# Analysis of Weibull Step-Stress Model In Presence of Competing Risk

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Abstract

In this paper we mainly consider the inference of a simple step-stress model based on complete sample, when the stress changes after a pre-fixed number of failures. It is assumed that there are more than one cause of failure, and the lifetime of the experimental units at each stress level follows Weibull distribution with the same shape parameter and different scale parameters. The distribution function under different stress levels are connected through the generalized Khamis Higgins model. The maximum likelihood estimates of the model parameters and the associated asymptotic confidence intervals are obtained. Further, we consider the Bayesian inference of the unknown model parameters based on fairly general prior distributions. We have also provided the results for Type-I censored data also. We assess the performances of the estimators through extensive simulation study for complete sample, and the analyses of one complete (simulated) data set and one Type-I censored solar lighting device data set have been performed for illustrative purpose. We propose different classical and Bayesian optimal criteria, and based on them we obtain the optimum stress changing time. Finally we have indicated how the assumption on the common shape parameter of two competing causes can be relaxed.

**Key Words** Step-stress Life-tests; Competing Risk; Maximum Likelihood Estimator; Bayes Estimates; Optimality.

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# 1 Introduction

Analysis of time-to-failure data plays an important role to understand the life characteristics of the products. But sufficient amount of data may not be available due to high reliability of the products, small duration time between production and release in the market etc. One way to overcome such difficulty is to conduct an accelerated life testing (ALT) experiment where units are exposed to higher stress level than the usual one. Due to higher stress, units are failed earlier than the usual time, hence, an experimenter has more failure data in a shorter period of time. Key references on ALT model are Nelson [33] and Bagdonavicius and Nikulin [2], see also Kateri and Kamps [22], Ismail [19, 20, 21] and the references cited therein for some recent developments.

Stress can be applied in many ways such as constant stress, step-stress, random stress etc. One particular type of ALT model is known as the step-stress life testing (SSLT) model, where stress changes at pre-fixed time(s). In a SSLT experiment a pre-fixed number of experimental units are put into an experiment with an initial stress level and then the stress level increases to its next level at a pre-fixed time and so on. If there are only two stress levels then it is known as the simple step-stress model. Different distributions are assumed for the failure time under different stress levels. To relate those failure time distributions between two stress levels one needs to make some assumptions. The most popular model assumption for this purpose is the cumulative exposure model (CEM) introduced by Sediakin [38]. The SSLT under the assumption of CEM has been studied by many authors such as Balakrishnan and Xie [9, 8], Balakrishnan [5]. Interested readers are also referred to a review article by Balakrishnan [5] on exact inference of the model parameters for exponential distributions, under different censoring schemes or see the recent monograph on this topic by Kundu and Ganguly [29]. An alternative model mainly for the Weibull lifetime distribution has been proposed by Khamis and Higgins [23], and it is popularly known as the KH model. The KH model is mathematically more tractable than the CEM in case of the Weibull lifetime distribution.

In many life testing experiment the experimental units may fail due to more than one cause. In such a situation one needs to study each cause of failure separately in presence of the other risk factors. Therefore, each failure data consists of two components; one is the time to failure and the other one is the cause of failure. Competing risk data is analyzed by Cox [13], David and Moeschberger [15] and Crowder [14] by assuming a parametric distribution for lifetime under each cause of failure. Key references on the analysis of step-stress competing risk data are by Beltrami [10, 11], Balakrishnan and Han [6], Liu and Shi [30], Liu and Qiu [31], Srivastava and Sharma [39], Xu et al. [41], Zhang et al. [42], and see the references cited therein. Recently Ganguly and Kundu [17] developed the exact distribution of the maximum likelihood estimators (MLEs) for exponential step-stress model in presence of competing risks.

In addition to multiple cause of failure, another common problem is to choose an optimal stress changing time. Running an experiment at the higher stress level results in more failures but it is also important to consider the performance of the parameter estimates under all the stress levels. Therefore, the problem of optimal design of the SSLT experiment is important for estimation precision of all the parameters. Bai et al. [3] obtained the optimum test plans to minimize the asymptotic variance of maximum likelihood estimators of the mean life at the designed stress level. Other key references on optimal step-stress models are Alhadeed and Yang [1], Balakrishnan and Han [7], see the references cited therein for some recent developments.

In this article we mainly consider the analysis of a SSLT model for complete sample in the presence of two different stress levels, say, $s_1$ and $s_2$ and two causes of failure only, say, $Cause-1$ and $Cause-2$. Although, in this article we have considered only two causes of failure for brevity, more than two causes also can be handled along the same manner. Here, it is assumed that the stress level changes from $s_1$ to $s_2$ after a pre-fixed number of failures, say $r$. Therefore, the stress changing time is random, which has been first considered by Xiong and Milliken [40]. Later the step-stress model with random stress changing time has been studied by several authors, see for example Balakrishan et al. [4], Kundu and Balakrishnan

[26] and Ganguly and Kundu [17].

In this article, the failure time distribution at the stress level $s_i$ and for $Cause - j$ is assumed to be a Weibull distribution with the shape parameter $\alpha_i$ and the scale parameter $\theta_{ij}$ $(i, j = 1, 2)$. It is further assumed that the lifetime distributions of the competing causes follow the latent failure time model assumptions of Cox [13]. To relate distributions under two different stress levels we propose a generalized KH model. In this case if we take $\alpha_1 = \alpha_2$ then for each cause of failure, we have a KH model. We provide the MLEs and Bayes estimators of the model parameters. Different confidence/credible intervals of the parameters have been obtained and their performances are evaluated through extensive simulation study. We further consider the case when the data are Type-I censored. We have provided both the classical and Bayesian inference of the unknown parameters in this case also. We have analyzed one complete (simulated) data set and one Type-I censored solar lighting device data set for illustrative purposes.

It may be mentioned that although the equality of the shape parameter of the competing causes looks quite restrictive, in practice it may not be so. Note that under this assumption, and based on the latent failure time model, it implies that the lifetime distribution of the experimental units at the $i$-th stress level, without the competing causes, follows Weibull distribution with the shape parameter $\alpha_i$ and scale parameter $\theta_{i1} + \theta_{i2}$, which is a quite reasonable assumption. Due to this assumption the analysis becomes tractable to a great extent. This assumption is not very uncommon in the statistical literature, see for example Miyakawa [32], Roy and Mukhopadhyay [35], Kundu et al. [27] and the references cited therein. It is observed in Section 6 that the solar lighting device data set satisfies this assumption as well.

We have further discussed three optimality criteria for stress changing time. To compute the optimal stress changing time, we have minimized the sum of the coefficient of variations of all the parameter estimates, as it has been recently proposed by Samanta et al. [37]. Here we use asymptotic variances and posterior variances in two different optimality criteria to obtain coefficient of variation. Another optimality criteria is defined by maximizing the determinant of the Fisher information matrix. For certain parameter values and for a given prior we provide some optimal stress changing time for a given sample size. We have also indicated how the results can be extended if the shape parameters of the two competing causes are not assumed to be same at each stress level, finally we present some open problems and conclude the paper.

The rest of the article is organized as follows. In Section 2, we provide the model and describe the likelihood function. In Section 3 we provide the classical inference of the model parameters. We discuss the Bayesian inference of the model parameters in Section 4. The inference of Type-I censored data has been given in Section 5. In Section 6, an extensive simulation study and two data sets have been analyzed. Different optimal criteria for stress changing time are given in Section 7. In Section 8 we have indicated how the analysis can be extended if the shape parameters of the competing causes of failure are not assumed to be same at each stress level. Finally we conclude the article in Section 9.

## 2 Model Assumption and Likelihood Function

Assume that the experiment starts with $n$ number of identical units at the initial stress level $s_1$. Stress level is increased to $s_2$ after $r$-th failure occurs, and we continue the experiment till the last failure takes place. Here the number of failures at both the stress levels are fixed but the stress changing time is random. It is also assumed that a unit can fail due to one of the two causes and both failure time and cause of failure are recorded. Therefore, the failure data will be of the form; $t_{1:n} < t_{2:n} < \ldots < t_{r:n} < t_{r+1:n} < \ldots < t_{n:n}$. With each of the failure time data $t_{i:n}$ $(i = 1, 2, \ldots, n)$, the associated cause of failure is denoted by $\delta_i$. Here, $\delta_i = j$, if $i-th$ failure occurs due to $Cause - j$ $(j = 1, 2)$. Let us define $r_{ij}$ be the number of failures due to $Cause - j$ at the stress level $s_i$ $(i, j = 1, 2)$.

Based on the assumptions mentioned in the previous section, the probability density function (PDF) of the experimental unit at the stress level $s_i$ $(i = 1, 2)$ and due to $Cause - j$

$(j = 1, 2)$ is given by

$$f(t; \alpha_i, \theta_{ij}) = \alpha_i \theta_{ij} t^{\alpha_i - 1} e^{-\theta_{ij} t^{\alpha_i}}, \qquad 0 < t < \infty, \quad \alpha_i > 0, \quad \theta_{ij} > 0.$$

Now to connect the distribution functions at the two stress levels, we have used a generalized form of the KH model. Let us first consider the distribution due to $Cause - 1$, based on the generalized KH step-stress model. It is assumed that the hazard function for $Cause - 1$, conditioning on the event that the stress changes at $\tau$, is

$$h_1(t) = \begin{cases} \alpha_1 \theta_{11} t^{\alpha_1 - 1} & \text{if} \quad 0 < t \leq \tau \\ \alpha_2 \theta_{21} t^{\alpha_2 - 1} & \text{if} \quad \tau < t < \infty. \end{cases}$$

Hence, the cumulative hazard function and survival function for $Cause - 1$ are, respectively,

$$H_1(t) = \begin{cases} \int_0^t \alpha_1 \theta_{11} u^{\alpha_1 - 1} du & = \theta_{11} t^{\alpha_1} & \text{if} \quad 0 < t \leq \tau \\ \int_0^\tau \alpha_1 \theta_{11} u^{\alpha_1 - 1} du + \int_\tau^t \alpha_2 \theta_{21} u^{\alpha_2 - 1} du & = \theta_{11} \tau^{\alpha_1} + \theta_{21}(t^{\alpha_2} - \tau^{\alpha_2}) & \text{if} \quad \tau < t < \infty, \end{cases}$$

and

$$S_1(t) = e^{-H_1(t)} = \begin{cases} e^{-\theta_{11} t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ e^{-\theta_{21}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{11} \tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty. \end{cases} \tag{1}$$

Similarly we can obtain the survival function for $Cause - 2$ as

$$S_2(t) = \begin{cases} e^{-\theta_{12} t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ e^{-\theta_{22}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{12} \tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty. \end{cases} \tag{2}$$

Hence the PDF of the lifetimes for $Cause - 1$ and $Cause - 2$ are given by

$$f_1(t) = \begin{cases} \alpha_1 \theta_{11} t^{\alpha_1 - 1} e^{-\theta_{11} t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ \alpha_2 \theta_{21} t^{\alpha_2 - 1} e^{-\theta_{21}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{11} \tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty, \end{cases} \tag{3}$$

$$f_2(t) = \begin{cases} \alpha_1 \theta_{12} t^{\alpha_1 - 1} e^{-\theta_{12} t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ \alpha_2 \theta_{22} t^{\alpha_2 - 1} e^{-\theta_{22}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{12}\tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty, \end{cases} \tag{4}$$

respectively. Note that for each cause of failure if $\alpha_1 = \alpha_2$, then the model is the conventional KH model.

# 3 Classical Inference

## 3.1 Maximum Likelihood Estimation

In this section we provide the MLEs of the unknown parameters based on the observation $\{(t_{i:n}, \delta_i); i = 1, \ldots, n\}$. Here $t_{1:n} < \ldots < t_{n:n}$ and $\delta_1, \ldots, \delta_n$ denote the failure times and the corresponding causes of failure, respectively.

First let us observe that if $\delta = 1$, then the likelihood contribution of $(t, 1)$ is

$$L_1(t) = f_1(t) S_2(t) = \begin{cases} \alpha_1 \theta_{11} t^{\alpha_1 - 1} e^{-\theta_{11} t^{\alpha_1}} \times e^{-\theta_{12} t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ \alpha_2 \theta_{21} t^{\alpha_2 - 1} e^{-\theta_{21}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{11}\tau^{\alpha_1}} \times e^{-\theta_{22}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{12}\tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty, \end{cases}$$

conditioning on the event that the stress change at $\tau$. Further, if $\delta = 2$, then the likelihood contribution of any data $(t, 2)$ is

$$L_2(t) = f_2(t) S_1(t) = \begin{cases} \alpha_1 \theta_{12} t^{\alpha_1 - 1} e^{-\theta_{12} t^{\alpha_1}} \times e^{-\theta_{11} t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ \alpha_2 \theta_{22} t^{\alpha_2 - 1} e^{-\theta_{22}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{12}\tau^{\alpha_1}} \times e^{-\theta_{21}(t^{\alpha_2} - \tau^{\alpha_2}) - \theta_{11}\tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty. \end{cases}$$

Hence, the log-likelihood of the data without the additive constant is given by

$$\begin{aligned} l(Data; \alpha_1, \alpha_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}) &= r \ln(\alpha_1) + r_{11} \ln(\theta_{11}) + r_{12} \ln(\theta_{12}) + (\alpha_1 - 1) \sum_{i=1}^{r} \ln(t_{i:n}) \\ &\quad -(\theta_{11} + \theta_{12}) D_1(\alpha_1) + (n - r) \ln(\alpha_2) + r_{21} \ln(\theta_{21}) + r_{22} \ln(\theta_{22}) \\ &\quad +(\alpha_2 - 1) \sum_{i=r+1}^{n} \ln(t_{i:n}) - (\theta_{21} + \theta_{22}) D_2(\alpha_2), \end{aligned} \tag{5}$$

where

$$D_1(\alpha_1) \;=\; \sum_{i=1}^{r} t_{i:n}^{\alpha_1} + (n-r)t_{r:n}^{\alpha_1} \quad \text{and} \quad D_2(\alpha_2) \;=\; \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} - (n-r)t_{r:n}^{\alpha_2}.$$

