+1})$, the constants to construct the polynomial equation. Consider $(L+1) \times (L+1-p)$ matrix $\mathbf{B_1}$ as follows:

$$\mathbf{B_1} = [\mathbf{u_{2p+1}} : \cdots : \mathbf{u_{L+1}}] = \begin{bmatrix} b_{1,1} & \cdots & b_{1,L+1-2p} \\ \vdots & \vdots & \vdots \\ b_{L+1,1} & \cdots & b_{L+1,L+1-2p} \end{bmatrix}.$$

Now the aim is to obtain a basis of $\mathbf{B_1}$, which is of the same form as matrix $\mathbf{A}$ in equation (6). Partition the matrix $\mathbf{B_1}$ as follows:

$$\mathbf{B_1^T} = [\,\mathbf{B_{1k}^T} \;:\; \mathbf{B_{2k}^T} \;:\; \mathbf{B_{3k}^T}\,]$$

for $k = 0, 1, \ldots, L-2p$, where $\mathbf{B_{1k}^T}$, $\mathbf{B_{2k}^T}$ and $\mathbf{B_{3k}^T}$ are of the orders $(L+1-2p) \times k$, $(L+1-2p) \times (2p+1)$ and $(L+1-2p) \times (L-k-2p)$ respectively. Consider the matrix

$$[\,\mathbf{B_{1k}^T} \;:\; \mathbf{B_{3k}^T}\,].$$

Since it is a random matrix, it is of rank $(L-2p)$ (full rank) almost surely. Therefore, there exists an $L-2p+1$ column vector $\mathbf{X_{k+1}} \neq \mathbf{0}$, such that

$$\begin{bmatrix} \mathbf{B_{1k}} \\ \mathbf{B_{3k}} \end{bmatrix} \mathbf{X_{k+1}} = \mathbf{0}.$$

Consider the $(2p+1)$ vector $\hat{\mathbf{a}}^{k+1} = (\hat{a}_{k+1,1}, \ldots, \hat{a}_{k+1,2p+1})$, where

$$\left(\hat{\mathbf{a}}^{k+1}\right)^T = \mathbf{B_{2k}} \mathbf{X_{k+1}}.$$

By proper normalization, we can make $\hat{a}_{k+1,1} > 0$ and $||\hat{\mathbf{a}}^{k+1}||^2 = 1$ for $k = 0, 1, \ldots, L-2p$. Therefore, there exist vectors $\mathbf{X_1}, \ldots, \mathbf{X_{L-2p+1}}$ such that

$$\mathbf{B_1}\,[\,\mathbf{X_1}\quad:\quad\ldots\quad:\quad\mathbf{X_{L-2p+1}}\,]=\begin{bmatrix}\hat{a}_{1,1} & 0 & \cdots & 0\\ \vdots & \hat{a}_{2,1} & \cdots & \vdots\\ \hat{a}_{1,2p+1} & \vdots & \cdots & \hat{a}_{L-2p+1,1}\\ 0 & \hat{a}_{2,2p+1} & \cdots & \vdots\\ \vdots & 0 & \cdots & \vdots\\ 0 & 0 & \cdots & \hat{a}_{L-2p+1,2p+1}\end{bmatrix}.$$

In the noiseless situation, $\hat{\mathbf{a}}^1 = \ldots = \hat{\mathbf{a}}^{2p+1} = \hat{\mathbf{a}}$. So it is reasonable to use any one of $\hat{\mathbf{a}}^k,\ k = 0, 1, \ldots, L-2p$ or all of them to estimate $\delta_1, \ldots, \delta_{2p}$. One can consider the average of all the $\hat{\mathbf{a}}^k$s and use it as an estimate of $\mathbf{a}$. We have considered that $\hat{\mathbf{a}}$ for which the prediction error is minimum. To obtain the prediction errors we consider the following method:

From model (1), we obtain

$$\sum_{s=1}^{M} y(s,t) = \sum_{k=1}^{p}\sum_{s=1}^{M}\left\{A_k^0\cos(s\lambda_k^0 + t\mu_k^0) + B_k^0\sin(s\lambda_k^0 + t\mu_k^0)\right\} + \sum_{s=1}^{M} e(s,t), \tag{18}$$

and we also obtain

$$y_1(t) = \sum_{k=1}^{p}\left\{a_k^0\cos(t\mu_k^0) + b_k^0\sin(t\mu_k^0)\right\} + e_1(t) = m_2(t) + e_1(t),\ (say) \tag{19}$$

where

$$a_k^0 = A_k^0\sum_s\cos(s\lambda_k^0)+B_k^0\sum_s\sin(s\lambda_k^0),\ b_k^0 = -A_k^0\sum_s\sin(s\lambda_k^0)+B_k^0\sum_s\cos(s\lambda_k^0),$$

and $e_1(t) = \sum_s e(s,t)$. Similarly as the form in equation (3), $m_2(t)$ is also written as

$$m_2(n) = \sum_{k=1}^{2p} d_k^0 e^{it\delta_k^0},\quad d_{2k}^0 = \frac{a_k^0 + ib_k^0}{2},\quad d_{2k-1}^0 = \frac{a_k^0 - ib_k^0}{2}. \tag{20}$$

Now for all $i = 1, \ldots, 2p+1$, consider $\hat{\mathbf{a}}^i$ and solving polynomial equation (7) obtain the corresponding $\delta_1, \ldots, \delta_{2p}$. Then the linear parameters of the corresponding one dimensional model (19) are obtained and finally we obtain the prediction error of this model (19).

