Exact Inference for the Two-Parameter Exponential Distribution Under Type-II Hybrid Censoring *

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Abstract

Epstein [9] introduced the Type-I hybrid censoring scheme as a mixture of Type-I and Type-II censoring schemes. Childs *et al.* [5] introduced the Type-II hybrid censoring scheme as an alternative to Type-I hybrid censoring scheme, and provided the exact distribution of the maximum likelihood estimator of the mean of a one parameter exponential distribution based on Type-II hybrid censored samples. The associated confidence interval also has been provided. The main aim of this paper is to consider a two-parameter exponential distribution, and to derive the exact distribution of the maximum likelihood estimators of the unknown parameters based on Type-II hybrid censored samples. The marginal distributions and the exact confidence intervals are also provided. The results can be used to derive the exact distribution of the maximum likelihood estimator of the percentile point, and to construct the associated confidence interval. Different methods are compared using extensive simulations and one data analysis has been performed for illustrative purposes.

Key Words and Phrases: Hybrid censoring; two-parameter exponential distribution; joint moment generating function; maximum likelihood estimators; bootstrap confidence interval.

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1 Introduction

In life testing experiments, often the data are censored. Among the different censoring schemes, Type-I and Type-II censoring schemes are the two most popular censoring schemes. In Type-I censoring scheme, the experimental time is fixed, but the number of failures is random, whereas

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in Type-II censoring scheme, the experimental time is random but the number of failures is fixed. A mixture of Type-I and Type-II censoring scheme is known as the hybrid censoring scheme and it can be described as follows: Suppose a total of n units is placed on a life testing experiment, and the lifetimes of the sample units are independent and identically distributed (*i.i.d.*) random variables. Let the ordered lifetimes of these items are denoted by $T_{1:n}, \ldots, T_{n:n}$ respectively. The test is terminated when r, a pre-chosen number, out of n items are failed, or when a pre-determined time T on test has been reached, *i.e.*, the test is terminated at the time point $T_* = \min\{T_{r:n}, T\}$. This particular censoring scheme was introduced by Epstein [9], and it is popularly known now as the Type-I hybrid censoring scheme. It has been used quite extensively in reliability acceptance test in MIL-STD-781C [12].

Like conventional Type-I censoring scheme, the main disadvantage of Type-I hybrid censoring scheme is that most of the inferential results are obtained under the condition that the number of observed failures is at least one, and moreover, there may be very few failures at the termination point of the experiment. In that case the efficiency of the estimator(s) may be very low. Due to this reason, Childs *et al.* [5] introduced the Type-II hybrid censoring scheme, as an alternative to the Type-I hybrid censoring scheme, that would terminate the experiment at the time point $T^* = \max\{T_{r:n}, T\}$. It has the advantage of guaranteeing that at least r failures are observed at the end of the experiment. Childs *et al.* [5] considered the exact distribution of the maximum likelihood estimator (MLE) of the mean of an one-parameter exponential distribution based on a Type-II hybrid censored samples. The exact confidence interval of the mean also has been obtained based on the exact distribution. Recently, Banerjee and Kundu [3] considered the statistical inference of the two-parameter Weibull distribution based on Type-II hybrid censored samples. Very recently, Childs *et al.* [6] obtained the exact distributions of the MLEs of the scale and location parameters of a two-parameter exponential distribution, when the data are Type-I hybrid censored.

The purpose of this paper is to consider a Type-II hybrid censoring scheme when the lifetime distributions of the n experimental units are assumed to be *i.i.d.* two-parameter exponential distribution. It is assumed that the probability density function of the lifetime of an experimental unit, for $\theta > 0$, is

$$f(t;\mu,\theta) = \frac{1}{\theta}\, e^{-\frac{t-\mu}{\theta}} \quad \text{if} \quad t > \mu, \tag{1}$$

and 0 otherwise. First we obtain the MLEs of the unknown parameters of $\mu \in \mathbb{R}$ and $\theta > 0$, and provide the joint moment generating function. Based on the joint moment generating function (MGF), we obtain the marginal MGFs, and the marginal distribution functions of the MLEs. From the marginal distribution functions, using the same idea as in Chen and Bhattacharya [4] the confidence interval of θ is obtained by solving non-linear equations. Since the confidence interval based on the MLEs is quite difficult to implement, we propose to use different bootstrap confidence intervals, whose implementation are quite straight forward.

Rest of the paper is organized as follows. In Section 2, we present the maximum likelihood estimators (MLEs) of the unknown parameters, and the joint and marginal moment generating functions of the MLEs are derived in the same section. Different confidence intervals are proposed in Section 3. Simulation results and the analysis of a data set are presented in Section 4. Finally we conclude the paper in Section 5. Proof of the theorems are provided in the appendix.

2 Maximum Likelihood Estimators and Their Marginal Distributions

Let N be the number of units failed before the time T, then the likelihood of the observed data is given by

$$L\,(\mu,\,\theta\,|\,\text{Data}) \;=\; \begin{cases} \frac{n!}{(n-N)!\,\theta^N}\, e^{-\frac{1}{\theta}\sum_{i=1}^{N}(t_{i:n}-\mu)-\frac{1}{\theta}(n-N)(T-\mu)} & \text{if } t_{r:n} < T \\[2ex] \frac{n!}{(n-r)!\,\theta^r}\, e^{-\frac{1}{\theta}\sum_{i=1}^{r}(t_{i:n}-\mu)-\frac{1}{\theta}(n-r)(t_{r:n}-\mu)} & \text{if } t_{r:n} > T. \end{cases} \tag{2}$$

From (2), it is clear that MLE of θ does not exist when $r = 1$ and $N = 0$. So for $r = 1$, the conditional MLEs, conditioning on the event $\{N \geq 1\}$ is given by

$$\begin{aligned} \widehat{\mu} &= T_{1:n}, \\ \widehat{\theta} &= \frac{1}{N}\left\{\sum_{i=2}^{N} T_{i:n} - (n-1)\,T_{1:n} + (n-N)\,T\right\}. \end{aligned} \tag{3}$$

For $r \geq 2$, MLEs of the unknown parameters exist for all values of N and they are given by

$$\widehat{\mu} = T_{1:n},$$

$$\widehat{\theta} = \begin{cases} \frac{1}{N}\left\{\sum_{i=2}^{N} T_{i:n} - (n-1)\,T_{1:n} + (n-N)\,T\right\} & \text{if } T_{r:n} < T \\ \frac{1}{r}\left\{\sum_{i=2}^{r-1} T_{i:n} - (n-1)\,T_{1:n} + (n-r+1)\,T_{r:n}\right\} & \text{if } T_{r:n} > T. \end{cases} \tag{4}$$

Next we consider the distributions of the MLEs. When $r = 1$, MLE of θ does not exist if $N = 0$. So for $r = 1$, we consider the conditional distribution of $\widehat{\theta}$ conditioning on the event $\{N \geq 1\}$. For $r \geq 2$, MLE of θ exists for all values of N. In order to find the distribution of $\widehat{\mu}$ and $\widehat{\theta}$, we first find the joint moment generating function (MGF) of $\widehat{\theta}$ and $\widehat{\mu}$, and then inverting it to get the distribution of the MLEs.