The MLEs of the unknown parameters can be obtained by maximizing (5) with respect to the unknown parameters. For known $\alpha_1$ and $\alpha_2$, the MLEs of $\theta_{11}$, $\theta_{12}$, $\theta_{21}$ and $\theta_{22}$ are given by

$$\widehat{\theta}_{11(MLE)} = \frac{r_{11}}{D_1(\alpha_1)}, \quad \widehat{\theta}_{12(MLE)} = \frac{r_{12}}{D_1(\alpha_1)}, \quad \widehat{\theta}_{21(MLE)} = \frac{r_{21}}{D_2(\alpha_2)}, \quad \widehat{\theta}_{22(MLE)} = \frac{r_{22}}{D_2(\alpha_2)},$$

provided $r_{ij} > 0$, for $i, j = 1, 2$. If $\alpha_1$ and $\alpha_2$ are known then the exact distribution of $\widehat{\theta}_{ij(MLE)}$ can be obtained similarly as in Ganguly and Kundu [17]. In case of unknown $\alpha_1$ and $\alpha_2$, the explicit form of MLEs of $\alpha_1$ and $\alpha_2$ do not exist. Let us consider the profile log-likelihood of $\alpha_1$ and $\alpha_2$ without the additive constant

$$\begin{aligned} l_1(Data;\alpha_1,\alpha_2) \;=\;& rln(\alpha_1) - rln(D_1(\alpha_1)) + (\alpha_1 - 1)\sum_{i=1}^{r} ln(t_{i:n}) + (n-r)ln(\alpha_2) \\ & -(n-r)ln(D_2(\alpha_2)) + (\alpha_2 - 1)\sum_{i=r+1}^{n} ln(t_{i:n}). \end{aligned} \tag{6}$$

Differentiating the profile log-likelihood as given in (6), with respect to $\alpha_1$ and $\alpha_2$ and equating them to zero, we obtain

$$\frac{r}{\alpha_1} - \frac{r(\sum_{i=1}^{r} t_{i:n}^{\alpha_1}\ln(t_{i:n}) + (n-r)t_{r:n}^{\alpha_1} ln(t_{r:n}))}{D_1(\alpha_1)} + \sum_{i=1}^{r} ln(t_{i:n}) \;=\; 0, \tag{7}$$

$$\frac{n-r}{\alpha_2} - \frac{(n-r)(\sum_{i=r+1}^{n} t_{i:n}^{\alpha_2}\ln(t_{i:n}) - (n-r)t_{r:n}^{\alpha_2} ln(t_{r:n}))}{D_2(\alpha_2)} + \sum_{i=r+1}^{n} ln(t_{i:n}) \;=\; 0. \tag{8}$$

The MLEs of $\alpha_1$ and $\alpha_2$ can be obtained by solving (7) and (8), respectively. The following results will be useful for further development.

**Lemma 1.** *Unique solution of equation (7) exists which maximizes (6) with respect to* $\alpha_1$.

*Proof.* See Appendix A.1. □

**Lemma 2.** *Let us define*

$$\begin{aligned} u(\alpha_2) &= \frac{1}{\alpha_2^2} + \frac{A_2(\alpha_2)}{[D_2(\alpha_2)]^2}, \qquad \textit{where} \\ A_2(\alpha_2) &= \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} (\ln(t_{i:n}))^2 - \left( \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} \ln(t_{i:n}) \right)^2 \\ & \quad -(n-r) t_{r:n}^{\alpha_2} \Bigg[ (\ln(t_{r:n}))^2 \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} - 2 \ln(t_{r:n}) \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} \ln(t_{i:n}) \\ & \quad + \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} (\ln(t_{i:n}))^2 \Bigg]. \end{aligned} \tag{9}$$

*If $u(\alpha_2) > 0$, for all $\alpha_2 > 0$, then unique solution of equation (8) exists which maximizes (6) with respect to $\alpha_2$.*

*Proof.* See Appendix A.2. □

Hence, using Lemma 1 and Lemma 2, the MLEs of $\alpha_1$ and $\alpha_2$ can be obtained as the solutions of equation (7) and (8), respectively. Standard numerical methods like fixed point iteration, Newton-Raphson methods may be used to solve non-linear equations numerically.

## 3.2 Asymptotic Confidence Interval

Asymptotic confidence intervals of the unknown parameters can be obtained by using the observed Fisher information matrix and then using the asymptotic normality results of the MLEs. Consider the parameter vector as $\eta = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \alpha_1, \alpha_2)^\top$. Therefore the observed Fisher information matrix is given by

$$F = ((f_{ij})) = ((-\frac{\delta^2 l}{\delta\eta_i \delta\eta_j})).$$

The elements of the Fisher information matrix are given in Appendix A.3. The asymptotic distribution of $\widehat{\eta} = (\widehat{\theta}_{11}, \widehat{\theta}_{12}, \widehat{\theta}_{21}, \widehat{\theta}_{22}, \widehat{\alpha}_1, \widehat{\alpha}_2)^\top$ is given by $\widehat{\eta} - \eta \sim N_6(0, F^{-1})$. Therefore

$100(1-\alpha)\%$ asymptotic CI of $\eta_i$ is given by

$$\left[\widehat{\eta}_i \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{ii}}\right],$$

where $V_{ii}$ is $(i,i)^{th}$ element of the matrix $F^{-1}$ which is given in Appendix A.7.

# 4 Bayesian Inference

In this section we consider the Bayesian inference of the model parameters. We mainly consider the squared error loss function, although other loss functions can also be incorporated easily. We provide the Bayes estimates (BE) of the unknown parameters and the associated symmetric and highest posterior density (HPD) credible intervals are also obtained.

## 4.1 Prior Assumption and Posterior Analysis

Here we assume a log-concave prior for both $\alpha_1$ and $\alpha_2$. In particular, let the prior distribution of $\alpha_1$ be Gamma with parameters $a_0$ and $b_0$, i.e.,

$$\pi_0(\alpha_1|a_0,b_0) = \frac{a_0^{b_0}}{\Gamma(b_0)} e^{-a_0\alpha_1}\alpha_1^{b_0-1}; \qquad a_0>0, b_0>0.$$

Similarly, we assume that $\alpha_2$ follows $Gamma(a_3,b_3)$. It is assumed that $(\theta_{11},\theta_{12})$ has a Gamma-Dirichlet prior and similarly $(\theta_{21},\theta_{22})$ also has a Gamma-Dirichlet prior. The explicit form of the Gamma-Dirichlet prior is provided in Pena and Gupta [34]. This prior distribution leads to the prior dependency between scale parameters under each stress level. Moreover, for known $\alpha_1$ and $\alpha_2$ this family of priors turns out to be a conjugate family of priors also. The joint probability density function of $\theta_{11}$ and $\theta_{12}$ with parameters, $a_1>0$, $b_1>0$, $a_2>0$ and $b_2>0$ is given by

$$\pi_{12}(\theta_{11},\theta_{12}|a_1,b_1,a_2,b_2) = \frac{\Gamma(a_2+b_2)}{\Gamma(a_1)}(b_1\theta_1)^{a_1-a_2-b_2}\frac{b_1^{a_2}}{\Gamma(a_2)}\theta_{11}^{a_2-1}e^{-b_1\theta_{11}}\frac{b_1^{b_2}}{\Gamma(b_2)}\theta_{12}^{b_2-1}e^{-b_1\theta_{12}},$$

where $\theta_1 = \theta_{11} + \theta_{12}$. This distribution will be denoted by $GD(a_1, b_1, a_2, b_2)$. Though $\theta_{11}$ and $\theta_{12}$ have dependent prior distribution in general, but if $a_1 = a_2 + b_2$ then it becomes independent. Similarly, we assume that the joint prior distribution of $\theta_{21}$ and $\theta_{22}$ is $GD(a_4, b_4, a_5, b_5)$. Also we assume that the prior distribution of $\alpha_1$, $\alpha_2$, $(\theta_{11}, \theta_{12})$ and $(\theta_{21}, \theta_{22})$ are independent. Hence the joint prior distribution is given by

$$\begin{aligned}\widetilde{\pi}(\alpha_1, \theta_{11}, \theta_{12}, \alpha_2, \theta_{21}, \theta_{22}) \;\propto\; & e^{-a_0\alpha_1}\alpha_1^{b_0-1}(\theta_{11}+\theta_{12})^{a_1-a_2-b_2}e^{-b_1\theta_{11}}\theta_{11}^{a_2-1}e^{-b_1\theta_{12}}\theta_{12}^{b_2-1} \\ & e^{-a_3\alpha_2}\alpha_2^{b_3-1}(\theta_{21}+\theta_{22})^{a_4-a_5-b_5}e^{-b_4\theta_{21}}\theta_{21}^{a_5-1}e^{-b_4\theta_{22}}\theta_{22}^{b_5-1}. \quad (10)\end{aligned}$$

In general, an explicit form of the Bayes estimates (BEs) of the unknown parameters under the squared error loss function cannot be obtained. But it is observed by Kundu and Pradhan [28] that the generation from the joint posterior distribution is quite simple. Hence, we propose to use the Gibbs sampling method to compute the BEs and to construct the associated credible intervals (CRIs). Note that the joint posterior distribution of parameters can be written as

$$\pi(\alpha_1, \theta_{11}, \theta_{12}, \alpha_2, \theta_{21}, \theta_{22}|data) \propto \pi_1(\alpha_1)\pi_2(\theta_{11}, \theta_{12}|\alpha_1)\pi_3(\alpha_2)\pi_4(\theta_{21}, \theta_{22}|\alpha_2).$$

where

$$\begin{aligned}\pi_1(\alpha_1) \;&\propto\; e^{-a_0\alpha_1}\alpha_1^{r+b_0-1}\Big[b_1 + D_1(\alpha_1)\Big]^{-(a_1+r_{11}+r_{12})}\Big(\prod_{i=1}^{r} t_{i:n}\Big)^{\alpha_1-1}, \\ \pi_2(\theta_{11}, \theta_{12}|\alpha_1) \;&\propto\; \Big[(b_1 + D_1(\alpha_1))(\theta_{11}+\theta_{12})\Big]^{(a_1+r_{11}+r_{12})-(a_2+r_{11})-(b_2+r_{12})} \\ &\times\; \big[b_1 + D_1(\alpha_1)\big]^{r_{11}+r_{12}+a_2+b_2}\theta_{11}^{r_{11}+a_2-1}e^{-\theta_{11}(b_1+D_1(\alpha_1))}\theta_{12}^{r_{12}+b_2-1}e^{-\theta_{12}(b_1+D_1(\alpha_1))}, \\ \pi_3(\alpha_2) \;&\propto\; e^{-a_3\alpha_2}\alpha_2^{n-r+b_3-1}\Big[b_4 + D_2(\alpha_2)\Big]^{-(a_4+r_{21}+r_{22})}\Big(\prod_{i=r+1}^{n} t_{i:n}\Big)^{\alpha_2-1}, \\ \pi_4(\theta_{21}, \theta_{22}|\alpha_2) \;&\propto\; \big[(b_4 + D_2(\alpha_2))(\theta_{21}+\theta_{22})\big]^{(a_4+r_{21}+r_{22})-(a_5+r_{21})-(b_5+r_{22})} \\ &\times \Big[b_4 + D_2(\alpha_2)\Big]^{r_{21}+r_{22}+a_5+b_5} e^{-\theta_{21}(b_4+D_2(\alpha_2))}\theta_{22}^{r_{22}+b_5-1}e^{-\theta_{22}(b_4+D_2(\alpha_2))}.\end{aligned}$$

The following results will be useful for Gibbs sampling purposes.

**Lemma 3.** $\pi_1(\alpha_1)$ *is a log-concave density function.*

*Proof.* See Appendix A.4. □

**Lemma 4.** *Let*

$$\begin{aligned} u_2(\alpha_2) &= \frac{n-r+b_3-1}{\alpha_2^2} + \frac{(a_4+r_{21}+r_{22})(b_4 g_2''(\alpha_2) + A_2(\alpha_2))}{[g_2(\alpha_2)]^2}, \qquad \textit{where} \\ g_2(\alpha_2) &= b_4 + \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} - (n-r)t_{r:n}^{\alpha_2}, \\ g_2''(\alpha_2) &= \sum_{i=r+1}^{n} t_{i:n}^{\alpha_2}(ln(t_{i:n}))^2 - (n-r)t_{r:n}^{\alpha_2}(ln(t_{r:n}))^2. \end{aligned}$$

*$A_2(\alpha_2)$ has been defined in (9). If $u_2(\alpha_2) > 0$ then $\pi_3(\alpha_2)$ is a log-concave density function.*

*Proof.* See Appendix A.5. □

Generation of samples from a Gamma-Dirichlet distribution is quite straight-forward which is given explicitly in Kundu and Pradhan [28]. Since $\pi_1(\alpha_1)$ is log-concave and if $u_2(\alpha_2) > 0$, $\pi_3(\alpha_2)$ is also log-concave, using the method proposed by Devroye [16] or Kundu [25], we can generate $\alpha_1$ and $\alpha_2$ from $\pi_1(\alpha_1)$ and $\pi_3(\alpha_2)$, respectively. We propose the following algorithm to compute Bayes estimates of the unknown parameters and the associated credible intervals.

**Algorithm 1**

Step 1. Generate $\alpha_1$ and $\alpha_2$ from $\pi_1(\alpha_1)$ and $\pi_3(\alpha_2)$, respectively, using the method proposed by Devroye [16] or Kundu [25]. Alternatively, one can use the ratio-of-uniform method introduced by Kinderman and Monahan [24] to generate $\alpha_1$ and $\alpha_2$. Though the ratio-of-uniform method do not need the log-concavity property of $\alpha_1$ and $\alpha_2$, it might leads to the higher number of rejection during the sample generation.

Step 2. For a given $\alpha_1$ generate $(\theta_{11}, \theta_{12})$ from $GD\big(a_1+r_{11}+r_{12},\, b_1+D_1(\alpha_1),\, a_2+r_{11},\, b_2+r_{12}\big)$ and for a given $\alpha_2$ generate $(\theta_{21}, \theta_{22})$ from $GD\big(a_4+r_{21}+r_{22},\, b_4+D_2(\alpha_2),\, a_5+r_{21},\, b_5+r_{22}\big)$.

Step 3. Repeat Step 1 and Step 2, $M$ times to obtain $(\alpha_1^1, \theta_{11}^1, \theta_{12}^1, \alpha_2^1, \theta_{21}^1, \theta_{22}^1)$ , . . . , $(\alpha_1^M, \theta_{11}^M, \theta_{12}^M, \alpha_2^M, \theta_{21}^M, \theta_{22}^M)$.