Exactly in the same way $\mathbf{b} = (b_1, \ldots, b_{2p+1})$ can be estimated using the columns of the data matrix $\mathbf{Y}$ and from the roots of the polynomial equation (8), we obtain the estimates of $\gamma_1, \ldots, \gamma_{2p}$. Finally, from $\gamma_1, \ldots, \gamma_{2p}$ and $\delta_1, \ldots, \delta_{2p}$, the estimates of $\lambda_1, \ldots, \lambda_p$ and $\mu_1, \ldots, \mu_p$ are obtained.

4 Pairing Algorithm

In this section we propose two pairing algorithms to estimate the pairs $\{(\lambda_k, \mu_k); k = 1, \ldots, p\}$ for model (1). One algorithm is based on $p!$ search. It is computationally efficient for small values of p, say $p = 2, 3$ and the other is based on p^2-search, so it is efficient for large values of p, *i.e.* when p is greater than 3. Suppose the estimates obtained using the method in section 3 are $\{\hat{\lambda}_{(1)}, \ldots, \hat{\lambda}_{(p)}\}$ and $\{\hat{\mu}_{(1)}, \ldots, \hat{\mu}_{(p)}\}$.

4.1 Algorithm 1

Consider all possible $p!$ combination of pairs $\{(\hat{\lambda}_{(j)}, \hat{\mu}_{(j)}) : j = 1, \ldots, p\}$ and calculate the sum of the periodogram function for each combination as

$$I(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1}^{p} \frac{1}{MN} \left| \sum_{s=1}^{M} \sum_{t=1}^{N} y(s,t) e^{-i(s\lambda_k + t\mu_k)} \right|^2 .$$

Consider that combination as the paired estimates of $\{(\lambda_j, \mu_j) : j = 1, \ldots, p\}$ for which this $I(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is maximum.

4.2 Algorithm 2

Consider the periodogram function $I_1(\lambda, \mu)$ of each pair $(\lambda_j, \mu_k),\ j = 1, \ldots, p,\ k = 1, \ldots, p$

$$I_1(\lambda, \mu) = \frac{1}{MN} \left| \sum_{s=1}^{M} \sum_{t=1}^{N} y(s,t) e^{-i(s\lambda + t\mu)} \right|^2 .$$

Compute $I_1(\lambda, \mu)$ over $\{(\hat{\lambda}_{(j)}, \hat{\mu}_{(k)}), j, k = 1, \ldots, p\}$. Choose the largest p values of $I(\hat{\lambda}_{(j)}, \hat{\mu}_{(k)})$ and the corresponding $\{(\hat{\lambda}_{(k)}, \hat{\mu}_{(k)}), k = 1, \ldots, p\}$ are the paired estimates of $\{(\lambda_k, \mu_k), k = 1, \ldots, p\}$.

5 Consistency Results

In this section we establish the strong consistency of the frequency estimators obtained by 2-D NSD method. We prove the results by using form (3). To prove the strong consistency we need the following assumptions as in the line of Rao, Zhao and Zhou [25] or Kundu and Mitra [16] on the parameters of model (1).

Assumption 1 *$\{e(s,t)\}$ is an array of independent and identically distributed real valued random variables with mean zero and finite variance σ^2.*

Assumption 2 *$\lambda_1, \ldots, \lambda_p$ are distinct and so also are $\mu_1, \ldots, \mu_p$.*

Assumption 3 *$A_1^0, \ldots, A_p^0$ and $B_1^0, \ldots, B_p^0$ are arbitrary real numbers not identically equal to zero.*

THEOREM 1 *Under the Assumptions 1 and 2, the estimators $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$ and $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \ldots, \hat{\mu}_p)$ obtained by the method described in section 3 are strongly consistent estimators of $\boldsymbol{\lambda}^0 = (\lambda_1^0, \ldots, \lambda_p^0)$ and $\boldsymbol{\mu}^0 = (\mu_1^0, \ldots, \mu_p^0)$ respectively.*

To prove Theorem 1, we need the following lemmas.

Lemma 1 *Let $\mathbf{Q} = ((Q_{ik}))$ and $\mathbf{W} = ((W_{ik}))$ be two $r \times r$ Hermitian matrices with spectral decomposition*

$$\mathbf{Q} = \sum_{i=1}^{r} \gamma_i \mathbf{q}_i \mathbf{q}_i^H, \qquad \gamma_1 \geq \ldots \geq \gamma_r,$$

$$\mathbf{W} = \sum_{i=1}^{r} \delta_i \mathbf{w}_i \mathbf{w}_i^H, \qquad \delta_1 \geq \ldots \geq \delta_r,$$

where $\gamma_i s$ and $\delta_i s$ are eigen values of $\mathbf{Q}$ and $\mathbf{W}$ respectively and $\mathbf{q}_i$ and $\mathbf{w}_i$ are the orthonormal eigenvectors of $\mathbf{Q}$ and $\mathbf{W}$ associated with γ_i and δ_i respectively, for $i = 1, \ldots, r$. Further assume that