Theorem 2.1. For $r = 1$, the conditional joint MGF of $(\widehat{\theta}, \widehat{\mu})$ conditioning on the event $\{N \geq 1\}$ exists for all $-\infty < \omega_1 < \infty$ and $-\infty < \omega_2 < \infty$ and is given by

$$\begin{aligned} & E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \,|\, N \geq 1] \\ = \; & (1-q^n)^{-1}\left[c_{10}\frac{e^{\mu_{10}\omega_1 + \mu\,\omega_2}}{\left(1+\frac{\omega_1}{\lambda_{10}}-\frac{\omega_2}{\nu_0}\right)} - d_{10}\frac{e^{T\omega_2}}{\left(1+\frac{\omega_1}{\lambda_{10}}-\frac{\omega_2}{\nu_0}\right)} + \frac{e^{\mu\,\omega_2} - q^n e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} \right. \\ & + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\left\{ c_{ij}\frac{e^{\mu_{ij}\omega_1+\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)} - d_{ij}\frac{e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)} \right\} \\ & \left. + \sum_{j=0}^{n-2}\left\{ c_{nj}\frac{e^{\mu_{nj}\omega_1+\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)} - d_{nj}\frac{e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)} \right\} \right]. \end{aligned}$$

For $r \geq 2$, the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ exists for $\omega_1 < \dfrac{r}{\theta}$ and $\omega_2 < \dfrac{1}{\theta} + \dfrac{n-1}{r}\,\omega_1$ and is given by

$$
\begin{aligned}
E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}}]
&= \frac{q^n\, e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_1}\right)^{\alpha_1}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} + \frac{e^{\mu\,\omega_2} - q^n e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} c_{ij}\, \frac{e^{\mu_{ij}\,\omega_1+\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)} - \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} d_{ij}\, \frac{e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)} \\
&\quad + \sum_{j=0}^{n-2} c_{nj}\, \frac{e^{\mu_{nj}\,\omega_1+\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)} - \sum_{j=0}^{n-2} d_{nj}\, \frac{e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)}
\end{aligned}
$$

when $\mu < T$ and for $\mu \geq T$

$$
E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}}] = \frac{e^{\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_1}\right)^{\alpha_1}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)}, \tag{5}
$$

where

$$
q = e^{-\frac{T-\mu}{\theta}}
$$

$$
\mu_{ij} = \begin{cases} \dfrac{1}{r}\,(n-j-1)\,(T-\mu) & \text{for } j=0,\cdots,i-1,\ i=1,\cdots,r-1 \\[2ex] \dfrac{1}{i}\,(n-j-1)\,(T-\mu) & \text{for } j=0,\cdots,i-1,\ i=r,\cdots,n. \end{cases} \tag{6}
$$

$$
\begin{aligned}
\alpha_i &= \begin{cases} r-1 & \text{for } i=1,\cdots,r-1 \\ i-1 & \text{for } i=r,\cdots,n \end{cases} \\
\nu_i &= \frac{i+1}{\theta} \qquad \text{for } i=0,\cdots,n-1 \\
\lambda_i &= \begin{cases} \dfrac{r}{\theta} & \text{for } i=1,\cdots,r-1 \\[2ex] \dfrac{i}{\theta} & \text{for } i=r,\cdots,n \end{cases} \\
\lambda_{ij} &= \begin{cases} \dfrac{r\,(j+1)}{(n-j-1)\,\theta} & \text{for } j=0,\cdots,i-1,\ i=1,\cdots,r-1 \\[2ex] \dfrac{i\,(j+1)}{(n-j-1)\,\theta} & \text{for } j=0,\cdots,i-1,\ i=r,\cdots,n \end{cases}
\end{aligned} \tag{7}
$$

$$
c_{ij} = (-1)^{i-j-1}\binom{n}{i}\binom{i}{j+1} q^{n-j-1} \qquad \text{for } j=0,\cdots,i-1,\ i=1,\cdots,n
$$

$$
d_{ij} = (-1)^{i-j-1}\binom{n}{i}\binom{i}{j+1} q^{n} \qquad \text{for } j=0,\cdots,i-1,\ i=1,\cdots,n
$$

∎

Remark 1. Note that as $T \to \infty$, the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ at (ω_1, ω_2) reduces to

$$\frac{e^{\mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1 - \frac{\omega_2}{\nu_{n-1}}\right)}$$

which is the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ in case of complete sample. That means as $T \to \infty$, $2n\widehat{\theta}/\theta$ is distributed as χ^2 random variable with $2n-2$ degrees of freedom, $n(\widehat{\mu} - \mu)/\theta$ is a standard exponential random variable, and they are independently distributed.

Remark 2. For $r \geq 2$, if T is smaller than the left end point of the support μ, *i.e.*, $T \leq \mu$, the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ at (ω_1, ω_2), as given in (5), becomes the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ in case of ordinary Type-II censoring scheme. In this case $2r\widehat{\theta}/\theta$ is distributed as χ^2 random variable with $2r-2$ degrees of freedom, $n(\widehat{\mu} - \mu)/\theta$ is a standard exponential random variable, and they are independently distributed.

Remark 3. It is obvious from the expression of the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ that the marginal distributions belong to the generalized mixtures of the well known distributions.

From the MGF of $\widehat{\theta}$, PDF of $\widehat{\theta}$ can be obtained by using the inversion technique as suggested by Chen and Bhattacharya [4], and it is as follows:

Theorem 2.2. For $r = 1$, conditional PDF of $\widehat{\theta}$ conditioned on $\{N \geq 1\}$ for $-\infty < t < \infty$ is

$$\begin{aligned} f_{\widehat{\theta}}(t) \;=\; & g_1(t; \alpha_n, \lambda_n) + (1-q^n)^{-1}\Bigg[c_{10}\, g_1(-t+\mu_{10}; 1, \lambda_{10}) - d_{10}\, g_1(-t; 1, \lambda_{10}) \\ & + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\, g_2(t-\mu_{ij}; \alpha_i, \lambda_i, \lambda_{ij}) - d_{ij}\, g_2(t; \alpha_i, \lambda_i, \lambda_{ij})\right\} \\ & + \sum_{j=0}^{n-2}\left\{c_{nj}\, g_2(t-\mu_{nj}; \alpha_n, \lambda_n, \lambda_{nj}) - d_{nj}\, g_2(t; \alpha_n, \lambda_n, \lambda_{nj})\right\}\Bigg]. \end{aligned} \tag{8}$$

For $r \geq 2$, the PDF of $\widehat{\theta}$ for $-\infty < t < \infty$ is

$$f_{\widehat{\theta}}(t) = \begin{cases} g_3(t) & \text{if } \mu < T \\ \\ g_1(t; \alpha_1, \lambda_1) & \text{if } \mu \geq T \end{cases} \tag{9}$$

where,

$$g_1(t;\alpha,\lambda) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda t} t^{\alpha-1} & \text{if } t \in (0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(t;\alpha,\lambda_1,\lambda_2) = \sum_{k=0}^{\alpha-1} p_k\, g_1(t;\alpha-k,\lambda_1) + \left(1-\sum_{k=0}^{\alpha-1} p_k\right) g_1(-t,1,\lambda_2) \quad \text{for } t \in \mathbb{R},$$

$p_k = \frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k.$

$$\begin{aligned} g_3(t) &= q^n\, g_1(t;\alpha_1,\lambda_1) + (1-q^n)\, g_1(t;\alpha_n,\lambda_n) \\ &+ \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} \{c_{ij}\, g_2(t-\mu_{ij};\alpha_i,\lambda_i,\lambda_{ij}) - d_{ij}\, g_2(t;\alpha_i,\lambda_i,\lambda_{ij})\} \\ &+ \sum_{j=0}^{n-2} \{c_{nj}\, g_2(t-\mu_{nj};\alpha_n,\lambda_n,\lambda_{nj}) - d_{nj}\, g_2(t;\alpha_n,\lambda_n,\lambda_{nj})\}\,. \end{aligned}$$

∎

Note that $g_3(t)$ depends on μ_{ij}, α_i, λ_i, and λ_{ij}; $i,j = 1,\ldots,n$, however, for brevity we do not write it explicitly.