Step 4. Bayes estimates of $\alpha_1$, $\theta_{11}$, $\theta_{12}$, $\alpha_2$, $\theta_{21}$ and $\theta_{22}$ with respect to squared error loss function are given by

$$\widehat{\alpha}_{1(B)} = \frac{1}{M}\sum_{k=1}^{M} \alpha_1^k, \qquad \widehat{\theta}_{11(B)} = \frac{1}{M}\sum_{k=1}^{M} \theta_{11}^k, \qquad \widehat{\theta}_{12(B)} = \frac{1}{M}\sum_{k=1}^{M} \theta_{12}^k,$$
$$\widehat{\alpha}_{2(B)} = \frac{1}{M}\sum_{k=1}^{M} \alpha_2^k, \qquad \widehat{\theta}_{21(B)} = \frac{1}{M}\sum_{k=1}^{M} \theta_{21}^k, \qquad \widehat{\theta}_{22(B)} = \frac{1}{M}\sum_{k=1}^{M} \theta_{22}^k.$$

Step 5. The corresponding posterior variance can be obtained as

$$V_{post}(\alpha_1) = \frac{1}{M}\sum_{k=1}^{M}(\alpha_1^k - \widehat{\alpha}_{1(B)})^2, \qquad V_{post}(\theta_{11}) = \frac{1}{M}\sum_{k=1}^{M}(\theta_{11}^k - \widehat{\theta}_{11(B)})^2,$$
$$V_{post}(\theta_{12}) = \frac{1}{M}\sum_{k=1}^{M}(\theta_{12}^k - \widehat{\theta}_{12(B)})^2, \qquad V_{post}(\alpha_2) = \frac{1}{M}\sum_{k=1}^{M}(\alpha_2^k - \widehat{\alpha}_{2(B)})^2,$$
$$V_{post}(\theta_{21}) = \frac{1}{M}\sum_{k=1}^{M}(\theta_{21}^k - \widehat{\theta}_{21(B)})^2, \qquad V_{post}(\theta_{22}) = \frac{1}{M}\sum_{k=1}^{M}(\theta_{22}^k - \widehat{\theta}_{22(B)})^2.$$

Step 6. To obtain the credible interval of $\alpha_1$, we order $\alpha_1^1, \ldots, \alpha_1^M$ as $\alpha_1^{(1)} < \ldots < \alpha_1^{(M)}$. Then $100(1-\alpha)\%$ symmetric credible interval of $\alpha_1$ is given by $(\alpha_1^{([\frac{\alpha}{2}M])}, \alpha_1^{([(1-\frac{\alpha}{2})M])})$.

Step 7. To construct $100(1-\alpha)\%$ HPD credible interval of $\alpha_1$, consider the set of credible intervals $(\alpha_1^{(j)}, \alpha_1^{([j+(1-\alpha)M])})$, $j = 1, \ldots, [\alpha M]$. Therefore, a $100(1-\alpha)\%$ HPD credible interval of $\alpha_1$ is $(\alpha_1^{(j^*)}, \alpha_1^{([j^*+(1-\alpha)M])})$, where $j^*$ is such that

$$\alpha_1^{([j^*+(1-\alpha)M])} - \alpha_1^{(j^*)} < \alpha_1^{([j+(1-\alpha)M])} - \alpha_1^{(j)} \quad \text{for all} \quad j = 1 \ldots [\alpha M].$$

Following the method of Step 6 and Step 7 we can obtain the symmetric and HPD credible intervals for other parameters.

# 5 Inference for Type-I censoring

So far we have considered the case when complete data are available. In this section we will provide the inference of the same model based on Type-I censored data. It is assumed that the experiment is terminated at a prefixed time $T$. Let $r_{00} = r_{11} + r_{12} + r_{21} + r_{22}$ denote the total number of failures before the experiment stops at the time $T$. It should be mentioned that the censoring time $T$ could be small enough that $r_{21} = r_{22} = 0$: this case occurs when

$T < t_{r:n}$. In that case, no estimation of the parameters are possible. Therefore, it is assumed that $T > t_{r:n}$ and $r_{ij} > 0$, for $i, j = 1, 2$.

The log-likelihood without the additive constant of the observed Type-I censored data $\{(t_{i:n}, \delta_i); i = 1, \ldots, r_{00}\}$ is given by

$$\begin{aligned} l(Data; \alpha_1, \alpha_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}) &= r \ln(\alpha_1) + r_{11} \ln(\theta_{11}) + r_{12} \ln(\theta_{12}) + (\alpha_1 - 1) \sum_{i=1}^{r} \ln(t_{i:n}) \\ &\quad -(\theta_{11} + \theta_{12}) D_1(\alpha_1) + (r_{00} - r) \ln(\alpha_2) + r_{21} \ln(\theta_{21}) + r_{22} \ln(\theta_{22}) \\ &\quad +(\alpha_2 - 1) \sum_{i=r+1}^{r_{00}} \ln(t_{i:n}) - (\theta_{21} + \theta_{22}) D_2^*(\alpha_2), \end{aligned} \tag{11}$$

where, $D_2^*(\alpha_2) = \sum_{i=r+1}^{r_{00}} t_{i:n}^{\alpha_2} - (n-r) t_{r:n}^{\alpha_2} + (n - r_{00}) T^{\alpha_2}$. For known $\alpha_1$ and $\alpha_2$, the MLEs of $\theta_{11}$, $\theta_{12}$, $\theta_{21}$ and $\theta_{22}$ are given by

$$\widehat{\theta}_{11(MLE)} = \frac{r_{11}}{D_1(\alpha_1)}, \quad \widehat{\theta}_{12(MLE)} = \frac{r_{12}}{D_1(\alpha_1)}, \quad \widehat{\theta}_{21(MLE)} = \frac{r_{21}}{D_2^*(\alpha_2)}, \quad \widehat{\theta}_{22(MLE)} = \frac{r_{22}}{D_2^*(\alpha_2)}.$$

The MLEs of $\alpha_1$ and $\alpha_2$ can be obtained by solving below two nonlinear equations:

$$\begin{aligned} \frac{r}{\alpha_1} - \frac{r(\sum_{i=1}^{r} t_{i:n}^{\alpha_1} \ln(t_{i:n}) + (n-r) t_{r:n}^{\alpha_1} ln(t_{r:n}))}{D_1(\alpha_1)} + \sum_{i=1}^{r} ln(t_{i:n}) &= 0, \\ \frac{r_{00}-r}{\alpha_2} - \frac{(r_{00}-r)\left(\sum_{i=r+1}^{r_{00}} t_{i:n}^{\alpha_2} \ln(t_{i:n}) - (n-r) t_{r:n}^{\alpha_2} \ln(t_{r:n}) + (n-r_{00}) T^{\alpha_2} \ln T\right)}{D_2^*(\alpha_2)} + \sum_{i=r+1}^{r_{00}} ln(t_{i:n}) &= 0. \end{aligned}$$

In this case also, similar to the complete sample case we need to use some iterative technique to solve the above two non-linear equations. Since, the likelihood function of the Type-I censored data is very similar to the complete data, therefore, under the same set of prior assumptions the BEs and the associated CRIs can be obtained similarly, the details are avoided. We have analyzed one Type-I censored solar lighting device data set for illustrative purposes in the next section.

# 6 Simulation and Data Analysis

## 6.1 Simulation

In this section an extensive simulation study has been performed to asses the model performances. We consider two sets of parameter values, in $Set-1: \alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{12} = 1.2$, $\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{22} = 1.7$ and in $Set-2: \alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{12} = 1.6$, $\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{22} = 2.2$. We have taken $n$ = 40, 50, 60, 100 and $r = 0.4n$, $0.5n$, $0.6n$. In each case we have computed the average estimates (AEs) and the associated mean squared errors (MSEs) based on 1000 replications. The following algorithm can be used for calculating the AEs and MSEs of the unknown parameters. It must be noted that this data generation, given in Step 1 to Step 3 of the following Algorithm 2, is valid only for a common shape parameter of the competing Weibull lifetimes.

**Algorithm 2**

Step 1. First generate data from the proposed model as follows. For given $n$, $r$ and the parameter values, generate $n$ observations from $U(0,1)$ distribution and order them. Let the ordered observations be $u_{1:n} < \ldots < u_{n:n}$.

Step 2. For $i = 1, \ldots, r$; $t_{i:n} = \left[-\frac{1}{\theta_{11}+\theta_{12}}\ln(1-u_{i:n})\right]^{\frac{1}{\alpha_1}}$. To assign the cause of failure for the $i$-th observation generate $v_i$ from $U(0,1)$. If $v_i < \frac{\theta_{11}}{\theta_{11}+\theta_{12}}$, then $\delta_i = 1$, otherwise $\delta_i = 2$.

Step 3. For $i = r+1, \ldots, n$; $t_{i:n} = \left[-\frac{1}{\theta_{21}+\theta_{22}}\ln(1-u_{i:n}) - \frac{\theta_{11}+\theta_{12}}{\theta_{21}+\theta_{22}}t_{r:n}^{\alpha_1} + t_{r:n}^{\alpha_2}\right]^{\frac{1}{\alpha_2}}$. Next generate $v_i$ from $U(0,1)$. If $v_i < \frac{\theta_{21}}{\theta_{21}+\theta_{22}}$, then $\delta_i = 1$, otherwise $\delta_i = 2$.

Step 4. Given the generated data $\{(t_{i:n}, \delta_i); i = 1, \ldots, n\}$, obtain the MLEs of the unknown parameters.

Step 5. Repeat Step 1 to Step 4, $M$ times and take average of $M$ estimates and calculate the MSEs for each of the unknown parameters.

The AEs of MLEs along with mean square errors (MSEs) are given in Table 1 and 2 for $Set-1$ and $Set-2$, respectively. The coverage percentage (CP) and average length (AL) of

95% asymptotic CIs are provided in Table 3 and 4, respectively. We have also considered the Bayes estimates (BEs) and the associated credible intervals (CRIs) under squared error loss function. In the simulation experiment we have considered almost non-informative proper priors, as suggested by Congdon [12]; the hyper parameters are $a_0 = b_0 = a_1 = b_1 = a_3 = b_3 = a_4 = b_4 = 0.0001$ and $a_2 = b_2 = a_5 = b_5 = 1$. Bayes estimates are given in Table 5, 6; CP and AL of 95% symmetric and HPD CRIs are given in Table 7, 9 and Table 8, 10, respectively.

The following points are observed from the simulation results. The performances of the MLEs and BEs with respect to the almost non-informative priors, are very similar in terms of AEs and MSEs. Moreover, in both the cases the average lengths of the confidence and credible intervals and the associated coverage percentages are very close to each other. In both the cases, as $n$ increases and $r/n$ remains fixed, the biases and MSEs decrease for each parameter. It indicates the consistency property of the estimators. Further, for fixed $n$ as $r$ increases, the biases and MSEs of the MLEs and BEs of $\alpha_1$ ($\alpha_2$), $\theta_{11}$ ($\theta_{21}$) and $\theta_{12}$ ($\theta_{22}$) decrease (increase), as expected. The performances of the asymptotic confidence intervals, symmetric and HPD credible intervals are quite satisfactory as simulation results show that coverage percentages are very close to the corresponding nominal level and the average confidence (credible) lengths gradually decrease, as $n$ increases. The normality of the MLEs that can be empirically observed on the 1000 simulations. We had also carried out the bootstrap confidence intervals of the unknown parameters, but the performances are not very satisfactory, hence not reported here. We have conducted some more simulations with different sets of parameters. It is observed that the results (AEs, MSEs, the lengths of the confidence and credible intervals) affect more due to shape parameters than the scale parameters. They are not reported here due to paucity of space.

**Table 1:** AE and MSEs of MLEs based on 1000 replications ($\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{12} = 1.2$, $\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{22} = 1.7$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AE | MSE | AE | MSE | AE | MSE | AE | MSE | AE | MSE | AE | MSE |
| 40 | 16 | 0.6757 | 0.0365 | 1.5345 | 2.3462 | 1.8308 | 4.1525 | 0.9088 | 0.1103 | 1.7267 | 0.5173 | 1.9496 | 0.5584 |
| 40 | 20 | 0.6618 | 0.0268 | 1.3204 | 0.7720 | 1.5729 | 1.1365 | 0.9922 | 0.2183 | 1.8266 | 0.9326 | 2.0379 | 1.3075 |
| 40 | 24 | 0.6480 | 0.0193 | 1.1874 | 0.3517 | 1.4683 | 0.5876 | 1.1050 | 0.4069 | 1.8143 | 1.2755 | 2.0135 | 1.8557 |
| 50 | 20 | 0.6591 | 0.0261 | 1.3996 | 2.2568 | 1.6705 | 3.6891 | 0.9043 | 0.0857 | 1.6640 | 0.3464 | 1.8953 | 0.4166 |
| 50 | 25 | 0.6457 | 0.0187 | 1.2263 | 0.4922 | 1.4538 | 0.6427 | 0.9299 | 0.1471 | 1.7043 | 0.6927 | 1.9119 | 0.8442 |
| 50 | 30 | 0.6347 | 0.0130 | 1.1555 | 0.2077 | 1.3724 | 0.2801 | 1.0391 | 0.3321 | 1.7663 | 0.9666 | 1.9676 | 1.1964 |
| 60 | 24 | 0.6482 | 0.0198 | 1.3024 | 0.7861 | 1.5518 | 1.2703 | 0.8772 | 0.0647 | 1.6268 | 0.2161 | 1.8418 | 0.2612 |
| 60 | 30 | 0.6424 | 0.0147 | 1.1825 | 0.3210 | 1.4304 | 0.4452 | 0.9214 | 0.1271 | 1.6951 | 0.7162 | 1.9053 | 0.7280 |
| 60 | 36 | 0.6322 | 0.0104 | 1.1191 | 0.1624 | 1.3411 | 0.2127 | 0.9917 | 0.2531 | 1.7549 | 1.3220 | 1.9958 | 1.1589 |
| 100 | 40 | 0.6246 | 0.0097 | 1.1399 | 0.2205 | 1.3674 | 0.2760 | 0.8429 | 0.0343 | 1.5637 | 0.1025 | 1.7785 | 0.1210 |
| 100 | 50 | 0.6233 | 0.0073 | 1.0874 | 0.1146 | 1.3227 | 0.1794 | 0.8628 | 0.0632 | 1.6260 | 0.1775 | 1.7953 | 0.1843 |
| 100 | 60 | 0.6169 | 0.0057 | 1.0688 | 0.0768 | 1.2714 | 0.1102 | 0.9217 | 0.1293 | 1.6158 | 0.2974 | 1.8348 | 0.5539 |