$$\delta_{n_{h-1}+1} = \cdots = \delta_{n_h} = \tilde{\delta}_h; \qquad 0 = n_0 \ < \ n_1 \ < \ \cdots \ < \ n_s = p; \quad h = 1, \ldots, s;$$

$$\tilde{\delta}_1 \ > \ \tilde{\delta}_2 \ > \ \cdots \tilde{\delta}_s$$

and that $|q_{ik} - w_{ik}| \ < \ \alpha$, $i, k = 1, \ldots, r$, then there exists a constant K independent of α such that

$$(1) \quad |\gamma_i - \delta_i| \ < \ K\alpha, \qquad i = 1, \ldots, r,$$

$$(2) \sum_{i=n_{h-1}+1}^{n_h} \mathbf{q}_i \mathbf{q}_i^H \ = \ \sum_{i=n_{h-1}+1}^{n_h} \mathbf{w}_i \mathbf{w}_i^H + \mathbf{C}^{(\mathbf{h})},$$

with

$$\mathbf{C}^{(\mathbf{h})} = ((C_{lk}^{(h)})), \qquad |C_{lk}^{(h)}| \leq K\alpha.$$

Proof of Lemma 1 The proof mainly follows from von Neumann's [26] inequality. For details see Bai, Miao and Rao [2].

Lemma 2 *Let $g_n(x)$ be a sequence of polynomials of degree l, with roots $x_1^{(n)}, \dots, x_l^{(n)}$ for each n. Let $g(x)$ be a polynomial of degree l, with roots $x_1, \dots, x_l$. If $g_n(x) \to g(x)$ as $n \to \infty$ for all x, then with proper rearrangement the roots of $g_n(x)$, $x_k^{(n)}$ converge to the roots of $g(x)$, i.e. to x_k, $k = 1, \dots, l$.*

Proof of Lemma 2 See Bai [1].

Proof of Theorem 1 We note that using form (3)

$$\begin{aligned} y(s,t) &= \sum_{k=1}^{2p} C_k^0 e^{i(s\gamma_k^0 + t\delta_k^0)} + e(s,t) \\ &= \sum_{k=1}^{2p} g_{ks}^0 e^{it\delta_k^0} + e(s,t), \quad t = 1, \dots, N, \end{aligned}$$

where $g_{ks}^0 = C_k^0 e^{is\gamma_k^0}$, $k = 1, \dots, 2p$ and $s = 1, \dots, M$. Now consider the $(u,v)^{th}$ element of the matrix $\frac{1}{N-L}\mathbf{A_s^H A_s}$ for any s. The $y(s,t)$ and $e(s,t)$ being real-valued, we have

$$\begin{aligned} &\left(\left(\frac{1}{N-L}\mathbf{A_s^H A_s}\right)\right)_{u,v} \\ = &\frac{1}{N-L}\sum_{w=1}^{N-L+1} \bar{y}(s,u+w)y(s,v+w) \\ = &\frac{1}{N-L}\sum_{w=1}^{N-L+1}\left(\sum_{k=1}^{2p}\bar{g}_{ks}^0 e^{-i(u+w)\delta_k^0} + \bar{e}(s,u+w)\right)\left(\sum_{k=1}^{2p} g_{ks}^0 e^{i(v+w)\delta_k^0} + e(s,v+w)\right) \\ = &\frac{1}{N-L}\sum_{w=1}^{N-L+1}\left(\sum_{k=1}^{2p}\bar{g}_{ks}^0 e^{-i(u+w)\delta_k^0}\right)\left(\sum_{k=1}^{2p} g_{ks}^0 e^{i(v+w)\delta_k^0}\right) \\ &\quad + \frac{1}{N-L}\sum_{w=1}^{N-L+1}\bar{e}(s,u+w)\sum_{k=1}^{2p} g_{ks}^0 e^{i(v+w)\delta_k^0} \\ &\quad + \frac{1}{N-L}\sum_{w=1}^{N-L+1} e(s,v+w)\sum_{k=1}^{2p}\bar{g}_{ks}^0 e^{-i(u+w)\delta_k^0} \\ &\quad + \frac{1}{N-L}\sum_{w=1}^{N-L+1}\bar{e}(s,u+w)e(s,v+w), \\ = &T_1(s) + T_2(s) + T_3(s) + T_4(s). \end{aligned}$$

Now we observe that

$$
\begin{aligned}
T_1(s) &= \frac{1}{N-L}\left\{\sum_{w=1}^{N-L+1}\left(\sum_{k=1}^{2p}|g_{ks}^0|^2 e^{i\delta_k^0(v-u)}\right)\right. \\
&\qquad \left. + \sum_{w=1}^{N-L+1}\left(\sum_{k\neq l}^{2p}\bar{g}_{ks}^0 g_{lm}^0 e^{i\delta_l^0(v+w)-i\delta_k^0(u+w)}\right)\right\} \\
&= \sum_{k=1}^{2p}|g_{ks}^0|^2 e^{i\delta_k^0(v-u)} + O\left(\frac{1}{N-L}\right) \\
&\qquad\qquad \text{(For fixed } L \text{ and large } N).
\end{aligned}
$$