Since the integration of density function over the whole range is one, we have, from the Theorem 2.2, the following identity.

$$\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}(c_{ij}-d_{ij}) + \sum_{j=0}^{n-2}(c_{nj}-d_{nj}) = 0\,.$$

Theorem 2.3. When $-\infty < t < \infty$, for $r=1$, conditional PDF of $\widehat{\mu}$ conditioning on $\{N \geq 1\}$ is given by

$$\begin{aligned} f_{\widehat{\mu}}(t) &= (1-q^n)^{-1}\Bigg[c_{10}\, g_1(t-\mu;1,\nu_0) - d_{10} g_1(t-T;1,\nu_0) + g_1(t-\mu;1,\nu_{n-1}) \\ &-q^n g_1(t-T;1,\nu_{n-1}) + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\{c_{ij}\, g_1(t-\mu;1,\nu_j) - d_{ij}\, g_1(t-T;1,\nu_j)\} \\ &+ \sum_{j=0}^{n-2}\{c_{nj}\, g_1(t-\mu;1,\nu_j) - d_{nj}\, g_1(t-T;1,\nu_j)\}\Bigg], \end{aligned}$$

and for $r \geq 2$, PDF of $\widehat{\mu}$ is

$$f_{\widehat{\mu}}(t) = \begin{cases} g_4(t-\mu) & \text{if } \mu < T \\ g_1(t-\mu; 1, \nu_{n-1}) & \text{if } \mu \geq T, \end{cases}$$

where

$$\begin{aligned} g_4(t) &= g_1(t; 1, \nu_{n-1}) + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\, g_1(t; 1, \nu_j) - d_{ij}\, g_1(t+\mu-T; 1, \nu_j)\right\} \\ &\quad + \sum_{j=0}^{n-2}\left\{c_{nj}\, g_1(t; 1, \nu_j) - d_{nj}\, g_1(t+\mu-T; 1, \nu_j)\right\}. \end{aligned}$$

∎

Note that $g_4(t)$ depends on ν_i; $i = 1, \ldots, n$, however, for brevity we do not write it explicitly. From PDFs of $\widehat{\theta}$, the corresponding moments can be easily obtained. The first two moments of $\widehat{\theta}$ are as follows. For $r = 1$,

$$\begin{aligned} E[\widehat{\theta}] &= \theta + \theta A_1(\mu, \theta) + (1-q^n)^{-1} B_1(\mu, \theta), \\ E[\widehat{\theta}^2] &= \theta^2 + \theta^2 C_1(\mu, \theta) + \theta D_1(\mu, \theta) + (1-q^n)^{-1} E_1(\mu, \theta). \end{aligned}$$

For $r \geq 2$,

$$E[\widehat{\theta}] = \begin{cases} \theta + \theta A_2(\mu, \theta) + B_1(\mu, \theta) & \text{if } \mu < T \\ \left(1 - \frac{1}{r}\right)\theta & \text{if } \mu \geq T, \end{cases}$$

$$E[\widehat{\theta}^2] = \begin{cases} \theta^2 + \theta^2 C_2(\mu, \theta) + \theta D_2(\mu, \theta) + E_1(\mu, \theta) & \text{if } \mu < T \\ \left(1 - \frac{1}{r}\right)\theta^2 & \text{if } \mu \geq T, \end{cases}$$

where

$$A_1(\mu,\theta) = \frac{1}{1-q^n}\Bigg[\sum_{i=2}^{n}\sum_{\substack{j=0\\ j\neq n-1}}^{i-1}(c_{ij}-d_{ij})\left\{\sum_{k=0}^{i-2}p_{jk}\frac{i-k-1}{i}-\left(1-\sum_{k=0}^{i-2}p_{jk}\right)\frac{n-j-1}{i(j+1)}\right\}$$
$$-\frac{1}{r}(c_{10}-d_{10})(n-1)-\frac{1-q^n}{n}\Bigg],$$

$$B_1(\mu,\theta) = \sum_{i=1}^{n}\sum_{j=0}^{i-1}c_{ij}\mu_{ij},$$

$$C_1(\mu,\theta) = \Bigg[\sum_{i=1}^{n}\sum_{\substack{j=0\\ j\neq n-1}}^{i-1}(c_{ij}-d_{ij})\left\{\sum_{k=0}^{i-2}p_{jk}\frac{i-k-1}{(i-k)^{-1}i^2}+2\left(1-\sum_{k=0}^{i-2}p_{jk}\right)\frac{n-j-1}{i^2(j+1)^2}\right\}$$
$$+2(n-1)(c_{10}-d_{10})-\frac{1-q^n}{n}\Bigg](1-q^n)^{-1},$$

$$D_1(\mu,\theta) = 2(1-q^n)^{-1}\Bigg[\sum_{i=1}^{n}\sum_{\substack{j=0\\ j\neq n-1}}^{i-1}c_{ij}\mu_{ij}\left\{\sum_{k=0}^{i-2}p_{jk}\frac{i-k-1}{i}-\left(1-\sum_{k=0}^{i-2}p_{jk}\right)\frac{n-j-1}{i(j+1)}\right\}$$
$$+(n-1)c_{10}\mu_{10}\Bigg],$$

$$E_1(\mu,\theta) = \sum_{i=1}^{n}\sum_{j=0}^{i-1}c_{ij}\mu_{ij}^2,$$

$$A_2(\mu,\theta) = q^n\left(\frac{1}{n}-\frac{1}{r}\right)-\frac{1}{n}$$
$$+\sum_{i=1}^{r-1}\sum_{j=0}^{i-1}(c_{ij}-d_{ij})\left\{\sum_{k=0}^{r-2}p_{jk}\frac{r-k-1}{r}-\left(1-\sum_{k=0}^{r-2}p_{jk}\right)\frac{n-j-1}{r(j+1)}\right\}$$
$$+\sum_{i=r}^{n}\sum_{\substack{j=0\\ j\neq n-1}}^{i-1}(c_{ij}-d_{ij})\left\{\sum_{k=0}^{i-2}p_{jk}\frac{i-k-1}{i}-\left(1-\sum_{k=0}^{i-2}p_{jk}\right)\frac{n-j-1}{i(j+1)}\right\},$$