**Table 2:** AE and MSEs of MLEs based on 1000 replications ($\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{12} = 1.6$, $\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{22} = 2.2$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AE | MSE | AE | MSE | AE | MSE | AE | MSE | AE | MSE | AE | MSE |
| 40 | 16 | 1.3383 | 0.1179 | 2.2329 | 4.3938 | 2.5206 | 5.4111 | 1.6047 | 0.3684 | 2.3805 | 1.2131 | 2.6407 | 1.7305 |
| 40 | 20 | 1.3113 | 0.0940 | 1.9343 | 2.4683 | 2.2189 | 2.3494 | 1.7121 | 0.7059 | 2.6680 | 3.8362 | 2.7589 | 3.7938 |
| 40 | 24 | 1.2960 | 0.0734 | 1.7731 | 1.1256 | 1.9824 | 1.1355 | 1.8605 | 1.1805 | 2.7851 | 5.7036 | 3.0196 | 6.2516 |
| 50 | 20 | 1.3007 | 0.0895 | 1.9866 | 3.0518 | 2.2897 | 3.9339 | 1.5740 | 0.2876 | 2.2942 | 0.7126 | 2.5305 | 0.9362 |
| 50 | 25 | 1.2930 | 0.0712 | 1.7686 | 1.1389 | 2.0711 | 1.5798 | 1.6488 | 0.5123 | 2.4052 | 1.3901 | 2.6500 | 1.6900 |
| 50 | 30 | 1.2724 | 0.0547 | 1.6598 | 0.6838 | 1.9162 | 0.8422 | 1.8137 | 0.9978 | 2.5750 | 3.3605 | 2.8316 | 3.7153 |
| 60 | 24 | 1.2984 | 0.0797 | 1.9016 | 1.8962 | 2.1913 | 2.5100 | 1.5257 | 0.2193 | 2.2376 | 0.5380 | 2.4557 | 0.5999 |
| 60 | 30 | 1.2703 | 0.0563 | 1.7098 | 0.8715 | 1.9256 | 1.0379 | 1.6027 | 0.3876 | 2.3142 | 0.8238 | 2.5332 | 0.9037 |
| 60 | 36 | 1.2648 | 0.0424 | 1.6179 | 0.4339 | 1.8652 | 0.6520 | 1.7461 | 0.7556 | 2.4278 | 2.3292 | 2.6832 | 2.3805 |
| 100 | 40 | 1.2486 | 0.0405 | 1.6284 | 0.4985 | 1.8656 | 0.7546 | 1.5048 | 0.1342 | 2.1015 | 0.2178 | 2.3740 | 0.2929 |
| 100 | 50 | 1.2508 | 0.0331 | 1.5786 | 0.3423 | 1.8047 | 0.3993 | 1.5147 | 0.2066 | 2.1344 | 0.2621 | 2.4026 | 0.3502 |
| 100 | 60 | 1.2368 | 0.0237 | 1.5144 | 0.2059 | 1.7276 | 0.2391 | 1.5984 | 0.3831 | 2.1884 | 0.4346 | 2.4272 | 0.5283 |

## 6.2 Data Analysis

### 6.2.1 Simulated Data Set

In this subsection we analyze a complete (simulated) data set of size 40. The data set is generated with $r = 16$, $\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{12} = 1.6$, $\alpha_2 = 1.4$, $\theta_{21} = 2.0$, and $\theta_{22} = 2.2$. The data set $(t, \delta)$ is provided in Table 11. Here $r_{11} = 7$, $r_{12} = 9$, $r_{21} = 10$ and $r_{22} = 14$. The MLEs and the Bayes estimates of $\alpha_1$, $\theta_{11}$, $\theta_{12}$, $\alpha_2$, $\theta_{21}$ and $\theta_{22}$ are given in Table 12. In Figure 1 we have plotted the profile log-likelihood of $\alpha_2$ for the data which is an unimodal function of $\alpha_2$. In the same figure we have also shown that the data satisfies the conditions stated in

**Table 3:** Average length and coverage percentage of 95% asymptotic CI of MLEs based on 1000 replications ($\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{12} = 1.2$, $\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{22} = 1.7$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| 40 | 16 | 0.5958 | 96.00 | 2.3625 | 94.50 | 2.6755 | 94.30 | 1.1196 | 94.90 | 1.9582 | 96.70 | 2.0978 | 96.20 |
| 40 | 20 | 0.5226 | 96.60 | 1.8866 | 94.90 | 2.1805 | 95.80 | 1.4248 | 95.50 | 2.2371 | 94.50 | 2.4071 | 94.90 |
| 40 | 24 | 0.9145 | 94.70 | 2.3716 | 95.20 | 2.6455 | 94.40 | 3.3633 | 94.10 | 3.6141 | 95.30 | 3.8115 | 95.20 |
| 50 | 20 | 0.5340 | 94.70 | 2.0745 | 94.30 | 2.4740 | 95.30 | 0.9944 | 94.30 | 1.7146 | 96.00 | 1.8381 | 95.80 |
| 50 | 25 | 0.4531 | 95.20 | 1.6355 | 94.40 | 1.8899 | 96.20 | 1.2920 | 94.00 | 1.9194 | 95.00 | 2.0675 | 94.90 |
| 50 | 30 | 0.4048 | 95.60 | 1.3515 | 96.50 | 1.5284 | 95.50 | 1.6813 | 94.00 | 2.2562 | 93.90 | 2.4867 | 93.80 |
| 60 | 24 | 0.4796 | 95.30 | 1.9163 | 94.80 | 2.2390 | 95.00 | 0.9076 | 94.20 | 1.5581 | 94.80 | 1.6696 | 96.50 |
| 60 | 30 | 0.4161 | 94.60 | 1.5023 | 95.50 | 1.6708 | 95.70 | 1.1749 | 94.90 | 1.7548 | 95.80 | 1.8973 | 95.80 |
| 60 | 36 | 0.7326 | 94.80 | 1.8808 | 95.40 | 2.0826 | 95.20 | 2.7476 | 94.10 | 2.8843 | 96.30 | 3.0766 | 96.10 |
| 100 | 40 | 0.7235 | 95.20 | 2.1779 | 94.70 | 2.4515 | 96.10 | 1.2942 | 94.10 | 1.5967 | 96.50 | 1.6770 | 96.00 |
| 100 | 50 | 0.6332 | 95.60 | 1.7458 | 95.40 | 1.9942 | 95.20 | 1.6612 | 95.60 | 1.7851 | 96.70 | 1.8860 | 95.40 |
| 100 | 60 | 0.5603 | 94.30 | 1.4334 | 96.10 | 1.6096 | 95.90 | 2.1598 | 94.90 | 2.0017 | 94.70 | 2.1345 | 96.10 |

**Table 4:** Average length and coverage percentage of 95% asymptotic CI of MLEs based on 1000 replications ($\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{12} = 1.6$, $\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{22} = 2.2$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| 40 | 16 | 1.1878 | 97.90 | 3.5307 | 94.40 | 3.8999 | 95.00 | 2.0821 | 94.20 | 2.7808 | 97.10 | 2.9308 | 97.10 |
| 40 | 20 | 1.0206 | 96.30 | 2.9374 | 94.50 | 3.2413 | 94.90 | 2.6192 | 94.40 | 3.1449 | 96.00 | 3.2902 | 95.60 |
| 40 | 24 | 0.9188 | 94.70 | 2.4283 | 96.00 | 2.6996 | 94.90 | 3.3950 | 93.70 | 3.6089 | 96.30 | 3.9664 | 95.20 |
| 50 | 20 | 1.0554 | 95.90 | 3.1994 | 95.60 | 3.6710 | 94.40 | 1.8493 | 93.80 | 2.4103 | 97.00 | 2.5633 | 97.70 |
| 50 | 25 | 0.9137 | 96.30 | 2.5926 | 95.70 | 2.9082 | 95.20 | 2.3350 | 94.90 | 2.6839 | 95.00 | 2.8255 | 97.40 |
| 50 | 30 | 0.8087 | 96.00 | 2.0788 | 95.70 | 2.3348 | 95.80 | 3.0563 | 95.10 | 3.1714 | 95.70 | 3.4033 | 96.20 |
| 60 | 24 | 0.9515 | 94.90 | 2.9978 | 95.00 | 3.3795 | 95.20 | 1.6695 | 94.60 | 2.1460 | 95.70 | 2.2529 | 97.10 |
| 60 | 30 | 0.8285 | 93.90 | 2.3245 | 96.00 | 2.5842 | 95.00 | 2.1468 | 95.30 | 2.4542 | 97.00 | 2.5422 | 96.80 |
| 60 | 36 | 0.7347 | 95.80 | 1.8959 | 95.30 | 2.0981 | 94.90 | 2.7933 | 93.80 | 2.7992 | 95.80 | 2.9663 | 94.50 |
| 100 | 40 | 0.3630 | 94.80 | 1.3991 | 95.80 | 1.6163 | 95.20 | 0.6966 | 95.30 | 1.1692 | 96.40 | 1.2473 | 95.50 |
| 100 | 50 | 0.3149 | 94.50 | 1.1259 | 95.60 | 1.3070 | 95.60 | 0.8931 | 94.70 | 1.3217 | 94.70 | 1.4121 | 94.70 |
| 100 | 60 | 0.2808 | 94.60 | 0.9371 | 95.90 | 1.0625 | 95.40 | 1.1972 | 95.80 | 1.6027 | 95.10 | 1.7445 | 94.60 |

Lemma 2 and Lemma 4. The Asymptotic CIs, Symmetric CRIs and HPD CRIs are given in Table 13. All the CIs/CRIs contain the respective true parameter values. The length of the asymptotic CIs of the parameters under first stress level is shorter than the symmetric and HPD CRIs, whereas the HPD CRIs provides the shortest length for the parameters under second stress level.

#### 6.2.2 Solar Lighting Device Data Set

In this subsection we have analyzed a solar lighting devices data set taken from Han and Kundu [18]. Thirty five solar lighting devices are put into a step-stress experiment. Here

**Table 5:** Bayes estimates and MSEs based on 1000 replications ($\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{12} = 1.2$, $\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{22} = 1.7$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | BE | MSE | BE | MSE | BE | MSE | BE | MSE | BE | MSE | BE | MSE |
| 40 | 16 | 0.6786 | 0.0295 | 1.7433 | 2.3083 | 2.0162 | 2.6648 | 0.7674 | 0.0817 | 1.9454 | 0.6776 | 2.1327 | 0.7281 |
| 40 | 20 | 0.6760 | 0.0253 | 1.5257 | 1.3735 | 1.8042 | 1.9451 | 0.8010 | 0.1230 | 2.0937 | 1.1466 | 2.3791 | 1.3403 |
| 40 | 24 | 0.6582 | 0.0205 | 1.3025 | 0.6745 | 1.5573 | 0.9132 | 0.8720 | 0.2136 | 2.4575 | 2.5295 | 2.7627 | 2.8566 |
| 50 | 20 | 0.6697 | 0.0232 | 1.5872 | 1.3650 | 1.9456 | 2.1361 | 0.7875 | 0.0646 | 1.8396 | 0.5030 | 2.0528 | 0.5386 |
| 50 | 25 | 0.6544 | 0.0182 | 1.3716 | 0.7732 | 1.6196 | 1.0388 | 0.7761 | 0.1049 | 2.0465 | 0.8994 | 2.2982 | 1.0689 |
| 50 | 30 | 0.6530 | 0.0146 | 1.2403 | 0.2953 | 1.4639 | 0.3987 | 0.8398 | 0.1638 | 2.3212 | 1.7527 | 2.6202 | 2.1482 |
| 60 | 24 | 0.6613 | 0.0200 | 1.5034 | 1.3355 | 1.7911 | 1.8393 | 0.7840 | 0.0577 | 1.7831 | 0.3435 | 2.0373 | 0.4568 |
| 60 | 30 | 0.6424 | 0.0138 | 1.2694 | 0.4133 | 1.4898 | 0.4779 | 0.7902 | 0.0840 | 1.9488 | 0.6181 | 2.2320 | 0.8684 |
| 60 | 36 | 0.6324 | 0.0101 | 1.1726 | 0.2071 | 1.3800 | 0.2296 | 0.8131 | 0.1430 | 2.2814 | 1.5792 | 2.5533 | 1.7718 |
| 100 | 40 | 0.6345 | 0.0106 | 1.2275 | 0.2885 | 1.5032 | 0.4732 | 0.7889 | 0.0345 | 1.6580 | 0.1448 | 1.8590 | 0.1652 |
| 100 | 50 | 0.6310 | 0.0078 | 1.1719 | 0.1698 | 1.3816 | 0.2068 | 0.7744 | 0.0590 | 1.8175 | 0.3671 | 2.0544 | 0.4058 |
| 100 | 60 | 0.6248 | 0.0059 | 1.0953 | 0.0844 | 1.3073 | 0.1032 | 0.7725 | 0.0873 | 2.0938 | 0.8911 | 2.3621 | 1.1485 |