Now by the strong law of large numbers (see Chung; 1974, Theorem 5.4.1), we say that

$$
T_2(s) = o(1) = T_3(s),
$$

$$
T_4(s) = \begin{cases} \sigma^2 & \text{if } u = v \\ 0 & \text{if } u \neq v, \end{cases} \qquad s = 1, \ldots, M.
$$

Hence

$$
\lim_{N\to\infty}\frac{1}{N-L}\mathbf{A_s^H}\mathbf{A_s} = \sigma^2\mathbf{I}_{L+1} + \mathbf{\Omega^{(L)}}^{\mathbf{H}}\mathbf{D_s}\mathbf{\Omega^{(L)}} \qquad \text{a.s.,}
$$

where

$$
\mathbf{\Omega^{(L)}} = \begin{bmatrix} e^{-i\delta_1^0} & \cdots & e^{-i(L+1)\delta_1^0} \\ \vdots & \vdots & \vdots \\ e^{-i\delta_{2p}^0} & \cdots & e^{-i(L+1)\delta_{2p}^0} \end{bmatrix}
$$

and

$$
\mathbf{D_s} = \text{diag}\{|g_{1s}|^2, |g_{2s}|^2, \cdots |g_{2ps}|^2\}.
$$

$\mathbf{I}_{L+1}$ is the identity matrix of order $L+1$. Therefore, averaging over all rows,

$$
\begin{aligned}
\lim_{N\to\infty}\frac{1}{M(N-L)}\sum_{s=1}^{M}\mathbf{A_s^H}\mathbf{A_s} &= \sigma^2\mathbf{I}_{L+1} + \mathbf{\Omega^{(L)}}^{\mathbf{H}}\mathbf{D}\mathbf{\Omega^{(L)}} \\
&= \mathbf{S} \quad \text{(say),}
\end{aligned}
$$

where $\mathbf{D} = \sum_{s=1}^{M}\mathbf{D_s}$. Note that due to Assumption 2, the rank of the matrix $\mathbf{\Omega^{(L)}}$ is $2p$ and due to Assumption 3, the rank of $\mathbf{D}$ is also $2p$. Let the ordered eigen values of $\mathbf{S}$ be

$$
\lambda_{(1)} \geq \lambda_{(2)} \geq \cdots \lambda_{(p)} > \lambda_{(p+1)} = \cdots \lambda_{(L+1)} = \sigma^2
$$

and suppose the singular value decomposition of $\mathbf{S}$ is

$$\mathbf{S} = \sum_{k=1}^{L+1} \lambda_{(k)}^2 \mathbf{s_k} \mathbf{s_k^H}.$$

Here $\mathbf{s_k}$ is the orthonormal eigen vector corresponding to the eigen value $\lambda_{(k)}$. Therefore using Lemma 1, we have

$$\sum_{k=2p+1}^{L+1} \mathbf{\hat{u}_k} \mathbf{\hat{u}_k^H} \longrightarrow \sum_{k=2p+1}^{L+1} \mathbf{s_k} \mathbf{s_k^H}$$

i.e. the vector space generated by $(\mathbf{\hat{u}_{2p+1}}, \ldots, \mathbf{\hat{u}_{L+1}})$ converges to the vector space generated by $(\mathbf{s_{2p+1}}, \ldots, \mathbf{s_{L+1}})$. Now similarly as Kundu and Mitra (1995), it can be shown that the vector space generated by $(\mathbf{\hat{u}_{2p+1}}, \ldots, \mathbf{\hat{u}_{L+1}})$ has a unique basis of the form

$$\begin{bmatrix} \hat{a}_{1,1} & 0 & \cdots & 0 \\ \vdots & \hat{a}_{2,1} & \cdots & \vdots \\ \hat{a}_{1,2p+1} & \vdots & \cdots & \hat{a}_{L-2p+1,1} \\ 0 & \hat{a}_{2,2p+1} & \cdots & \vdots \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & \hat{a}_{L-2p+1,2p+1} \end{bmatrix},$$

with $\hat{a}_{k,1} > 0$ and $||\hat{\mathbf{a}}_k|| = 1$, where $\hat{\mathbf{a}}_k = (\hat{a}_{k,1}, \ldots, \hat{a}_{k,2p+1})$ for $k = 1, \ldots, L-2p+1$. Similarly the vector space generated by $(\mathbf{s_{2p+1}}, \ldots, \mathbf{s_{L+1}})$ has a unique basis of the form

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & a_1 & \cdots & \vdots \\ a_{2p+1} & \vdots & \cdots & a_1 \\ 0 & a_{2p+1} & \cdots & \vdots \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & a_{2p+1} \end{bmatrix},$$

with $a_1 > 0$ and $||\mathbf{a}_k|| = 1$, $\mathbf{a}_k = (a_1, \ldots, a_{2p+1})$. This implies that

$$\hat{\mathbf{a}}_k \to \mathbf{a} \quad \text{a.s.}, \quad k = 1, \ldots, L - p + 1.$$

Therefore, using Lemma 2, we can conclude that the roots obtained from the polynomial equation (7) with $\hat{\mathbf{a}}_k$, are consistent estimators of $\delta_1^0, \ldots, \delta_{2p}^0$ for $k = 1, \ldots, L - 2p + 1$.