$$C_2(\mu,\theta) = \sum_{i=1}^{r-1}\sum_{j=0}^{i-1}(c_{ij}-d_{ij})\left\{\sum_{k=0}^{r-2}p_{jk}\frac{r-k-1}{r^2(r-k)^{-1}}+2\left(1-\sum_{k=0}^{r-2}p_{jk}\right)\frac{(n-j-1)^2}{r^2(j+1)^2}\right\}$$
$$+\sum_{i=r}^{n}\sum_{\substack{j=0\\ j\neq n-1}}^{i-1}(c_{ij}-d_{ij})\left\{\sum_{k=0}^{i-2}p_{jk}\frac{(i-k-1)}{i^2(i-k)^{-1}}+2\left(1-\sum_{k=0}^{i-2}p_{jk}\right)\frac{(n-j-1)^2}{i^2(j+1)^2}\right\}$$
$$-\frac{q^n}{r}-\frac{1-q^n}{n},$$

$$
\begin{aligned}
D_2(\mu,\theta) &= 2\sum_{i=1}^{r-1}\sum_{j=0}^{i-1} c_{ij}\mu_{ij}\left\{\sum_{k=0}^{r-2} p_{jk}\frac{r-k-1}{r} - \left(1-\sum_{k=0}^{r-2} p_{jk}\right)\frac{n-j-1}{r(j+1)}\right\} \\
&\quad +2\sum_{i=r}^{n}\sum_{\substack{j=0\\ j\neq n-1}}^{i-1} c_{ij}\mu_{ij}\left\{\sum_{k=0}^{i-2} p_{jk}\frac{i-k-1}{i} - \left(1-\sum_{k=0}^{i-2} p_{jk}\right)\frac{n-j-1}{i(j+1)}\right\}.
\end{aligned}
$$

The CDFs of $\widehat{\theta}$ and $\widehat{\mu}$ can be easily obtained from their respective PDFs. The CDFs are as follows. When $-\infty < t < \infty$, the CDF of $\widehat{\theta}$ for $r = 1$ is

$$
\begin{aligned}
F_{\widehat{\theta}}(t) &= \Gamma^*(\lambda_n t, \alpha_n) + (1-q^n)^{-1}\Bigg[c_{10}\, e^{-\lambda_{10}\langle \mu_{10}-t\rangle} - d_{10}\, e^{-\lambda_{10}\langle -t\rangle} \\
&\quad + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\{c_{ij}\, G(t;\mu_{ij},\alpha_i,\lambda_i,\lambda_{ij}) - d_{ij}\, G(t;0,\alpha_i,\lambda_i,\lambda_{ij})\} \\
&\quad + \sum_{j=0}^{n-2}\{c_{nj}\, G(t;\mu_{nj},\alpha_n,\lambda_n.\lambda_{nj}) - d_{nj}\, G(t;0,\alpha_n,\lambda_n,\lambda_{nj})\}\Bigg],
\end{aligned}
$$

and for $r \geq 2$, the CDF is

$$
F_{\widehat{\theta}}(t) = \begin{cases}
q^n\,\Gamma^*(\lambda_1 t,\alpha_1) + (1-q^n)\ \Gamma^*(\lambda_n t,\alpha_n) \\
\quad + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\{c_{ij}\, G(t;\mu_{ij},\alpha_i,\lambda_i,\lambda_{ij}) - d_{ij}\, G(t;0,\alpha_i,\lambda_i,\lambda_{ij})\} \\
\quad + \sum_{j=0}^{n-2}\{c_{nj}\, G(t;\mu_{nj},\alpha_n,\lambda_n.\lambda_{nj}) - d_{nj}\, G(t;0,\alpha_n,\lambda_n,\lambda_{nj})\} & \text{if } \mu < T \\
\Gamma^*(\lambda_1 t,\alpha_1) & \text{if } \mu \geq T,
\end{cases}
$$

where, $\langle x\rangle$ denotes the $\max\{0,\, x\}$, and

$$
\Gamma^*(t;\,\alpha) = \begin{cases} 0 & \text{if } t \leq 0 \\ \int_0^t \frac{1}{\Gamma(\alpha)}\, e^{-x}x^{\alpha-1}dx & \text{if } t > 0, \end{cases}
$$

$$
G(t;\,\mu,\,\alpha,\,\lambda_1,\,\lambda_2) = \begin{cases} \left(1-\sum_{k=0}^{\alpha-1} p_k\right)(1-\Gamma^*(-\lambda_2 t;\,1)) & \text{if } t \leq 0 \\ 1-\left(\sum_{k=0}^{\alpha-1} p_k(1-\Gamma^*(\lambda_1 t;\,\alpha-k)\right) & \text{if } t \leq 0. \end{cases}
$$

The CDF of $\widehat{\mu}$ for $r = 1$ is

$$
F_{\widehat{\mu}}(t) = \begin{cases} 0 & \text{if } t < \mu \\ (1-q^n)^{-1}\left[1-e^{-\frac{n}{\theta}(t-\mu)}\right] & \text{if } \mu \leq t < T \\ 1 & \text{if } t \geq T, \end{cases}
$$

and for $r \geq 2$, the same is

$$F_{\widehat{\mu}}(t) \quad = \quad \begin{cases} 0 & \text{if } t < \mu \\ 1 - e^{-\frac{n}{\theta}(t-\mu)} & \text{if } t \geq \mu\,. \end{cases}$$

Note that the p^{th} quantile of the exponential distribution given in (1) is $\eta_p = \mu + a_p\,\theta$, where $a_p = -ln(1-p)$. Therefore the MLE of η_p can be found by replacing μ and θ by their respective MLEs. The moment generating function of the MLE of η_p, say $\widehat{\eta}_p$ can be obtained easily from the joint MGF of $\widehat{\mu}$ and $\widehat{\theta}$ in Theorem 2.1.

3 Confidence Intervals:

Note that the distribution of $\widehat{\mu}$ is same as that of the lowest order statistic and this has been extensively studied in the literature, hence we do not pursue it here. In this section, we present different methods for construction of confidence intervals (CIs) for the unknown parameter θ. From the Theorem 2.2, we can find the approximate CI for θ. However, as PDF of $\widehat{\theta}$ is quite complicated, we also present the bootstrap CI for the same parameter.

3.1 Approximate Confidence interval:

From CDF of $\widehat{\theta}$, approximate CI can be found, based on the assumption that CDF of $\widehat{\theta}$ is a strictly decreasing function of θ. Several authors including Chen and Bhattacharyya [4], Gupta and Kundu [10], Kundu and Basu [11], Childs *et al.* [5], and Balakrishnan *et al.* [2] used this method to find the CI of θ. Though it is difficult to verify the assumption, an extensive numerical study supports the monotonicity assumption.

Suppose $\widehat{\theta}_{\text{obs}}$ is the estimate of θ. Then a two-sided $100(1-\alpha)\%$ approximate CI, say (θ_L, θ_U), for θ can be constructed by solving the equations

$$F_{\theta_L}(\widehat{\theta}_{obs}) = 1 - \frac{\alpha}{2} \quad \text{and} \quad F_{\theta_U}(\widehat{\theta}_{obs}) = \frac{\alpha}{2}$$

for θ_L(the lower bound of θ) and θ_U(the upper bound of θ) replacing μ by its MLE. However, they are nonlinear equations, and we need to solve them by some numerical procedure, *e.g.*, bisection method or Newton-Raphson method.