**Table 6:** Bayes estimates and MSEs based on 1000 replications ($\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{12} = 1.6$, $\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{22} = 2.2$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | BE | MSE | BE | MSE | BE | MSE | BE | MSE | BE | MSE | BE | MSE |
| 40 | 16 | 1.2793 | 0.0795 | 2.3911 | 3.9807 | 2.6509 | 4.4080 | 1.3223 | 0.2642 | 2.7175 | 1.5442 | 2.9830 | 1.7552 |
| 40 | 20 | 1.2747 | 0.0712 | 2.0188 | 1.9632 | 2.2874 | 2.4948 | 1.3033 | 0.3587 | 3.2216 | 3.2037 | 3.4957 | 3.6644 |
| 40 | 24 | 1.2724 | 0.0626 | 1.8502 | 1.2713 | 2.0829 | 1.4636 | 1.3882 | 0.4924 | 3.7099 | 5.5347 | 3.9995 | 5.9731 |
| 50 | 20 | 1.2697 | 0.0606 | 2.1417 | 2.5192 | 2.3825 | 2.9148 | 1.2897 | 0.2068 | 2.6123 | 1.1956 | 2.8364 | 1.2617 |
| 50 | 25 | 1.2610 | 0.0581 | 1.8867 | 1.4233 | 2.1283 | 1.5361 | 1.3096 | 0.3092 | 3.0312 | 2.4976 | 3.2820 | 2.7827 |
| 50 | 30 | 1.2484 | 0.0463 | 1.7042 | 0.6747 | 1.9607 | 0.9734 | 1.3479 | 0.4200 | 3.5341 | 4.5570 | 3.8187 | 5.1618 |
| 60 | 24 | 1.2671 | 0.0562 | 2.0075 | 1.7856 | 2.3144 | 2.2743 | 1.3011 | 0.1846 | 2.4494 | 0.7271 | 2.7054 | 0.9518 |
| 60 | 30 | 1.2561 | 0.0503 | 1.8186 | 1.2199 | 2.0663 | 1.5745 | 1.3026 | 0.2712 | 2.8667 | 1.7528 | 3.1460 | 2.0964 |
| 60 | 36 | 1.2340 | 0.0355 | 1.6617 | 0.5755 | 1.8214 | 0.5733 | 1.3166 | 0.3561 | 3.3693 | 3.6960 | 3.7081 | 4.3320 |
| 100 | 40 | 1.2434 | 0.0379 | 1.7939 | 1.1295 | 2.0337 | 1.3689 | 1.3202 | 0.1087 | 2.2174 | 0.2832 | 2.4466 | 0.3194 |
| 100 | 50 | 1.2265 | 0.0250 | 1.5862 | 0.3377 | 1.8071 | 0.3872 | 1.2818 | 0.1738 | 2.4952 | 0.8088 | 2.7299 | 0.9059 |
| 100 | 60 | 1.2233 | 0.0203 | 1.5390 | 0.2244 | 1.7433 | 0.2597 | 1.2761 | 0.2765 | 3.0175 | 2.1966 | 3.3397 | 2.6916 |

the operating temperature has been used as a stress factor which was initially set as $293K$ and after 16-th failure it has been increased to $353K$. The experiment has been terminated at time $T = 6$ (in hundred hours). The failure of a device occurs due to one of the two causes: capacitor failure or controller failure. The failure time along with the cause of failure are given in Table 14. Thirty one failures have been observed before the termination of the experiment. We have analyzed this Type-I censored data by assuming the above model assumptions. The MLEs and the Bayes estimates of $\alpha_1$, $\theta_{11}$, $\theta_{12}$, $\alpha_2$, $\theta_{21}$ and $\theta_{22}$ are reported in Table 15. The plot $(a)$ of the Figure 2 provides the shape of the profile log-likelihood of $\alpha_2$ for the data set. The conditions on Lemmas 2 and 4 are satisfied by the data which

**Table 7:** Average length and coverage percentage of 95% symmetric CRIs based on 1000 replications ($\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{12} = 1.2$, $\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{22} = 1.7$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| 40 | 16 | 0.6606 | 97.50 | 3.1907 | 97.20 | 3.7262 | 98.30 | 1.1133 | 95.70 | 2.7377 | 94.00 | 2.9684 | 95.40 |
| 40 | 20 | 0.5876 | 96.70 | 2.4259 | 96.00 | 2.7747 | 96.10 | 1.3197 | 95.80 | 3.7055 | 94.50 | 4.0644 | 95.40 |
| 40 | 24 | 0.5264 | 96.50 | 1.8232 | 95.40 | 2.0524 | 95.70 | 1.6330 | 97.50 | 5.3370 | 94.30 | 6.0110 | 95.10 |
| 50 | 20 | 0.6018 | 96.90 | 2.8574 | 95.30 | 3.2199 | 97.10 | 1.0477 | 95.40 | 2.2704 | 93.70 | 2.5000 | 95.20 |
| 50 | 25 | 0.5225 | 97.90 | 2.0339 | 97.00 | 2.2936 | 96.40 | 1.2236 | 94.60 | 3.3017 | 95.10 | 3.6676 | 95.30 |
| 50 | 30 | 0.4778 | 97.20 | 1.6368 | 96.80 | 1.8465 | 97.40 | 1.5336 | 97.60 | 4.8580 | 94.20 | 5.5709 | 94.00 |
| 60 | 24 | 0.5441 | 97.30 | 2.4623 | 97.60 | 2.8309 | 97.90 | 0.9694 | 96.30 | 1.9518 | 94.50 | 2.1523 | 92.20 |
| 60 | 30 | 0.4781 | 97.40 | 1.7789 | 97.20 | 2.0280 | 97.60 | 1.1845 | 96.80 | 2.9015 | 95.40 | 3.1975 | 93.00 |
| 60 | 36 | 0.4336 | 98.20 | 1.4049 | 95.90 | 1.6239 | 97.80 | 1.3988 | 96.40 | 4.6253 | 94.50 | 5.1670 | 95.10 |
| 100 | 40 | 0.4257 | 97.40 | 1.6894 | 98.00 | 2.0241 | 97.30 | 0.7843 | 96.60 | 1.2979 | 95.90 | 1.3804 | 95.50 |
| 100 | 50 | 0.3814 | 98.10 | 1.3839 | 97.20 | 1.5707 | 97.50 | 0.9719 | 96.00 | 1.9414 | 94.70 | 2.1367 | 94.70 |
| 100 | 60 | 0.3455 | 98.60 | 1.0718 | 97.40 | 1.2263 | 97.70 | 1.2102 | 96.50 | 3.7263 | 95.30 | 4.1506 | 94.20 |

**Table 8:** Average length and coverage percentage of 95% HPD CRIs based on 1000 replications ($\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{12} = 1.2$, $\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{22} = 1.7$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| 40 | 16 | 0.6519 | 97.90 | 2.8032 | 96.60 | 3.2400 | 97.60 | 1.0842 | 91.40 | 2.4741 | 97.60 | 2.6819 | 97.30 |
| 40 | 20 | 0.5821 | 97.50 | 2.2252 | 97.60 | 2.5485 | 98.10 | 1.2560 | 93.20 | 3.2535 | 96.90 | 3.5433 | 98.30 |
| 40 | 24 | 0.5222 | 97.50 | 1.7190 | 96.50 | 1.9311 | 97.60 | 1.5012 | 94.50 | 4.5518 | 98.40 | 5.1633 | 98.30 |
| 50 | 20 | 0.5948 | 97.40 | 2.5465 | 96.60 | 2.8738 | 98.00 | 1.0246 | 91.90 | 2.0703 | 96.40 | 2.2799 | 97.90 |
| 50 | 25 | 0.5183 | 97.60 | 1.8892 | 98.00 | 2.1411 | 97.50 | 1.1687 | 91.00 | 2.8580 | 98.20 | 3.1986 | 97.60 |
| 50 | 30 | 0.4744 | 97.90 | 1.5495 | 97.50 | 1.7557 | 97.90 | 1.4080 | 94.50 | 4.1341 | 98.50 | 4.7333 | 98.00 |
| 60 | 24 | 0.5385 | 97.50 | 2.2271 | 98.70 | 2.5912 | 98.50 | 0.9617 | 93.20 | 1.8204 | 95.90 | 2.0205 | 95.60 |
| 60 | 30 | 0.4744 | 97.80 | 1.6595 | 97.60 | 1.9057 | 97.80 | 1.1491 | 93.20 | 2.5405 | 98.20 | 2.7515 | 97.50 |
| 60 | 36 | 0.4305 | 98.20 | 1.3471 | 96.40 | 1.5595 | 97.40 | 1.3103 | 93.10 | 3.8760 | 97.60 | 4.3879 | 97.70 |
| 100 | 40 | 0.4219 | 97.30 | 1.5883 | 97.80 | 1.9125 | 98.20 | 0.7810 | 94.80 | 1.2647 | 96.60 | 1.3483 | 97.10 |
| 100 | 50 | 0.3785 | 98.10 | 1.3258 | 96.80 | 1.5074 | 97.60 | 0.9632 | 92.40 | 1.7180 | 97.60 | 1.9017 | 97.80 |
| 100 | 60 | 0.3438 | 98.80 | 1.0446 | 97.50 | 1.1966 | 98.50 | 1.1647 | 92.40 | 3.0677 | 98.30 | 3.4001 | 98.30 |

have been shown in plots ($b$) and ($c$), respectively, of the Figure 2. Different CIs/CRIs of the parameters are given in Table 16.

Now we want to check whether the model assumptions are valid for this data set or not. We want to see whether generalized KH model with Weibull distributions at the two different stress levels without competing causes can be used or not. A goodness of fit test has been performed based on observed thirty one failure data, using Kolmogorov-Smirnov (K-S) test statistics. For the fitted model based on MLEs, the value of the K-S statistic is 0.1993 and the corresponding $p$-value is 0.1483. The K-S statistic and $p$-value for the model based on Bayes estimates are 0.1999 and 0.1463, respectively. The values in both cases indicate a

**Table 9:** Average length and coverage percentage of 95% symmetric CRIs based on 1000 replications ($\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{12} = 1.6$, $\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{22} = 2.2$).

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| 40 | 16 | 1.2788 | 98.50 | 5.4067 | 98.90 | 5.9528 | 98.10 | 2.1567 | 96.30 | 4.7084 | 96.90 | 4.9459 | 96.50 |
| 40 | 20 | 1.1323 | 97.70 | 3.6495 | 97.90 | 4.1371 | 97.60 | 2.5509 | 97.20 | 7.1774 | 93.90 | 7.4255 | 94.70 |
| 40 | 24 | 1.0118 | 98.60 | 2.7989 | 97.30 | 3.2151 | 96.30 | 2.9331 | 97.50 | 9.7346 | 94.10 | 10.7242 | 93.70 |
| 50 | 20 | 1.1576 | 98.50 | 4.3808 | 98.60 | 4.7631 | 98.50 | 2.0235 | 95.70 | 3.8435 | 94.90 | 4.1741 | 96.70 |
| 50 | 25 | 1.0141 | 98.10 | 3.1447 | 97.10 | 3.5262 | 98.00 | 2.3714 | 95.50 | 5.9919 | 93.70 | 6.3209 | 95.40 |
| 50 | 30 | 0.9036 | 96.40 | 2.4287 | 96.60 | 2.6290 | 96.20 | 2.7432 | 97.00 | 9.0239 | 92.80 | 9.9153 | 93.80 |
| 60 | 24 | 1.0534 | 98.80 | 3.8656 | 98.30 | 4.3547 | 98.70 | 1.8725 | 95.60 | 3.0807 | 96.90 | 3.3209 | 95.60 |
| 60 | 30 | 0.9327 | 97.40 | 2.8092 | 97.10 | 3.1448 | 98.00 | 2.2311 | 96.80 | 5.2525 | 94.00 | 5.7216 | 94.60 |
| 60 | 36 | 0.8287 | 97.50 | 2.2171 | 96.80 | 2.4658 | 97.80 | 2.6042 | 97.00 | 8.4227 | 93.30 | 9.1679 | 93.40 |
| 100 | 40 | 0.8276 | 97.40 | 2.6679 | 96.60 | 3.0692 | 97.00 | 1.4788 | 96.60 | 1.8296 | 95.50 | 1.9671 | 96.40 |
| 100 | 50 | 0.7309 | 98.50 | 2.0553 | 97.70 | 2.2500 | 98.50 | 1.8394 | 95.10 | 2.8905 | 95.10 | 3.0250 | 95.90 |
| 100 | 60 | 0.6639 | 98.20 | 1.6569 | 96.60 | 1.8602 | 97.30 | 2.2565 | 95.50 | 6.3185 | 94.40 | 6.8442 | 94.20 |

**Table 10:** Average length and coverage percentage of 95% HPD CRIs based on 1000 replications ($\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{12} = 1.6$, $\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{22} = 2.2$).

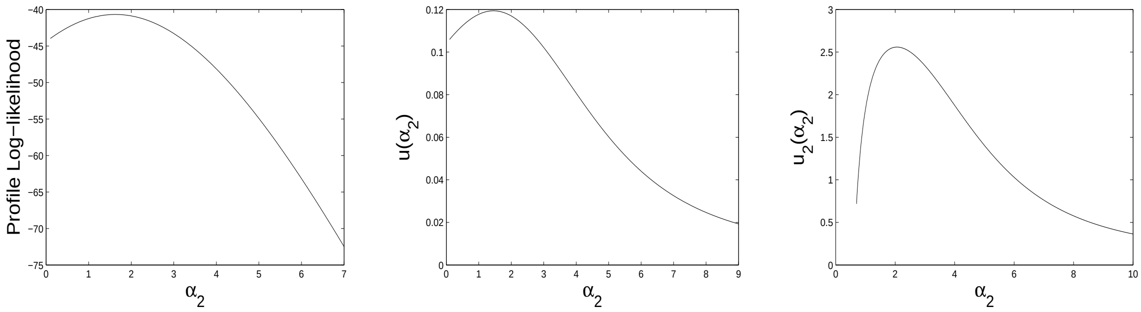
| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| 40 | 16 | 1.2600 | 98.40 | 4.4696 | 97.00 | 4.9884 | 96.30 | 2.0909 | 93.20 | 3.9696 | 98.50 | 4.2488 | 99.10 |
| 40 | 20 | 1.1175 | 98.00 | 3.2113 | 97.30 | 3.6447 | 97.10 | 2.3815 | 92.80 | 5.9042 | 99.20 | 6.2226 | 98.90 |
| 40 | 24 | 1.0011 | 98.40 | 2.5516 | 97.60 | 2.9603 | 96.20 | 2.6563 | 94.50 | 8.1332 | 98.90 | 8.9096 | 99.20 |
| 50 | 20 | 1.1426 | 98.50 | 3.7791 | 96.60 | 4.1326 | 96.80 | 1.9711 | 93.00 | 3.2629 | 98.80 | 3.5341 | 98.50 |
| 50 | 25 | 1.0048 | 98.10 | 2.8418 | 97.00 | 3.1904 | 97.90 | 2.2520 | 93.00 | 4.9425 | 98.30 | 5.1783 | 98.70 |
| 50 | 30 | 0.8959 | 96.50 | 2.2685 | 96.70 | 2.4532 | 96.20 | 2.5115 | 92.80 | 7.4840 | 99.00 | 8.1843 | 99.00 |
| 60 | 24 | 1.0417 | 98.80 | 3.3894 | 96.50 | 3.8094 | 98.20 | 1.8431 | 92.20 | 2.7579 | 98.10 | 2.9164 | 98.00 |
| 60 | 30 | 0.9256 | 97.60 | 2.5777 | 97.70 | 2.8668 | 97.40 | 2.1418 | 93.40 | 4.2490 | 98.80 | 4.6076 | 98.60 |
| 60 | 36 | 0.8218 | 97.50 | 2.0714 | 97.30 | 2.3170 | 97.60 | 2.4046 | 92.40 | 6.9381 | 98.90 | 7.4291 | 99.00 |
| 100 | 40 | 0.8236 | 97.40 | 2.4393 | 97.20 | 2.8297 | 97.50 | 1.4733 | 94.60 | 1.7622 | 97.50 | 1.8858 | 97.50 |
| 100 | 50 | 0.7274 | 98.40 | 1.9451 | 97.60 | 2.1275 | 98.00 | 1.8196 | 91.30 | 2.4252 | 98.30 | 2.5815 | 98.20 |
| 100 | 60 | 0.6597 | 98.00 | 1.5839 | 97.30 | 1.7887 | 97.50 | 2.1566 | 91.30 | 4.7023 | 98.70 | 5.1483 | 98.80 |