To prove the strong consistency of $\hat{\gamma}_1, \ldots, \hat{\gamma}_{2p}$, consider the t^{th} column in place of s^{th} row. Then using the same technique as above, it can be shown that $\hat{\gamma}_1, \ldots, \hat{\gamma}_{2p}$ are strongly consistent estimators of $\gamma_1^0, \ldots, \gamma_{2p}^0$.

6 Numerical Experiments

In section 3, we have developed a method to estimate the frequencies of the 2-D sinusoidal models. The large sample properties of the estimators are examined in the previous section. In this section, we study the small sample properties of the estimators using simulated data. All the computations are performed at the Indian Institute of Technology Kanpur on Sun Workstation using the random number generator of Press *et al.* [27]. NAG subroutines are used for eigen decomposition and to obtain the roots of different polynomial equations. The numerical experiments have been conducted for different values of σ^2, the error variance.

We consider the following model with two components for the simulation study:

$$\begin{aligned} y(m,n) &= A_1^0 \cos(m\lambda_1^0 + n\mu_1^0) + B_1^0 \sin(m\lambda_1^0 + n\mu_1^0) \\ &\quad + A_2^0 \cos(m\lambda_2^0 + n\mu_2^0) + B_2^0 \sin(m\lambda_2^0 + n\mu_2^0) + e(m,n), \qquad (21) \\ &\qquad\qquad m = 1, \ldots, 30; \quad n = 1, \ldots, 30; \end{aligned}$$

with

$$A_1^0 = 4.0, B_1^0 = 5.0, \lambda_1^0 = 2.0, \mu_1^0 = 1.0,$$
$$A_2^0 = 3.5, B_2^0 = 5.5, \lambda_2^0 = 2.5, \mu_2^0 = 1.5.$$

The error random variables $e(m,n)$s are *i.i.d.* normal random variables with mean zero and variance σ^2. We consider $\sigma = 0.01, 0.03, 0.05, 0.1, 0.3, 0.5, 1.0$ and 2.0. For each σ we generate 1000 different data sets using different sequences of $e(m,n)$ and the frequencies are estimated using the method described in section 3. We obtain the average estimates and the mean squared errors over one thousand replications. We have also calculated the ALSEs and the LSEs for the frequencies of model (21) maximizing locally the 2-D periodogram function and minimizing the residual sum of squares respectively. The mean squared errors (MSEs) of the LSEs and the ALSEs and the asymptotic variances (ASYVs) of the LSEs are also reported for comparison purposes. The asymptotic variances (ASYV) are obtained using the distribution obtained for model (1) in Kundu and Gupta [14]. All these results are reported in Table 1. For different σ, we have reported the ALSEs, the NSD estimators, the LSEs and the ASYVs in the columns. The mean squared errors of the corresponding estimators are given in the brackets in the next row. As model (21) has two components we have used Algorithm 2 described in section 4.2 to obtain the final estimates of the frequencies.

From the results of simulation of model (21), it is observed that the proposed 2-D NSD method works quite well for different values of σ^2, the error variance. For small values of σ, the NSD estimators work better than the ALSEs and for large σ both the ALSEs and the NSD estimators perform almost identically. In case of one dimensional model the performances of the NSD estimators depends on L, the extended order. But it has been observed in simulation study that the performances of the 2-D NSD estimators do not depend much on the choice of L. It works better for small values of L ($\leq M/3$ or $\leq N/3$). For all L, less than equal to 10, the NSD estimators work almost in a similar way in terms of MSEs. As L increases, its performances deteriorate. Also the computational cost is very high if L is large as compared to small L. Though the ALSEs are computed as the local maxima of the 2-D periodogram function, the computational cost is still very high as compared to the NSD estimators. For L less than 20 *i.e.* $2M/3$, the ALSEs are more computationally expensive than the NSD estimators.

Table 1: The ALSEs, the NSD estimators, the LSEs, the corresponding MSEs and the asymptotic variances of the different parameters of the 2-D sinusoidal model.