3.2 Bootstrap Confidence Interval:

The exact CIs presented in the previous section are computationally quite complicated, specially when sample size is large. So we consider the bootstrap CIs. Here we consider two types of bootstrap CIs, *viz.*, percentile bootstrap CI and Bias adjusted percentile (BCa) interval; see Efron and Tibshirani [8] for details.

3.2.1 Bootstrap sample:

Step 1. Given T, r, n, and the original Type-II sample, $\widehat{\mu}$ and $\widehat{\theta}$ are obtained from (3) or (4).

Step 2. Based on T, r, n, $\widehat{\mu}$, and $\widehat{\theta}$, a random sample of size n from Uniform(0,1) distribution is generated, and order them to get $(U_{1:n}, \ldots, U_{n:n})$.

Step 3. Let

$$t^*_{i:n} = \widehat{\mu} - \widehat{\theta}\ log\ U_{i:n}$$

Step 4. If $t^*_{r:n} < T$, then find N_1 such that

$$t^*_{N_1:n} < T \leq t^*_{N_1+1:n}\,, \quad \text{and set} \quad N^* = \begin{cases} N_1 & \text{if} \quad t^*_{r:n} < T \\ r & \text{if} \quad t^*_{r:n} \geq T \end{cases}.$$

Now, $\{t^*_{1:n}, \ldots, t^*_{N^*:n}\}$ is the bootstrap sample.

Step 5. Based on n, T, r, and the bootstrap sample, $\widehat{\mu}^*$ and $\widehat{\theta}^*$ are obtained form (3) or (4).

Step 6. Step 1-5 are repeated B times, and $\widehat{\theta}^*$'s are ordered in ascending order to obtain the bootstrap sample

$$\left\{\widehat{\theta}^{*[1]}, \widehat{\theta}^{*[2]}, \ldots, \widehat{\theta}^{*[B]}\right\}.$$

3.2.2 Percentile bootstrap CI:

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval for θ is

$$\left(\widehat{\theta}^{*[\frac{\alpha}{2}B]}, \widehat{\theta}^{*[(1-\frac{\alpha}{2})B]}\right),$$

where, $[x]$ denotes the largest integer less than or equal to x.

3.2.3 Bias adjusted percentile (BCa) interval:

A two-sided $100(1-\alpha)\%$ BCa bootstrap confidence interval for θ is

$$\left(\widehat{\theta}^{*[\alpha_1 B]}, \widehat{\theta}^{*[\alpha_2 B]}\right),$$

where,

$$\alpha_1 = \Phi\left\{\widehat{z}_0 + \frac{\widehat{z}_0 + z_{1-\alpha/2}}{1 - a(\widehat{z}_0 + z_{1-\alpha/2})}\right\} \quad \text{and} \quad \alpha_2 = \Phi\left\{\widehat{z}_0 + \frac{\widehat{z}_0 + z_{\alpha/2}}{1 - a(\widehat{z}_0 + z_{\alpha/2})}\right\}.$$

Here $\Phi(\cdot)$ is the CDF of the standard normal distribution, z_α is the upper α-point of standard normal distribution and

$$\widehat{z}_0 = \Phi^{-1}\left\{\frac{\#\text{ of } \widehat{\theta}^{*[j]} < \widehat{\theta}}{B}\right\}, \quad j = 1, \ldots, B.$$

A estimate of the acceleration a is

$$\widehat{a} = \frac{\sum_{i=1}^{N^*}\left[\widehat{\theta}^{(\cdot)} - \widehat{\theta}^{(i)}\right]^3}{6\left\{\sum_{i=1}^{N^*}\left[\widehat{\theta}^{(\cdot)} - \widehat{\theta}^{(i)}\right]^2\right\}^{3/2}},$$

where $\widehat{\theta}^{(i)}$ is the MLE of θ based on the original sample with the i^{th} observation deleted, and

$$\widehat{\theta}^{(\cdot)} = \frac{1}{N^*}\sum_{i=1}^{N^*}\widehat{\theta}^{(i)}.$$

4 Simulation Results and Data Analysis

In this section the results of Monte Carlo simulation is presented to study the performance of the inference procedures described in the Sections 3. We choose the value of the location parameter μ to be zero and different values for the scale parameter θ, *viz.*, 1.00, 2.00, 3.00, 4.00 and 5.00. We also take $n = 20$, $r = 16$ and different choices for T, *viz.*, 1.5, 2.5, 3.5. The coverage percentage of different confidence interval discussed in the section 3 are calculated based on the 10,000 Monte Carlo simulations and $B = 1000$. These values are presented in the Tables 2, 3 and 4.

It is clear from the Table 2 that the approximate method of constructing confidence interval is always maintaining its coverage percentage to its pre-fixed nominal level. Among the bootstrap methods for constructing confidence interval, adjusted percentile bootstrap method is better then percentile bootstrap method with respect to the coverage percentage. From the Table 3, we can see that the coverage percentage of the percentile bootstrap method is quite lower than its pre-fixed nominal level, while the same of the adjusted bootstrap method is somewhat close to its nominal level (see Table 4). Though approximate method of construction of confidence interval is always better with respect to coverage percentage and average length, it is very complicated to calculate specially when n is large. So we suggest to use the BCa bootstrap CI for the large values of n.

Next, we consider a data set to illustrate the procedures described in the previous sections. Here we consider the data provided in Bain [1]. Suppose a sample of 20 items are put on the test and the test is terminated after 150 hours. There are 13 failures within this time and the failures times are 3, 19, 23, 26, 37, 38, 41, 45 ,58, 84, 90, 109, and 138. To make an artificial hybrid Type-II data from this data set one can take any r less than or equal to 13. Here we take $r = 12$. Under the assumption that lifetime has two-parameters exponential distribution, MLEs of μ and θ are 3 and 130.85 respectively. Different types of confidence intervals are reported in the Table 1.

5 Conclusion

In this paper, we have considered the Hybrid Type-II censoring scheme, when life times have two-parameters exponential distribution, under frequentist approach. We have found the MLEs for both the parameters. We have considered different methods for construction of confidence interval. We have seen that the approximate and BCa bootstrap methods of construction of confidence interval are quite good. So we recommend to use BCa bootstrap method of construction of confidence interval specially when n is large.

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A Appendix

We need the the following lemmas to prove the theorems.

Lemma A.1. Let $T_{1:n} < \ldots < T_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$ and the corresponding CDF is $F(\cdot)$. Let T be a pre-fixed number such that $F(T)$ is positive and N denote the number of order statistics less than or equal to T. The conditional joint PDF of $T_{1:n}, \ldots, T_{N:n}$ conditioned on the event $N = i$ is given by

$$f\left(t_1, t_2, \ldots, t_i \,|\, N = i\right) = \frac{n!}{(n-i)! P[N = i]} \prod_{j=1}^{i} f\left(t_j\right) \left\{1 - F\left(T\right)\right\}^{n-i}$$

$$\text{if } t_1 < \ldots < t_i < T,$$

where $F(\cdot)$ is the distribution function of $f(\cdot)$.

Proof: See [5].