**Table 11:** Simulated data of size $n = 40$.

| Stress Level | Data | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | (0.0300,2) | (0.0369,2) | (0.0452,2) | (0.0618,2) | (0.0662,2) | (0.0790,1) | (0.0860,2) | (0.1110,1) |
| | (0.1137,1) | (0.1184,2) | (0.1297,2) | (0.1366,1) | (0.1986,2) | (0.2027,1) | (0.2122,1) | (0.2189,1) |
| $S_2$ | (0.2228,1) | (0.2324,1) | (0.2334,1) | (0.2421,2) | (0.2579,2) | (0.2821,2) | (0.2887,2) | (0.3339,1) |
| | (0.3432,1) | (0.3584,2) | (0.3767,1) | (0.4087,2) | (0.4121,2) | (0.4185,1) | (0.4340,2) | (0.5112,1) |
| | (0.5301,2) | (0.5747,2) | (0.7089,1) | (0.7104,1) | (0.7129,2) | (0.7151,2) | (0.7293,2) | (0.7927,2) |

**Table 12:** The MLEs and the Bayes estimates of the data in Table 11.

| Estimate | $\alpha_1$ | $\theta_{11}$ | $\theta_{12}$ | $\alpha_2$ | $\theta_{21}$ | $\theta_{22}$ |
|---|---|---|---|---|---|---|
| MLE | 1.3985 | 1.8862 | 2.4252 | 1.6284 | 1.9598 | 2.7438 |
| Bayes | 1.3684 | 2.2453 | 2.7939 | 1.5126 | 2.0409 | 2.7793 |




(a) Profile log-likelihood of $\alpha_2$ (b) Condition on Lemma 2 (c) Condition on Lemma 4

**Figure 1:** Plot of profile log-likelihood of $\alpha_2$ and the sufficient conditions for the data in Table 11.

**Table 13:** CI/CRI of the unknown parameters for data in Table 11.

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CI/CRI | Level | LL | UL | LL | UL | LL | UL | LL | UL | LL | UL | LL | UL |
| Asymptotic CI | 90% | 0.863 | 1.934 | 0.000 | 3.889 | 0.000 | 4.901 | 0.658 | 2.598 | 0.825 | 3.095 | 1.349 | 4.139 |
| | 95% | 0.758 | 2.039 | 0.000 | 4.280 | 0.000 | 5.384 | 0.469 | 2.787 | 0.603 | 3.316 | 1.076 | 4.411 |
| | 99% | 0.559 | 2.238 | 0.000 | 5.025 | 0.000 | 6.305 | 0.108 | 3.148 | 0.181 | 3.738 | 0.557 | 4.930 |
| Symmetric CRI | 90% | 0.857 | 2.060 | 0.570 | 6.062 | 0.759 | 7.480 | 0.794 | 2.494 | 1.088 | 3.325 | 1.632 | 4.314 |
| | 95% | 0.789 | 2.190 | 0.471 | 7.597 | 0.646 | 9.328 | 0.748 | 2.683 | 0.963 | 3.682 | 1.476 | 4.750 |
| | 99% | 0.664 | 2.459 | 0.319 | 11.904 | 0.464 | 14.280 | 0.709 | 3.067 | 0.759 | 4.520 | 1.211 | 5.762 |
| HPD CRI | 90% | 0.811 | 2.001 | 0.302 | 4.634 | 0.411 | 5.729 | 0.700 | 2.272 | 0.975 | 3.114 | 1.436 | 4.004 |
| | 95% | 0.742 | 2.124 | 0.224 | 6.092 | 0.360 | 7.531 | 0.700 | 2.494 | 0.831 | 3.425 | 1.275 | 4.428 |
| | 99% | 0.602 | 2.381 | 0.158 | 9.911 | 0.265 | 12.002 | 0.700 | 2.911 | 0.655 | 4.270 | 1.047 | 5.422 |

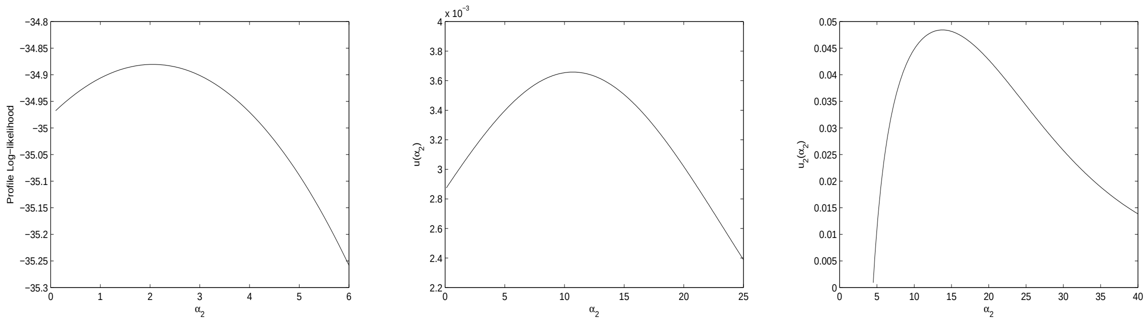
good fit of the proposed model to the solar lighting device data. The empirical and the fitted CDFs are shown in Figure 3. Therefore, for this data set we can conclude that the model assumptions are satisfied.

**Table 14:** Type-I censored Solar lighting device data set.

| Stress Level | Data | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | (0.140,1) | (0.783,2) | (1.324,2) | (1.582,1) | (1.716,2) | (1.794,2) | (1.883,2) | (2.293,2) |
| | (2.660,2) | (2.674,2) | (2.725,2) | (3.085,2) | (3.924,2) | (4.396,2) | (4.612,1) | (4.892,2) |
| $S_2$ | (5.002,1) | (5.022,2) | (5.082,2) | (5.112,1) | (5.147,1) | (5.238,1) | (5.244,1) | (5.247,1) |
| | (5.305,1) | (5.337,2) | (5.407,1) | (5.408,2) | (5.445,1) | (5.483,1) | (5.717,2) | |

**Table 15:** The MLEs and the Bayes estimates of the data in Table 14.

| Estimate | $\alpha_1$ | $\theta_{11}$ | $\theta_{12}$ | $\alpha_2$ | $\theta_{21}$ | $\theta_{22}$ |
|---|---|---|---|---|---|---|
| MLE | 1.3027 | 0.0145 | 0.0628 | 2.0578 | 0.0818 | 0.0409 |
| Bayes | 1.2687 | 0.0181 | 0.0634 | 2.0415 | 0.0820 | 0.0448 |




(a) Profile log-likelihood of $\alpha_2$ (b) Condition on Lemma 2 (c) Condition on Lemma 4

**Figure 2:** Plot of profile log-likelihood of $\alpha_2$ and the sufficient conditions for the data in Table 14.

**Table 16:** CI/CRI of the unknown parameters for data in Table 14.

| | | $\alpha_1$ | | $\theta_{11}$ | | $\theta_{12}$ | |
|---|---|---|---|---|---|---|---|
| CI/CRI | Level | LL | UL | LL | UL | LL | UL |
| Asymptotic CI | 90% | 0.8050 | 1.8004 | 0 | 0.0318 | 0.0088 | 0.1168 |
| | 95% | 0.7079 | 1.8975 | 0 | 0.0352 | 0 | 0.1273 |
| | 99% | 0.5227 | 2.0826 | 0 | 0.0416 | 0 | 0.1474 |
| Symmetric CRI | 90% | 0.7942 | 1.9048 | 0.0062 | 0.0352 | 0.0374 | 0.0950 |
| | 95% | 0.7302 | 2.0187 | 0.0050 | 0.0401 | 0.0337 | 0.1028 |
| | 99% | 0.6191 | 2.2783 | 0.0031 | 0.0512 | 0.0270 | 0.1178 |
| HPD CRI | 90% | 0.7672 | 1.8663 | 0.0046 | 0.0317 | 0.0347 | 0.0912 |
| | 95% | 0.6974 | 1.9657 | 0.0032 | 0.0361 | 0.0314 | 0.0991 |
| | 99% | 0.5729 | 2.2040 | 0.0019 | 0.0469 | 0.0256 | 0.1150 |

| | | $\alpha_2$ | | $\theta_{21}$ | | $\theta_{22}$ | |
|---|---|---|---|---|---|---|---|
| CI/CRI | Level | LL | UL | LL | UL | LL | UL |
| Asymptotic CI | 90% | 0.4105 | 3.7051 | 0 | 0.3744 | 0 | 0.1887 |
| | 95% | 0.0891 | 4.0265 | 0 | 0.4315 | 0 | 0.2176 |
| | 99% | 0 | 4.6393 | 0 | 0.5403 | 0 | 0.2725 |
| Symmetric CRI | 90% | 0.2414 | 6.5516 | 0.0450 | 0.1282 | 0.0193 | 0.0791 |
| | 95% | 0.2197 | 7.6800 | 0.0401 | 0.1402 | 0.0163 | 0.0888 |
| | 99% | 0.2034 | 9.3650 | 0.0314 | 0.1634 | 0.0112 | 0.1079 |
| HPD CRI | 90% | 0.2000 | 5.1449 | 0.0420 | 0.1232 | 0.0158 | 0.0735 |
| | 95% | 0.2000 | 6.5523 | 0.0365 | 0.1342 | 0.0129 | 0.0816 |
| | 99% | 0.2000 | 8.8063 | 0.0279 | 0.1577 | 0.0087 | 0.1012 |

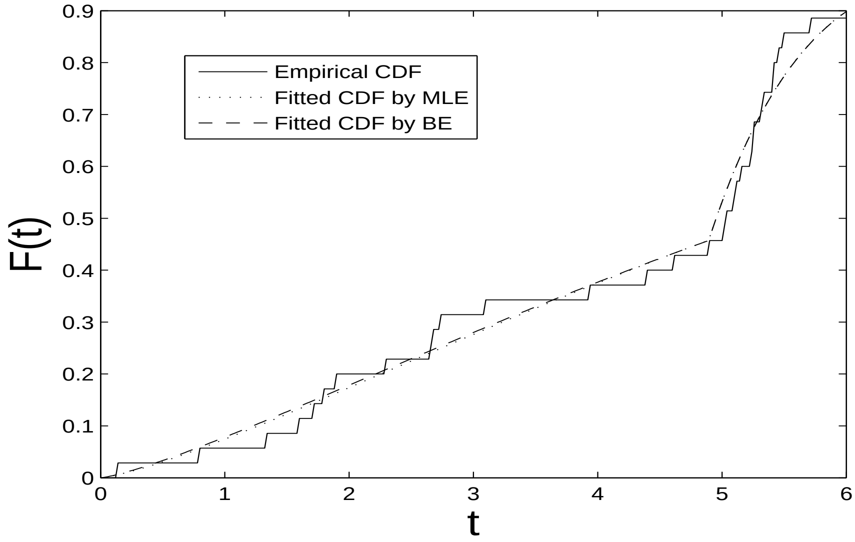



**Figure 3:** Empirical and the fitted CDFs for the data in Table 14.

# 7 Optimality of Test Plan

In this section we consider the problem of choosing $r$ by optimizing different optimality functions. Here we define three optimality criteria based on asymptotic coefficient of variances, Fisher information matrix and posterior coefficient of variances of the model parameters.

## 7.1 Criterion-1

In this subsection we will define optimality criterion based on asymptotic variances of the MLEs of parameters. Asymptotic variances are the diagonals of the inverse of Fisher information matrix. Here we have used sum of coefficient of variations as our optimal function instead of sum of the variances of the parameters, as given in Samanta et al. [37]. The sum of the variances may be dominated by the variance of any particular parameter, since the parameter values may be in different scale. Hence an optimal value of $r$ can be obtained by minimizing expected value of sum of coefficient of variation, i.e., by minimizing $E(\phi_1(r))$, where

$$\phi_1(r) = \frac{\sqrt{V_{11}}}{\hat{\theta}_{11(MLE)}} + \frac{\sqrt{V_{22}}}{\hat{\theta}_{12(MLE)}} + \frac{\sqrt{V_{33}}}{\hat{\theta}_{21(MLE)}} + \frac{\sqrt{V_{44}}}{\hat{\theta}_{22(MLE)}} + \frac{\sqrt{V_{55}}}{\hat{\alpha}_{1(MLE)}} + \frac{\sqrt{V_{66}}}{\hat{\alpha}_{2(MLE)}},$$

where $V_{ii}$ is the $(ii)-th$ element of inverse of the Fisher information matrix which are given in Appendix A.7.