σ	Parameter	ALSE	NSD estimator	LSE	ASYV
	λ_1	1.99699	2.00001	1.99999	
		(9.08479e-06)	(3.54279e-09)	(5.39972e-11)	7.2267387e-11
	μ_1	1.00029	1.00000	0.999994	
.01		(8.61867e-08)	(1.61219e-09)	(5.52318e-11)	7.2267387e-11
	λ_2	2.49917	2.50001	2.50002	
		(6.75802e-07)	(2.63496e-09)	(2.40655e-11)	6.9716774e-11
	μ_2	1.49932	1.50000	1.50001	
		(4.57080e-07)	(8.79792e-10)	(2.45539e-11)	6.9716774e-11
	λ_1	1.99698	2.00001	1.99999	
		(9.12686e-06)	(3.13326e-08)	(4.87039e-10)	6.5040645e-10
	μ_1	1.00029	1.00000	0.999999	
.03		(8.61407e-08)	(1.48746e-08)	(4.84410e-10)	6.5040645e-10
	λ_2	2.49917	2.50001	2.50001	
		(6.75478e-07)	(2.26150e-08)	(2.33181e-10)	6.2745098e-10
	μ_2	1.49932	1.50000	1.50000	
		(4.57967e-07)	(7.75861e-09)	(2.28176e-10)	6.2745098e-10
	λ_1	1.99698	2.00002	2.00000	
		(9.14752e-06)	(8.77772e-08)	(1.40601e-09)	1.8066845e-09
	μ_1	1.00029	1.00000	1.00000	
.05		(8.68718e-08)	(4.05905e-08)	(1.41209e-09)	1.8066845e-09
	λ_2	2.49918	2.50001	2.50001	
		(6.75214e-07)	(5.56449e-08)	(6.58749e-10)	1.7429194e-09
	μ_2	1.49933	1.50000	1.50000	
		(4.54628e-07)	(2.21050e-08)	(6.51090e-10)	1.7429194e-09
	λ_1	1.99697	2.00002	2.00000	
		(9.19954e-06)	(3.31397e-07)	(5.46929e-09)	7.2267388e-09
	μ_1	1.00029	1.00000	0.999999	
0.1		(9.84439e-08)	(1.59495e-07)	(5.61064e-09)	7.2267388e-09
	λ_2	2.49918	2.50001	2.50001	
		(6.83887e-07)	(2.26631e-07)	(2.43253e-09)	6.9716783e-09
	μ_2	1.49932	1.50000	1.50000	
		(4.64469e-07)	(9.38960e-08)	(2.43484e-09)	6.9716783e-09

(continued...)

σ	Parameter	ALSE	NSD estimator	LSE	ASYV
	λ_1	1.99696	1.99999	2.00000	
		(9.38446e-06)	(3.16364e-06)	(4.90887e-08)	6.5040652e-08
.3	μ_1	1.00029	0.99999	0.999999	
		(2.11111e-07)	(1.48538e-06)	(4.81352e-08)	6.5040652e-08
	λ_2	2.49918	2.50000	2.50001	
		(7.34819e-07)	(2.25009e-06)	(2.35643e-08)	6.2745102e-08
	μ_2	1.49932	1.49998	1.49999	
		(5.08247e-07)	(7.75322e-07)	(2.29973e-08)	6.2745102e-08
	λ_1	1.99696	2.00000	1.99998	
		(9.68857e-06)	(9.18530e-06)	(1.41363e-07)	1.8066847e-07
	μ_1	1.00029	1.00000	1.00002	
0.5		(4.36193e-07)	(4.08143e-06)	(1.41096e-07)	1.8066847e-07
	λ_2	2.49918	2.50001	2.50000	
		(8.28642e-07)	(5.46881e -06)	(6.58413e-08)	1.7429194e-07
	μ_2	1.49932	1.50003	1.50000	
		(6.02746e-07)	(2.19510e-06)	(6.57252e-08)	1.7429194e-07
	λ_1	1.99694	1.99969	1.99999	
		(1.11344e-05)	(3.84807e-05)	(5.33798e-07)	7.2267403e-07
	μ_1	1.00030	0.99991	1.00001	
1.0		(1.49700e-06)	(1.78891e-05)	(5.34425e-07)	7.2267403e-07
	λ_2	2.49918	2.50009	2.49999	
		(1.28682e-06)	(2.20381e-05)	(2.81330e-07)	6.9716793e-07
	μ_2	1.49931	1.49990	1.50000	
		(1.02375e-06)	(8.93373e-06)	(2.82741e-07)	6.9716793e-07
	λ_1	1.99690	1.99979	1.99993	
		(1.66416e-05)	(1.64697e-04)	(2.14585e-06)	2.8906954e-06
	μ_1	1.00033	0.99981	1.00005	
2.0		(5.70992e-06)	(7.68182e-05)	(2.15722e-06)	2.8906954e-06
	λ_2	2.49917	2.49967	2.50001	
		(3.10050e-06)	(9.20901e-05)	(1.04708e-06)	2.7886711e-06
	μ_2	1.49928	1.50013	1.49998	
		(2.68293e-06)	(4.02589e-05)	(1.03805E-06)	2.7886711E-06

7 Data Analysis

In this section we present the analysis of two data sets for illustrative purposes. (i) Synthesized Gray-scale texture, (ii) An real symmetric gray-scale texture.

7.1 A Synthesized Gray-scale Texture

Now we analyze a symmetric gray-scale texture generated from the following model for $m = 1, \cdots, 100$, $n = 1, \cdots, 100$,

$$\begin{aligned} y(m,n) &= 5.0\cos(1.5m + 1.0n) + 5.0\sin(1.5m + 1.0n) \\ &\quad +2.0\cos(1.4m + 0.9n) + 2.0\sin(1.4m + 0.9n) + e(m,n). \end{aligned} \tag{22}$$

Here $e(m,n)$'s are *i.i.d.* Gaussian random variables with mean 0 and variance 20. The noisy symmetric texture is plotted in Figure 3, and the original texture without the error component is plotted in Figure 1. The problem is to extract the original texture plotted in Figure 1 from the noisy texture plotted in Figure 3. Note that here the two frequency sets *viz*, (1.5, 1.0) and (1.4, 0.9) are very close to each other. When we plot the periodogram of the above data, see Figure 5, we observe a single peak. This obscures the fact that there are actually two frequency components, and thus makes it difficult to provide correct initial guesses of the frequencies. We use the 2-D NSD method with $L = 10$, and find the following estimates;

$$\begin{aligned} &\widehat{A}_1 = 4.7217, \quad \widehat{B}_1 = 4.8916, \quad \widehat{\lambda}_1 = 1.5110, \quad \widehat{\mu}_1 = 1.0112, \\ &\widehat{A}_2 = 2.0561, \quad \widehat{B}_2 = 1.9125, \quad \widehat{\lambda}_2 = 1.3887, \quad \widehat{\mu}_2 = 0.9005. \end{aligned}$$

We have plotted the estimated symmetric texture in Figure 4. It is clear that the method works quite well.