Lemma A.2. Let $T_{1:n} < \ldots < T_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$ and the corresponding CDF is $F(\cdot)$. Let T be a pre-fixed number such that $F(T)$ is positive and N denote the number of order statistics less than or equal to T. Let $r \in \{1, 2, \ldots, n\}$ be a pre-fixed integer. Then PDF of $T_{1:n}, T_{2:n}, \ldots, T_{r:n}$ conditioned on the event $N = 0$ is given by

$$f\left(t_1, t_2, \ldots, t_r \,|\, N = 0\right) = \frac{n!}{(n-r)! P[N = 0]} \prod_{j=1}^{r} f\left(t_j\right) \left\{1 - F\left(t_r\right)\right\}^{n-r}$$

$$\text{if } T < t_1 < \ldots < t_r < \infty.$$

For $i = 1, 2, \ldots, r-1$, PDF of $T_{1:n}, T_{2:n}, \ldots, T_{r:n}$ conditioned on the event $N = i$ is given by

$$f\left(t_1, t_2, \ldots, t_r \,|\, N = i\right) = \frac{n!}{(n-r)! P[N = i]} \prod_{j=1}^{r} f\left(t_j\right) \left\{1 - F\left(t_r\right)\right\}^{n-r}$$

$$\text{if } t_1 < \ldots < t_i < T < t_{i+1} < \ldots < t_r < \infty.$$

Proof: See [5].

Lemma A.3. Let X be a Gamma$(\alpha, 1)$ random variable, Y be a standard Exponential random variable, and they are independently distributed. The PDF of X is

$$f_1(x; \alpha, \lambda_1) = \begin{cases} \frac{1}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-x} & \text{for } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The PDF of Y is

$$f_2(y) = \begin{cases} e^{-y} & \text{for } y > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then for any arbitrary constant A, λ_1 and λ_2 the MGF of $A + \lambda_1 X + \lambda_2 Y$ is given by

$$M_{A+\lambda_1 X+\lambda_2 Y}(\omega) = e^{\omega A} (1 - \lambda_1 \omega)^{-\alpha} (1 - \lambda_2 \omega)^{-1} .$$

This MGF exists if

$$\omega \quad \in \quad \begin{cases} (\frac{1}{\lambda_2}, \frac{1}{\lambda_1}) & \text{if } \lambda_2 < 0 \\ (-\infty, \min\{\frac{1}{\lambda_1}, \frac{1}{\lambda 2}\}) & \text{if } \lambda_2 \geq 0 \,. \end{cases}$$

Proof: This lemma can be proved using the joint distribution of (X, Y), and therefore the proof is omitted.

Lemma A.4. Let X be a Gamma(α, λ_1)(with α integer) and Y be a Exponential(λ_2) random variable and they be independently distributed. Then the PDF of $X - Y$ is given by

$$g_2\,(t; \alpha, \lambda_1, \lambda_2) = \sum_{k=0}^{\alpha-1} p_k\, g_1\,(t; \alpha - k, \lambda_1) + \left(1 - \sum_{k=0}^{\alpha-1} p_k\right) g_1\,(-t, 1, \lambda_2) \quad \text{for } t \in \mathbb{R}\,,$$

where $p_k = \frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k$.

Proof: See [6].

Proof of Theorem 2.1:

Note that the number of unit failed before time T, N, is a non-negative random variable with the probability mass function (PMF)

$$P[N = i] \quad = \quad \binom{n}{i}\left(1 - e^{-\frac{T-\mu}{\theta}}\right)^i e^{-(n-i)\frac{T-\mu}{\theta}} \quad \text{if } i = 0, 1, \ldots, n\,. \tag{10}$$

Case-I: $r = 1$

The joint conditional MGF of $(\widehat{\theta}, \widehat{\mu})$ conditioned on the event $\{N \geq 1\}$ can be written as

$$E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \,|\, N \geq 1] \quad = \quad \sum_{i=1}^{n} E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \,|\, N = i] \times P[N = i \,|\, N \geq 1]\,. \tag{11}$$

Using (3) and Lemma A.1,

$$
\begin{aligned}
& E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \,|\, N=1] \\
= \;& \frac{1}{\theta\left(1-e^{-\frac{T-\mu}{\theta}}\right)} \int_{\mu}^{T} e^{\omega_1 (n-1)(T-t)+\omega_2\, t-\frac{1}{\theta}(t-\mu)} dt \\
= \;& \frac{e^{(n-1)T\omega_1+\frac{\mu}{\theta}}\left\{e^{-(n-1)\mu\,\omega_1+\mu\,\omega_2-\frac{\mu}{\theta}} - e^{-\frac{T}{\theta}-(n-1)T\omega_1+T\omega_2}\right\}}{\theta\left(1-e^{-\frac{T-\mu}{\theta}}\right)\left(\frac{1}{\theta}+(n-1)\,\omega_1-\omega_2\right)}.
\end{aligned}
$$

Using the above expression and (10)

$$
\begin{aligned}
& E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \,|\, N=1] \times P[N=1 \,|\, N\geq 1] \\
= \;& (1-q^n)^{-1}\left[c_{10}\,\frac{e^{\mu_{10}\,\omega_1+\mu\,\omega_2}}{(1+(n-1)\,\theta\omega_1-\theta\omega_2)} - d_{10}\,\frac{e^{T\omega_2}}{(1+(n-1)\,\theta\omega_1-\theta\omega_2)}\right], \qquad (12)
\end{aligned}
$$

where q, c_{10}, d_{10} and μ_{10} are as in (6) and (7). For $i=2,3,\ldots,n$, using (3) and Lemma A.1, we have

$$
\begin{aligned}
& E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \,|\, N=i] \\
= \;& \frac{i!}{\theta^i\left(1-e^{-\frac{T-\mu}{\theta}}\right)^i} \times \int_{\mu}^{T}\int_{t_1}^{T}\cdots\int_{t_{i-1}}^{T} e^{\frac{\omega_1}{i}\left\{\sum_{j=2}^{i} t_j-(n-1)t_1+(n-i)T\right\}+\omega_2 t_1-\frac{1}{\theta}\sum_{j=1}^{i}(t_j-\mu)} dt_i\ldots dt_1 \\
= \;& \frac{i!\,e^{-i\,\frac{T-\mu}{\theta}+T\omega_2}}{\theta^i\left(1-e^{-\frac{T-\mu}{\theta}}\right)^i}\int_{\mu}^{T}\int_{t_1}^{T}\cdots\int_{t_{i-1}}^{T} e^{-\left(\frac{1}{\theta}+\frac{n-1}{i}\omega_1-\omega_2\right)(t_1-T)-\left(\frac{1}{\theta}-\frac{\omega_1}{i}\right)\sum_{j=2}^{i}(t_j-T)} dt_i\ldots dt_1 \\
= \;& \frac{e^{-i\,\frac{T-\mu}{\theta}+T\omega_2}}{\left(1-e^{-\frac{T-\mu}{\theta}}\right)^i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i}{j+1} \times \frac{e^{\left(\frac{j+1}{\theta}+\frac{n-j-1}{\theta}\omega_1-\omega_2\right)(T-\mu)}-1}{\left(1-\frac{\theta}{i}\omega_1\right)^{i-1}\left(1+\frac{(n-j-1)\theta}{i(j+1)}\omega_1-\frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}
$$