## 7.2 Criterion-2

Here we try to find the optimal value of $r$ by maximizing the expected value of the determinant of Fisher information matrix (given in Appendix A.6). Hence our objective function is

$E(\phi_2(r))$, where

$$\begin{aligned}\phi_2(r) \;=\; & \tfrac{r_{11}}{\theta_{11}^2} \times \tfrac{r_{12}}{\theta_{12}^2} \times \tfrac{r_{21}}{\theta_{21}^2} \times \tfrac{r_{22}}{\theta_{22}^2} \times \Bigg[ \tfrac{r}{\alpha_1^2} + (\theta_{11} + \theta_{12}) \Bigg\{ \textstyle\sum_{i=1}^{r} t_{i:n}^{\alpha_1} (ln(t_{i:n}))^2 + (n-r) t_{r:n}^{\alpha_1} (ln(t_{r:n}))^2 \Bigg\} \\ & -(\tfrac{\theta_{11}^2}{r_{11}} + \tfrac{\theta_{12}^2}{r_{12}}) \Bigg\{ \textstyle\sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n}) + (n-r) t_{r:n}^{\alpha_1} ln(t_{r:n}) \Bigg\}^2 \Bigg] \\ & \times \Bigg[ \tfrac{n-r}{\alpha_2^2} + (\theta_{21} + \theta_{22}) \Bigg\{ \textstyle\sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} (ln(t_{i:n}))^2 - (n-r) t_{r:n}^{\alpha_2} (ln(t_{r:n}))^2 \Bigg\} \\ & -(\tfrac{\theta_{21}^2}{r_{21}} + \tfrac{\theta_{22}^2}{r_{22}}) \Bigg\{ \textstyle\sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} ln(t_{i:n}) - (n-r) t_{r:n}^{\alpha_2} ln(t_{r:n}) \Bigg\}^2 \Bigg].\end{aligned}$$

## 7.3 Criterion-3

Another optimal criterion can be defined by considering the posterior coefficients of variation of the parameters. Similar to the Criterion-1 here we use posterior variance and posterior mean of unknown parameters to obtain the coefficients of variation. Hence the optimal criterion is $E(\phi_3(r))$, where

$$\phi_3(r) = \frac{\sqrt{V_{post}(\alpha_1)}}{\hat{\alpha}_{1(B)}} + \frac{\sqrt{V_{post}(\alpha_2)}}{\hat{\alpha}_{2(B)}} + \frac{\sqrt{V_{post}(\theta_{11})}}{\hat{\theta}_{11(B)}} + \frac{\sqrt{V_{post}(\theta_{12})}}{\hat{\theta}_{12(B)}} + \frac{\sqrt{V_{post}(\theta_{21})}}{\hat{\theta}_{21(B)}} + \frac{\sqrt{V_{post}(\theta_{22})}}{\hat{\theta}_{22(B)}}.$$

Due to absence of closed form of the posterior variance, we use Gibbs sampling technique to calculate it. The expected value of $\phi_3(r)$ can be obtained by replicating it $M$ times and then by taking the average of the optimal values. Below is the algorithm to obtain an optimal $r$ for a given $n$ under different optimality criteria as discussed above.

**Algorithm 3**

Step 1. For a given $r$, $n$ and parameter values, generate the data $\{(t_{1:n}, \delta_1), \ldots, (t_{n:n}, \delta_n)\}$.

Step 2. Calculate the objective function ($\phi_1(r)$ or $\phi_2(r)$ or $\phi_3(r)$) under different optimality criteria.

Step 3. Replicate Step 1-2, $M$ times to get $\phi_1^1(r), \ldots, \phi_1^M(r)$ or $\phi_2^1(r), \ldots, \phi_2^M(r)$ or $\phi_3^1(r), \ldots, \phi_3^M(r)$. Calculate the median of the objective functions and denote them by $\phi_1^m(r)$, $\phi_2^m(r)$ and $\phi_3^m(r)$ respectively.

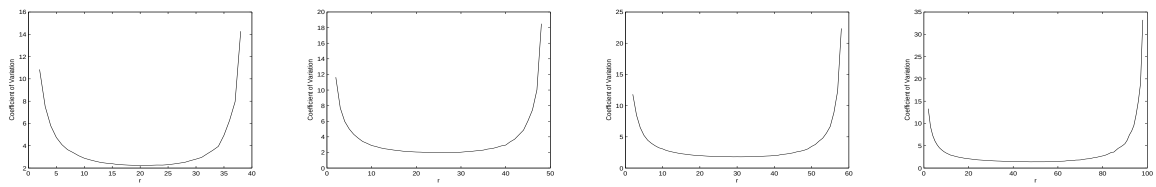
Step 4. Replicate Step 1-3 for all possible values of $r$.

Step 5. Choose $r$ for which $\phi_1^m(r)$ is minimum or $\phi_2^m(r)$ is maximum or $\phi_3^m(r)$ is minimum.

An optimal choice of $r$ for a given $n$ and given set of parameter values under different optimality criteria, are given in Table 17. For Criterion-3, which is based on Bayes estimates and the posterior variances of the parameters, we have used the same non-informative prior as given in simulation section. From the optimality results it has been observed that the optimal value of $r$ under Criterion-1 is almost half of the sample size whereas the optimal value of under Criterion-2 and Criterion-3 is approximately forty to forty five percent of the total sample size. Plots of the objective functions under different optimality criteria, with respect to different $r$ are given from Figure 4 to Figure 9.

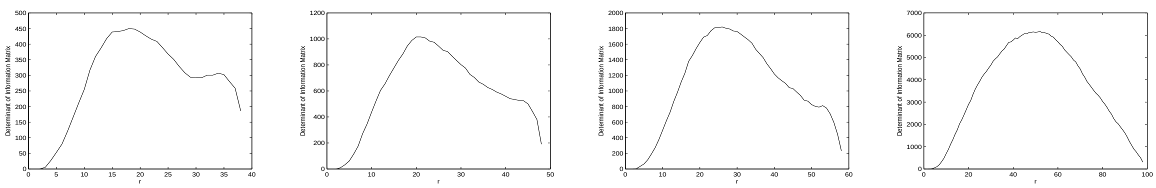
**Table 17:** Optimal choice of $r$ under different optimality criteria.

| | $\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{11} = 1.2$,$\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{11} = 1.7$ | | | $\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{11} = 1.6$,$\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{11} = 2.2$ | | |
|---|---|---|---|---|---|---|
| $n$ | A-optimality | D-optimality | Bayes optimality | A-optimality | D-optimality | Bayes optimality |
| 30 | 15 | 14 | 15 | 17 | 13 | 17 |
| 40 | 20 | 16 | 18 | 20 | 18 | 19 |
| 50 | 26 | 20 | 22 | 27 | 19 | 22 |
| 60 | 29 | 26 | 26 | 33 | 27 | 24 |
| 100 | 48 | 47 | 44 | 54 | 40 | 46 |



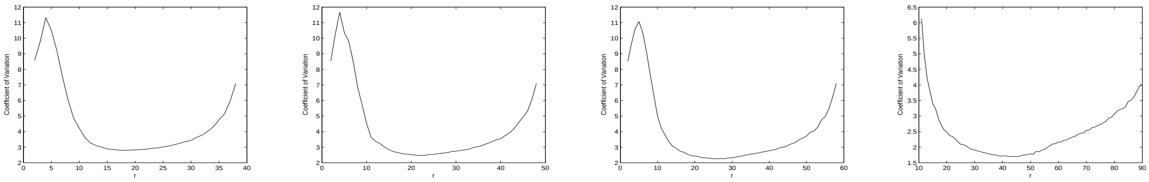
(a) $n = 40$ (b) $n = 50$ (c) $n = 60$ (d) $n = 100$

**Figure 4:** Plot of sum of coefficient of variation against different values of $r$ with parameter values $\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{11} = 1.2$,$\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{11} = 1.7$.



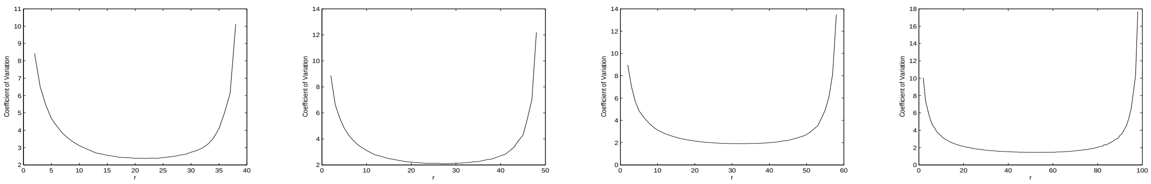
(a) $n = 40$ (b) $n = 50$ (c) $n = 60$ (d) $n = 100$

**Figure 5:** Plots of determinant of information matrix against different values of $r$ with parameter values $\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{11} = 1.2$,$\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{11} = 1.7$.



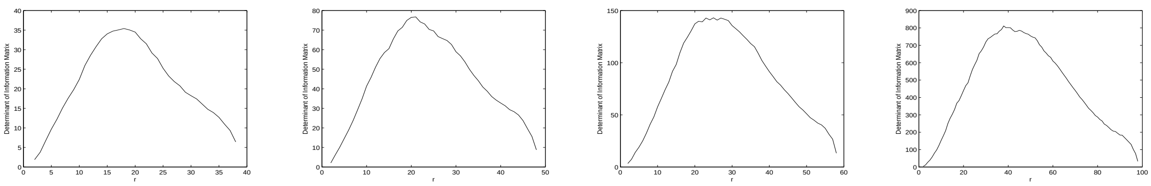
(a) $n = 40$ (b) $n = 50$ (c) $n = 60$ (d) $n = 100$

**Figure 6:** Plots of sum of posterior coefficient of variation against different values of $r$ with parameter values $\alpha_1 = 0.6$, $\theta_{11} = 1.0$, $\theta_{11} = 1.2$,$\alpha_2 = 0.8$, $\theta_{21} = 1.5$, $\theta_{11} = 1.7$.



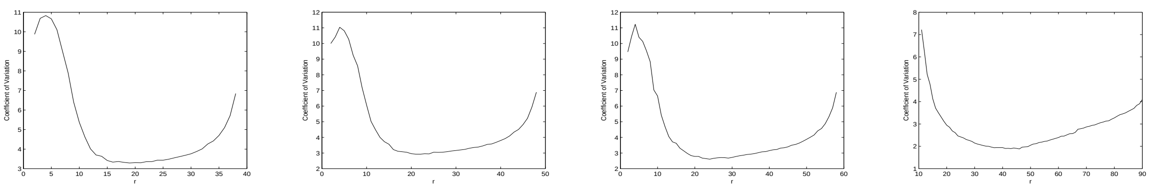
(a) $n = 40$ (b) $n = 50$ (c) $n = 60$ (d) $n = 100$

**Figure 7:** Plots of sum of coefficient of variation against different values of $r$ with parameter values $\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{11} = 1.6$,$\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{11} = 2.2$.



(a) $n = 40$ (b) $n = 50$ (c) $n = 60$ (d) $n = 100$

**Figure 8:** Plots of determinant of information matrix against different values of $r$ with parameter values $\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{11} = 1.6$,$\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{11} = 2.2$.



(a) $n = 40$ (b) $n = 50$ (c) $n = 60$ (d) $n = 100$

**Figure 9:** Plots of sum of posterior coefficient of variation against different values of $r$ with parameter values $\alpha_1 = 1.2$, $\theta_{11} = 1.4$, $\theta_{11} = 1.6$,$\alpha_2 = 1.4$, $\theta_{21} = 2.0$, $\theta_{11} = 2.2$.

# 8 Cause Specific Different Shape Parameters

So far we have performed all the analysis based on the assumption that the shape parameters of the competing causes are equal at each stress level. In this section we briefly indicate how this assumption can be relaxed, and it is observed that the analysis becomes more complicated. In this section it is assumed that at the stress level $s_i$ ($i = 1, 2$) and due to $Cause - j$ ($j = 1, 2$) the lifetime distribution of experimental unit is Weibull with the shape parameter $\alpha_{ij}$ and scale parameter $\theta_{ij}$. As before, for a given cause, to connect the distribution under two stress level we assume generalized KH model. Hence the PDF of the lifetimes for $Cause - 1$ and $Cause - 2$ are given by

$$f_1(t) = \begin{cases} \alpha_{11}\theta_{11}t^{\alpha_{11}-1}e^{-\theta_{11}t^{\alpha_{11}}} & \text{if} \quad 0 < t \leq \tau \\ \alpha_{21}\theta_{21}t^{\alpha_{21}-1}e^{-\theta_{21}(t^{\alpha_{21}}-\tau^{\alpha_{21}})-\theta_{11}\tau^{\alpha_{11}}} & \text{if} \quad \tau < t < \infty. \end{cases} \tag{12}$$

$$f_2(t) = \begin{cases} \alpha_{12}\theta_{12}t^{\alpha_{12}-1}e^{-\theta_{12}t^{\alpha_{12}}} & \text{if} \quad 0 < t \leq \tau \\ \alpha_{22}\theta_{22}t^{\alpha_{22}-1}e^{-\theta_{22}(t^{\alpha_{22}}-\tau^{\alpha_{22}})-\theta_{12}\tau^{\alpha_{12}}} & \text{if} \quad \tau < t < \infty, \end{cases} \tag{13}$$

respectively.