7.2 An Original Synthesized Gray-Scale Texture

We want to use model (1) to analyze the symmetric gray-scale texture Figure 2. To get an idea about the number of components, we have plotted the 2-D periodogram function of the observed texture in Figure 6. From the 2-D periodogram function, we note that there are many adjacent peaks, and it is not possible to guess the correct number of components from the periodogram. Here the number of components is chosen by minimizing the

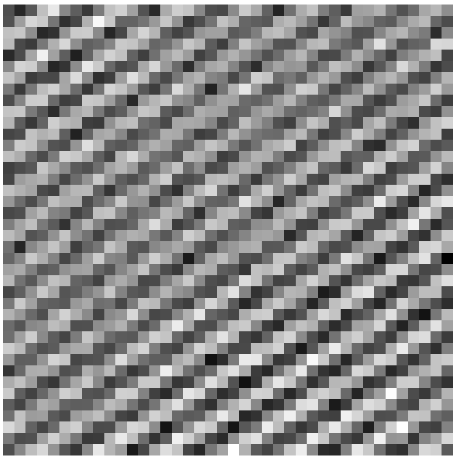

Figure 3: The image plot of the noisy model (22).

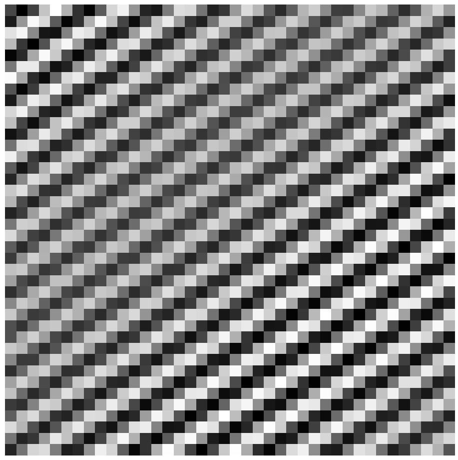

Figure 4: The image plot of the estimated texture of Figure 1.

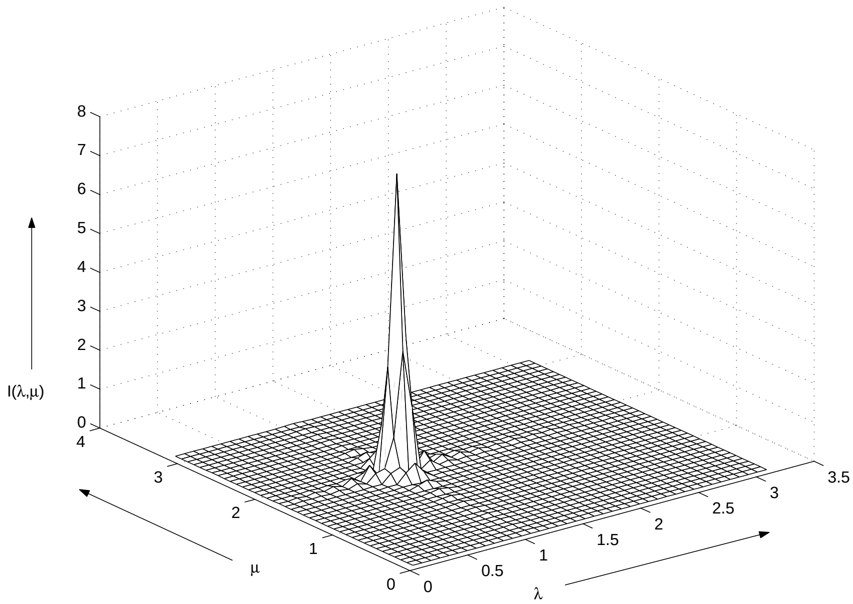

Figure 5: Periodogram of the synthesized texture.