Using the above expression and (10), one will get for $i=2,\cdots,n$

$$
\begin{aligned}
& E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \,|\, N=i] \times P[N=i \,|\, N\geq 1] \\
= \;& \frac{e^{-n\frac{T-\mu}{\theta}+T\omega_2}}{1-e^{-n\frac{T-\mu}{\theta}}}\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{n}{i}\binom{i}{j+1}\frac{e^{\left(\frac{j+1}{\theta}+\frac{n-j-1}{i}\omega_1-\omega_2\right)(T-\mu)}-1}{\left(1-\frac{\theta}{i}\omega_1\right)^{i-1}\left(1+\frac{(n-j-1)\theta}{i(j+1)}\omega_1-\frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}
$$

So using the above expression and (12) in (11), we have

$$
\begin{aligned}
& E[e^{\omega\widehat{\theta}} \,|\, N \geq 1] \\
= \; & (1-q^n)^{-1} \left[c_{10} \frac{e^{\mu_{10}\omega_1 + \mu\,\omega_2}}{\left(1 + \frac{\omega_1}{\lambda_{10}} - \frac{\omega_2}{\nu_0}\right)} - d_{10} \frac{e^{T\omega_2}}{\left(1 + \frac{\omega_1}{\lambda_{10}} - \frac{\omega_2}{\nu_0}\right)} + \frac{e^{\mu\,\omega_2} - q^n e^{T\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 - \frac{\omega_2}{\nu_{n-1}}\right)} \right. \\
& + \sum_{i=2}^{n-1} \sum_{j=0}^{i-1} \left\{ c_{ij} \frac{e^{\mu_{ij}\omega_1 + \mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i} \left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} - d_{ij} \frac{e^{T\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i} \left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} \right\} \\
& \left. + \sum_{j=0}^{n-2} \left\{ c_{nj} \frac{e^{\mu_{nj}\omega_1 + \mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)} - d_{nj} \frac{e^{T\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)} \right\} \right].
\end{aligned}
$$

where q, α_i, λ_i, μ_{ij}, λ_{ij}, c_{ij} and d_{ij} are as in (6) and (7). Now using Lemmas A.3 and A.4, one will get (8). ∎

Case-II: $r \geq 2$

In this case, the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ can be expressed as

$$
E[e^{\omega_1\widehat{\theta} + \omega_2\widehat{\mu}}] \;=\; \sum_{i=0}^{n} E[e^{\omega_1\widehat{\theta} + \omega_2\widehat{\mu}} \,|\, N = i] \times P[N = i]\,. \tag{13}
$$

Now using (4) and Lemma A.2, for $\omega_1 < \frac{r}{\theta}$ and $\omega_2 < \frac{1}{\theta} + \frac{n-1}{r}\,\omega_1$

$$
\begin{aligned}
& E[e^{\omega_1\widehat{\theta} + \omega_2\widehat{\mu}} \,|\, N = 0] \\
= \; & \frac{n!}{\theta^r\,(n-r)!P[N=0]} \\
& \times \int_{\max\{T,\mu\}}^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_{r-1}}^{\infty} e^{\frac{\omega_1}{r}\left\{\sum_{j=2}^{r-1} t_j - (n-1)t_1 + (n-r+1)t_r\right\} + \omega_2\,t_1 - \sum_{j=1}^{r}\left(\frac{t_j-\mu}{\theta}\right) - (n-r)\left(\frac{t_r-\mu}{\theta}\right)} dt_r \ldots dt_1 \\
= \; & \frac{e^{-n\frac{\max\{T,\mu\}-\mu}{\theta} + \omega_2 \max\{T,\mu\}}}{P[N=0]\left(1 - \frac{\theta}{r}\omega_1\right)^{r-1}\left(1 - \frac{\theta}{n}\omega_2\right)}\,.
\end{aligned}
$$

Hence for $\omega_1 < \frac{r}{\theta}$ and $\omega_2 < \frac{1}{\theta} + \frac{n-1}{r}\,\omega_1$

$$
E[e^{\omega_1\widehat{\theta} + \omega_2\widehat{\mu}} \,|\, N = 0] \times P[N = 0] = e^{-n\frac{\max\{T,\mu\}-\mu}{\theta} + \omega_2 \max\{T,\mu\}} \times \frac{1}{\left(1 - \frac{\theta}{r}\omega_1\right)^{r-1}\left(1 - \frac{\theta}{n}\omega_2\right)}\,. \tag{14}
$$

Now for $i = 1, 2, \ldots, r-1$ and $\omega_1 < \frac{r}{\theta}$, using (4) and Lemma A.2

$$\begin{aligned}
& E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \\
= \; & \frac{n!}{(n-r)! P[N=i] \theta^r} \\
& \times \int_{\mu}^{T} \!\!\cdots\! \int_{t_{i-1}}^{T} \int_{T}^{\infty} \!\!\cdots\! \int_{t_{r-1}}^{\infty} e^{\frac{\omega_1}{r}\left\{\sum_{j=2}^{r-1} t_j - (n-1)t_1 + (n-r+1)t_r\right\} + \omega_2 t_1 - \sum_{j=1}^{r}\left(\frac{t_j-\mu}{\theta}\right) - (n-r)\left(\frac{t_r-\mu}{\theta}\right)} dt_r \ldots dt_1 \\
= \; & \frac{n!\, e^{\frac{n\mu}{\theta} - \frac{iT}{\theta} - \frac{n-i}{r} T\omega_1 + \omega_2 T}}{(n-r)! P[N=i] \theta^r} \\
& \times \left\{ \int_{\mu}^{T} \int_{t_1}^{T} \!\!\cdots\! \int_{t_{i-1}}^{T} e^{-\left(\frac{1}{\theta} + \frac{n-1}{r}\omega_1 - \omega_2\right)(t_1 - T) - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=2}^{i}(t_j - T)} dt_i \ldots dt_1 \right\} \\
& \times \left\{ \int_{T}^{\infty} \int_{t_{i+1}}^{\infty} \!\!\cdots\! \int_{t_{r-1}}^{\infty} e^{-\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=i+1}^{r-1} t_j - (n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) t_r} dt_r \ldots dt_{i+1} \right\} \\
= \; & \frac{e^{-n\frac{T-\mu}{\theta} + T\omega_2}}{P[N=i]} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i} \binom{i}{j+1} \times \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)(T-\mu)} - 1}{\left(1 - \frac{\theta}{r}\omega_1\right)^{r-1} \left(1 + \frac{(n-j-1)\theta}{r(j+1)}\omega_1 - \frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}$$

Hence for $i = 1, 2, \ldots, r-1$ and $\omega < \frac{r}{\theta}$,

$$\begin{aligned}
& E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \times P[N=i] \\
= \; & e^{-n\frac{T-\mu}{\theta} + T\omega_2} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i} \binom{i}{j+1} \times \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)(T-\mu)} - 1}{\left(1 - \frac{\theta}{r}\omega_1\right)^{r-1} \left(1 + \frac{(n-j-1)\theta}{r(j+1)}\omega_1 - \frac{\theta}{j+1}\omega_2\right)}. \qquad (15)
\end{aligned}$$