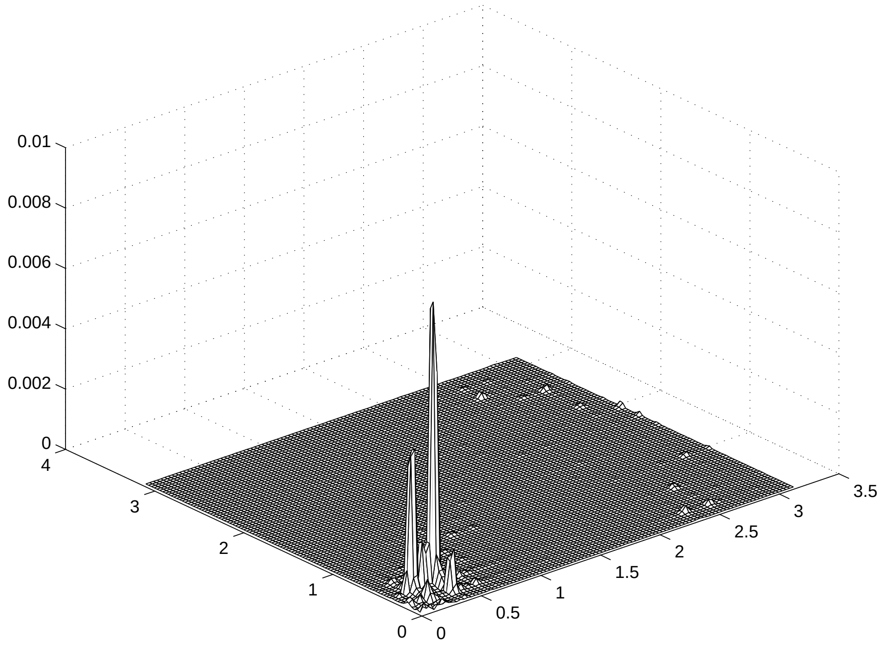

Figure 6: Periodogram of the symmetric gray-scale of Figure 2.

Bayesian Information Criterion (BIC), as it has been suggested in Nandi, Prasad and Kundu [23]. Plot of BIC values for different model order k is provided in Figure 7. In this case the estimate of q is 7. We have used $L = 15$, when the model order is less than 10, and $L = 20$, when the model order is greater than 10. We have provided the plot of the estimated texture in Figure 8. To test the randomness of the noise patterns, we have used Hopkins' test (see for example Zeng and Dubes [35]) with the number of sampling origins as 20, similarly as in Nandi, Prasad and Kundu [23]. The value of the Hopkins' statistic is 0.5231. Since the p value is greater than 0.25, we cannot reject the null hypothesis that the noise pattern is random. From the Figure 8, it is clear that the 2-D sinusoidal model provides a good fit to the symmetric gray-scale texture data, and moreover the proposed 2-D NSD method is also working quite well.

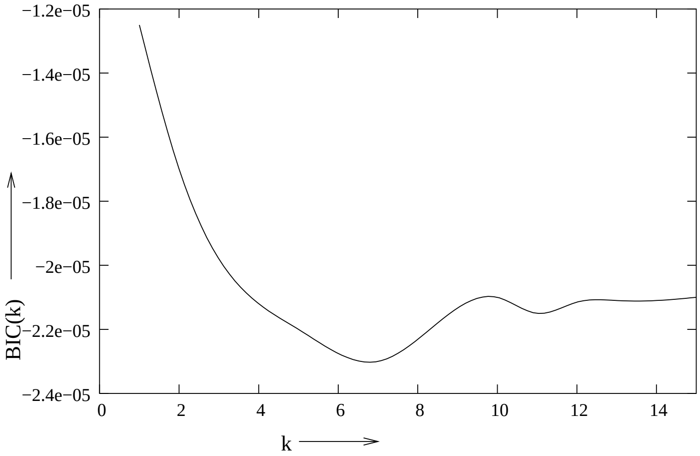

Figure 7: Plot of BIC for different values of model order.

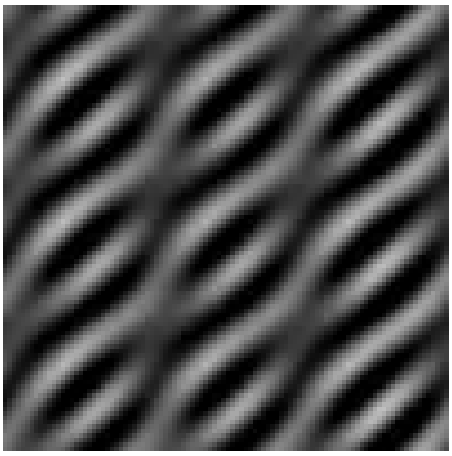

Figure 8: The estimated gray-scale texture.

8 Conclusions

In this paper, we have considered the estimation of the frequencies of the the 2-D sinusoidal model under the assumption of *i.i.d.* errors. Though the LSE is the most reasonable estimator, it is well known that obtaining the LSEs even in 1-D, is a difficult problem. We have developed a consistent non-iterative procedure to estimate the unknown parameters of the 2-D sinusoidal model (1) which is an extension of the 1-D NSD method. We have proposed two pairing algorithms to estimate the final set of frequencies. It has been observed that the proposed method provides consistent estimators. Numerical results indicate that the 2-D NSD estimators can be used as the starting values to obtain the LSEs for the sinusoidal model (1) in most of the cases. Also for the 2-D sinusoidal model the proposed estimators work better than the ALSEs. Recently, Prasad and Kundu [24] used three dimensional superimposed sinusoidal model to analyze colored textures. It seems the proposed 2-D NSD method can be extended to three dimension also. Work is in progress, it will be reported later.

In this paper we have considered model (1) when the errors are *i.i.d.* random variables. The natural question is whether the method is going to work when the errors are from a stationary sequences. In this case, it is immediate that for fixed M and N, if σ^2 goes to zero, the 2-D NSD estimates are strongly consistent. But for fixed σ^2, as M and N approach ∞, we could not establish the strong consistency of the proposed estimators in this case. Further work is needed in that direction.

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