Now for $i = r, r+1, \ldots, n$, using (4) and Lemma A.1

$$\begin{aligned}
& E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \\
= \; & \frac{n!}{(n-i)! \theta^i P[N=i]} \\
& \times \int_{\mu}^{T} \int_{t_1}^{T} \!\!\cdots\! \int_{t_{i-1}}^{T} e^{\frac{\omega_1}{i}\left\{\sum_{j=2}^{i} t_j - (n-1)t_1 + (n-i)T\right\} + \omega_2 t_1 - \sum_{j=1}^{i}\left(\frac{t_j-\mu}{\theta}\right) - (n-i)\left(\frac{T-\mu}{\theta}\right)} dt_i \ldots dt_1 \\
= \; & \frac{e^{-n\frac{T-\mu}{\theta} + T\omega_2}}{P[N=i]} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i} \binom{i}{j+1} \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{i}\omega_1 - \omega_2\right)(T-\mu)} - 1}{\left(1 - \frac{\theta}{i}\omega_1\right)^{i-1} \left(1 + \frac{(n-j-1)\theta}{i(j+1)}\omega_1 - \frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}$$

Hence for $i = r, r+1, \ldots, n$,

$$\begin{aligned}
& E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \times P[N=i] \\
= \; & e^{-n\frac{T-\mu}{\theta} + T\omega_2} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i} \binom{i}{j+1} \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{i}\omega_1 - \omega_2\right)(T-\mu)} - 1}{\left(1 - \frac{\theta}{i}\omega_1\right)^{i-1} \left(1 + \frac{(n-j-1)\theta}{i(j+1)}\omega_1 - \frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}$$

So using the above expression, (14) and (15) in (13), we have for $\omega_1 < \frac{r}{\theta}$ and $\omega_2 < \frac{n}{\theta}$

$$
\begin{aligned}
& E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}}] \\
= \; & \frac{q^n \, e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_1}\right)^{\alpha_1}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} + \frac{e^{\mu\,\omega_2} - q^n e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} \\
& + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} c_{ij} \frac{e^{\mu_{ij}\,\omega_1 + \mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)} - \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} d_{ij} \frac{e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)} \\
& + \sum_{j=0}^{n-2} c_{nj} \frac{e^{\mu_{nj}\,\omega_1 + \mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)} - \sum_{j=0}^{n-2} d_{nj} \frac{e^{T\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)},
\end{aligned}
$$

where q, μ_{ij}, α_i, λ_i, λ_{ij}, c_{ij} and d_{ij} are as in (6) and (7). Now using Lemmas A.3, A.4, one will get (9). ∎

Table 1: Lower and upper limits of different confidence intervals for θ

	Approx. CI		Per Boot. CI		BCa Boot. CI	
Level	LL	UL	LL	UL	LL	UL
90%	91.75	241.02	75.61	185.33	91.80	221.37
95%	84.97	270.12	70.57	199.49	86.23	263.28
99%	73.57	342.61	60.82	220.36	75.86	274.07

Table 2: Coverage percentage and average length of approximate confidence intervals based on 10000 simulations with $\mu = 0$, $n = 20$ and $r = 16$.

		90% C.I.		95% C.I.		99% C.I.	
T	θ	Cov. Per.	Ave. Len.	Cov. Per.	Ave. Len.	Cov. Per.	Ave. Len.
	1.00	89.98	0.93	95.35	1.14	99.27	1.60
	2.00	89.87	1.87	95.44	2.29	99.26	3.23
1.50	3.00	89.63	2.79	95.38	3.44	99.16	4.85
	4.00	89.18	3.69	95.15	4.59	98.96	6.48
	5.00	88.51	4.55	94.63	5.77	98.51	8.10
	1.00	89.66	0.87	95.03	1.07	99.20	1.49
	2.00	90.13	1.87	95.46	2.29	99.26	3.22
2.50	3.00	90.00	2.80	95.43	3.44	99.28	4.84
	4.00	90.00	3.73	95.40	4.59	99.16	6.46
	5.00	89.60	4.65	95.38	5.73	99.14	8.08
	1.00	89.88	0.83	95.11	1.02	99.02	1.42
	2.00	89.85	1.84	95.22	2.25	99.21	3.16
3.50	3.00	90.18	2.80	95.51	3.44	99.22	4.84
	4.00	90.01	3.74	95.36	4.58	99.27	6.46
	5.00	89.83	4.67	95.28	5.73	99.23	8.07

Table 3: Coverage percentage and average length of bootstrap confidence intervals based on 10000 simulations with $\mu = 0$, $n = 20$, $r = 16$ and $B = 1000$.

		90% C.I.		95% C.I.		99% C.I.	
T	θ	Cov. Per.	Ave. Len.	Cov. Per.	Ave. Len.	Cov. Per.	Ave. Len.
1.50	1.00	81.47	0.72	88.02	0.85	95.00	1.12
	2.00	78.28	1.47	84.78	1.74	92.83	2.27
	3.00	78.28	2.21	84.78	2.63	92.84	3.43
	4.00	78.28	2.94	84.78	3.50	92.84	4.58
	5.00	78.28	3.68	84.78	4.38	92.84	5.73
2.50	1.00	81.93	0.71	88.26	0.85	95.04	1.12
	2.00	79.68	1.44	86.53	1.71	94.39	2.24
	3.00	78.28	2.20	84.78	2.61	92.85	3.40
	4.00	78.28	2.94	84.78	3.50	92.84	4.57
	5.00	78.28	3.68	84.78	4.38	92.84	5.72
3.50	1.00	80.87	0.70	87.44	0.83	94.56	1.10
	2.00	82.47	1.43	88.70	1.71	95.39	2.24
	3.00	79.05	2.17	86.02	2.58	94.04	3.36
	4.00	78.28	2.92	84.78	3.47	92.96	4.53
	5.00	78.28	3.67	84.78	4.36	92.84	5.70

Table 4: Coverage percentage and average length of BCa bootstrap confidence intervals based on 10000 simulations with $\mu = 0$, $n = 20$, $r = 16$ and $B = 1000$.

		90% C.I.		95% C.I.		99% C.I.	
T	θ	Cov. Per.	Ave. Len.	Cov. Per.	Ave. Len.	Cov. Per.	Ave. Len.
1.50	1.00	87.61	0.87	93.67	1.03	97.81	1.23
	2.00	85.85	1.77	92.15	2.09	96.56	2.48
	3.00	85.70	2.67	92.05	3.15	96.39	3.71
	4.00	85.57	3.58	91.92	4.23	96.18	4.92
	5.00	85.54	4.50	91.72	5.33	95.73	6.09
2.50	1.00	85.36	0.87	91.70	1.03	97.14	1.23
	2.00	87.03	1.76	93.22	2.08	97.47	2.46
	3.00	85.72	2.67	92.10	3.15	96.57	3.70
	4.00	85.58	3.58	91.93	4.23	96.20	4.92
	5.00	85.54	4.50	91.72	5.33	95.73	6.09
3.50	1.00	85.44	0.85	91.60	1.00	96.91	1.21
	2.00	87.20	1.76	93.26	2.07	97.69	2.46
	3.00	86.74	2.66	93.03	3.14	97.24	3.68
	4.00	85.60	3.57	92.06	4.23	96.45	4.91
	5.00	85.55	4.50	91.73	5.33	95.76	6.08

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