## 8.1 Classical Inference

Based on the above assumptions and proceeding the same way as given in Section 3.1, the log-likelihood of the data $\{(t_{i:n}, \delta_i); i = 1, \ldots, n\}$ without the additive constant is given by

$$
\begin{aligned}
l(Data;\alpha_{ij},\theta_{ij};i,j=1,2) &= r_{11}\ln(\alpha_{11})+r_{11}\ln(\theta_{11})+r_{12}\ln(\alpha_{12})+r_{12}\ln(\theta_{12}) \\
&+(\alpha_{11}-1)\sum_{i=1}^{r}(2-\delta_i)\ln(t_{i:n})+(\alpha_{12}-1)\sum_{i=1}^{r}(\delta_i-1)\ln(t_{i:n}) \\
&-\theta_{11}D_{11}(\alpha_{11})-\theta_{12}D_{12}(\alpha_{12})+r_{21}\ln(\alpha_{21})+r_{22}\ln(\alpha_{22}) \\
&+r_{21}\ln(\theta_{21})+r_{22}\ln(\theta_{22})+(\alpha_{21}-1)\sum_{i=r+1}^{n}(2-\delta_i)\ln(t_{i:n}) \\
&(\alpha_{22}-1)\sum_{i=r+1}^{n}(\delta_i-1)\ln(t_{i:n})-\theta_{21}D_{21}(\alpha_{21})-\theta_{22}D_{22}(\alpha_{22}),
\end{aligned}
\tag{14}
$$

where,

$$
\begin{aligned}
D_{11}(\alpha_{11}) &= \textstyle\sum_{i=1}^{r} t_{i:n}^{\alpha_{11}}+(n-r)t_{r:n}^{\alpha_{11}}, \quad & D_{12}(\alpha_{12}) &= \textstyle\sum_{i=1}^{r} t_{i:n}^{\alpha_{12}}+(n-r)t_{r:n}^{\alpha_{12}}, \\
D_{21}(\alpha_{21}) &= \textstyle\sum_{i=r+1}^{n} t_{i:n}^{\alpha_{21}}-(n-r)t_{r:n}^{\alpha_{21}}, \quad & D_{22}(\alpha_{22}) &= \textstyle\sum_{i=r+1}^{n} t_{i:n}^{\alpha_{22}}-(n-r)t_{r:n}^{\alpha_{22}}.
\end{aligned}
$$

For known $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$ and $\alpha_{22}$, the MLEs of $\theta_{11}$, $\theta_{12}$, $\theta_{21}$ and $\theta_{22}$ are given by

$$
\widehat{\theta}_{11(MLE)}=\frac{r_{11}}{D_{11}(\alpha_{11})},\quad \widehat{\theta}_{12(MLE)}=\frac{r_{12}}{D_{12}(\alpha_{12})},\quad \widehat{\theta}_{21(MLE)}=\frac{r_{21}}{D_{21}(\alpha_{21})},\quad \widehat{\theta}_{22(MLE)}=\frac{r_{22}}{D_{22}(\alpha_{22})}.
$$

The MLEs of $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$ and $\alpha_{22}$ can be obtained in a similar way using profile log-likelihood method. The profile log-likelihood function of $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$ and $\alpha_{22}$ without the additive constant is given by

$$
\begin{aligned}
l_1(Data;\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22}) &= r_{11}\ln(\alpha_1)-r_{11}\ln(D_{11}(\alpha_{11}))+r_{12}\ln(\alpha_{12})-r_{12}\ln(D_{12}(\alpha_{12})) \\
&+(\alpha_{11}-1)\sum_{i=1}^{r}(2-\delta_i)\ln(t_{i:n})+(\alpha_{12}-1)\sum_{i=1}^{r}(\delta_i-1)\ln(t_{i:n}) \\
&+r_{21}\ln(\alpha_{21})-r_{21}\ln(D_{21}(\alpha_{21}))+r_{22}\ln(\alpha_{22})-r_{22}\ln(D_{22}(\alpha_{22})) \\
&+(\alpha_{21}-1)\sum_{i=r+1}^{n}(2-\delta_i)\ln(t_{i:n})+(\alpha_{22}-1)\sum_{i=r+1}^{n}(\delta_i-1)\ln(t_{i:n}).
\end{aligned}
\tag{15}
$$

Therefore, the MLEs of $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$ and $\alpha_{22}$ can be obtain by solving the following four nonlinear equations.

$$\frac{r_{11}}{\alpha_{11}} - \frac{r_{11}(\sum_{i=1}^{r} t_{i:n}^{\alpha_{11}} \ln(t_{i:n}) + (n-r)t_{r:n}^{\alpha_{11}} \ln(t_{r:n}))}{D_{11}(\alpha_{11})} + \sum_{i=1}^{r}(2-\delta_i)\ln(t_{i:n}) = 0, \quad (16)$$

$$\frac{r_{12}}{\alpha_{12}} - \frac{r_{12}(\sum_{i=1}^{r} t_{i:n}^{\alpha_{12}} \ln(t_{i:n}) + (n-r)t_{r:n}^{\alpha_{12}} \ln(t_{r:n}))}{D_{12}(\alpha_{12})} + \sum_{i=1}^{r}(\delta_i - 1)\ln(t_{i:n}) = 0, \quad (17)$$

$$\frac{r_{21}}{\alpha_{21}} - \frac{r_{21}(\sum_{i=r+1}^{n} t_{i:n}^{\alpha_{21}} \ln(t_{i:n}) - (n-r)t_{r:n}^{\alpha_{21}} \ln(t_{r:n}))}{D_{21}(\alpha_{21})} + \sum_{i=r+1}^{n}(2-\delta_i)\ln(t_{i:n}) = 0, \quad (18)$$

$$\frac{r_{22}}{\alpha_{22}} - \frac{r_{22}(\sum_{i=r+1}^{n} t_{i:n}^{\alpha_{22}} \ln(t_{i:n}) - (n-r)t_{r:n}^{\alpha_{22}} \ln(t_{r:n}))}{D_{22}(\alpha_{22})} + \sum_{i=r+1}^{n}(\delta_i - 1)\ln(t_{i:n}) = 0. \quad (19)$$

Clearly, they cannot be obtained in explicit forms. Some iterative procedures like Newton-Raphson or Gauss-Newton are needed to solve these non-linear equations. Asymptotic CIs can be obtained using the observed Fisher information matrix along the same way as before. Details are avoided.

### 8.2 Bayesian Inference

In this subsection we provide the Bayesian inference of the unknown parameters of the above model. The prior distributions of $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$, and $\alpha_{22}$ are assumed to be $Gamma(a_0, b_0)$, $Gamma(a_1, b_1)$, $Gamma(a_4, b_4)$, $Gamma(a_5, b_5)$, respectively. The priors on $(\theta_{11}, \theta_{12})$ and $(\theta_{21}, \theta_{22})$ are $GD(a_2, b_2, a_3, b_3)$ and $GD(a_6, b_6, a_7, b_7)$, respectively. Moreover, the prior distributions of $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$, $\alpha_{22}$, $(\theta_{11}, \theta_{12})$ and $(\theta_{21}, \theta_{22})$ are independent. Hence the joint prior distribution is given by

$$\begin{aligned}\widetilde{\pi}(\alpha_{ij}, \theta_{ij}; i, j = 1, 2) \propto\ & e^{-a_0\alpha_{11}} \alpha_{11}^{b_0-1} e^{-a_1\alpha_{12}} \alpha_{12}^{b_1-1} (\theta_{11}+\theta_{12})^{a_2-a_3-b_3} e^{-b_2\theta_{11}} \theta_{11}^{a_3-1} e^{-b_2\theta_{12}} \theta_{12}^{b_3-1} \\ & e^{-a_4\alpha_{21}} \alpha_{21}^{b_4-1} e^{-a_5\alpha_{22}} \alpha_{22}^{b_5-1} (\theta_{21}+\theta_{22})^{a_6-a_7-b_7} e^{-b_6\theta_{21}} \theta_{21}^{a_7-1} e^{-b_6\theta_{22}} \theta_{22}^{b_7-1}.\end{aligned} \quad (20)$$

As before we will not be able to obtain the Bayes estimates in explicit form. Note that the joint posterior distribution of parameters can be written as

$$\pi(\alpha_{ij}, \theta_{ij}; i,j = 1,2|data) \propto \pi_1(\alpha_{11})\pi_2(\alpha_{12})\pi_3(\theta_{11}|\alpha_{11})\pi_4(\theta_{12}|\alpha_{12})$$
$$\times \pi_5(\alpha_{21})\pi_6(\alpha_{22})\pi_7(\theta_{21}|\alpha_{21})\pi_7(\theta_{22}|\alpha_{22}),$$

where

$$\begin{aligned}
\pi_1(\alpha_{11}) &\propto e^{-a_0\alpha_{11}}\alpha_{11}^{r_{11}+b_0-1}\Big[b_2 + D_{11}(\alpha_{11})\Big]^{-(a_3+r_{11})}\left(\prod_{i=1}^{r} t_{i:n}^{(2-\delta_i)}\right)^{\alpha_{11}-1}, \\
\pi_2(\alpha_{12}) &\propto e^{-a_1\alpha_{12}}\alpha_{12}^{r_{12}+b_1-1}\Big[b_2 + D_{12}(\alpha_{12})\Big]^{-(b_3+r_{12})}\left(\prod_{i=1}^{r} t_{i:n}^{(\delta_i-1)}\right)^{\alpha_{12}-1}, \\
\pi_3(\theta_{11}|\alpha_{11}) &\propto \frac{\Big[b_2+D_{11}(\alpha_{11})\Big]^{a_3+r_{11}}}{\Gamma(a_3+r_{11})} e^{-\theta_{11}(b_2+D_{11}(\alpha_{11}))}\theta_{11}^{a_3+r_{11}-1}, \\
\pi_4(\theta_{12}|\alpha_{12}) &\propto \frac{\Big[b_2+D_{12}(\alpha_{12})\Big]^{b_3+r_{12}}}{\Gamma(b_3+r_{12})} e^{-\theta_{12}(b_2+D_{12}(\alpha_{12}))}\theta_{12}^{b_3+r_{12}-1}, \\
\pi_5(\alpha_{21}) &\propto e^{-a_4\alpha_{21}}\alpha_{21}^{r_{21}+b_4-1}\Big[b_6 + D_{21}(\alpha_{21})\Big]^{-(a_7+r_{21})}\left(\prod_{i=r+1}^{n} t_{i:n}^{(2-\delta_i)}\right)^{\alpha_{21}-1}, \\
\pi_6(\alpha_{22}) &\propto e^{-a_5\alpha_{22}}\alpha_{22}^{r_{22}+b_5-1}\Big[b_6 + D_{22}(\alpha_{22})\Big]^{-(b_7+r_{22})}\left(\prod_{i=1}^{r} t_{i:n}^{(\delta_i-1)}\right)^{\alpha_{22}-1}, \\
\pi_7(\theta_{21}|\alpha_{21}) &\propto \frac{\Big[b_6+D_{21}(\alpha_{21})\Big]^{a_7+r_{21}}}{\Gamma(a_7+r_{21})} e^{-\theta_{21}(b_6+D_{21}(\alpha_{21}))}\theta_{21}^{a_7+r_{21}-1}, \\
\pi_8(\theta_{22}|\alpha_{22}) &\propto \frac{\Big[b_6+D_{22}(\alpha_{22})\Big]^{b_7+r_{22}}}{\Gamma(b_7+r_{22})} e^{-\theta_{22}(b_6+D_{22}(\alpha_{22}))}\theta_{22}^{b_7+r_{22}-1}, \\
w(\alpha_{ij}, \theta_{ij}; i,j = 1,2) &= (\theta_{11}+\theta_{12})^{a_2-a_3-b_3}(\theta_{21}+\theta_{22})^{a_6-a_7-b_7}.
\end{aligned}$$

Similar to Lemmas 3 and 4 it can be shown that $\pi_1(\alpha_{11})$, $\pi_2(\alpha_{12})$ and under certain conditions $\pi_5(\alpha_{21})$ and $\pi_6(\alpha_{22})$ are also log-concave density functions. Therefore, we can use importance sampling technique as before to compute Bayes estimates and to construct associated credible intervals. Note that if $a_2 = a_3 + b_3$ and $a_6 = a_7 + b_7$, i.e., if we assume independent gamma prior for $(\theta_{11}, \theta_{12})$ and $(\theta_{21}, \theta_{22})$ then the Bayes estimates and the CRIs of unknown parameters can be obtained using Gibbs sampling technique.

### 8.3 Inference for Type-I censoring

Observe that the log-likelihood function without the additive constant, based on the Type-I censored data is given by

$$\begin{aligned} l(Data;\alpha_{ij},\theta_{ij};i,j=1,2) \;=\;& r_{11}\ln(\alpha_{11})+r_{11}\ln(\theta_{11})+r_{12}\ln(\alpha_{12})+r_{12}\ln(\theta_{12}) \\ & +(\alpha_{11}-1)\sum_{i=1}^{r}(2-\delta_i)\ln(t_{i:n})+(\alpha_{12}-1)\sum_{i=1}^{r}(\delta_i-1)\ln(t_{i:n}) \\ & -\theta_{11}D_{11}(\alpha_{11})-\theta_{12}D_{12}(\alpha_{12})+r_{21}\ln(\alpha_{21})+r_{22}\ln(\alpha_{22}) \\ & +r_{21}\ln(\theta_{21})+r_{22}\ln(\theta_{22})+(\alpha_{21}-1)\sum_{i=r+1}^{r_{00}}(2-\delta_i)\ln(t_{i:n}) \\ & (\alpha_{22}-1)\sum_{i=r+1}^{r_{00}}(\delta_i-1)\ln(t_{i:n})-\theta_{21}D^*_{21}(\alpha_{21})-\theta_{22}D^*_{22}(\alpha_{22}), \end{aligned} \tag{21}$$

where, $D^*_{21}(\alpha_{21})=\sum_{i=r+1}^{r_{00}}t_{i:n}^{\alpha_{21}}-(n-r)t_{r:n}^{\alpha_{21}}+(n-r_{00})T^{\alpha_{21}}$ and $D^*_{22}(\alpha_{22})=\sum_{i=r+1}^{r_{00}}t_{i:n}^{\alpha_{22}}-(n-r)t_{r:n}^{\alpha_{22}}+(n-r_{00})T^{\alpha_{22}}$.

Since the log-likelihood function (21) is quite similar to the complete data log-likelihood function (14), the classical and the Bayesian inferences can be carried out along the same line as before.

# 9 Conclusion

In this article we provide a method of analyzing competing risk failure data under SSLT experiment. We assume two independent competing risks acting simultaneously on the units under consideration. Also we consider that the lifetime of experimental units for a given cause follow Weibull distribution. To connect the distribution under two stress levels we assume generalized KH model. We provide MLEs and Bayes estimates of model parameters. The performance of asymptotic confidence intervals, symmetric credible intervals and HPD credible intervals are assessed by extensive simulation study. From the simulation results it has been observed that performance of MLEs and Bayes estimates are quite satisfactory even if we consider the almost non-informative prior for Bayesian analysis. Asymptotic confidence intervals and different credible intervals are performing very well. We also provide the analysis under Type-I censoring scheme. We define three optimality criteria for the choice of stress changing time and it has been seen that an unique optimal stress changing time exists. The main objective of a step-stress experiment is to increase the stress level for rapid failures of the experimental units. Therefore, it is expected that the lifetime is lower in the higher stress level, although we have not incorporated this information here. It would be interesting to provide some order restricted inference on model parameters similarly as in Samanta and Kundu [36]. More work is needed along that direction.

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# A Appendix

## A.1 Proof of lemma 1

$$\frac{\delta^2 l_1(Data;\alpha_1,\alpha_2)}{\delta\alpha_1^2} = -\frac{r}{\alpha_1^2} - \frac{rA_1(\alpha_1)}{[D_1(\alpha_1)]^2}$$

where

$$\begin{aligned}
A_1(\alpha_1) &= \left[\sum_{i=1}^{r} t_{i:n}^{\alpha_1} + (n-r)t_{r:n}^{\alpha_1}\right] \times \left[\sum_{i=1}^{r} t_{i:n}^{\alpha_1}(ln(t_{i:n}))^2 + (n-r)t_{r:n}^{\alpha_1}(ln(t_{r:n}))^2\right] \\
&\quad -\left[\sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n}) + (n-r)t_{r:n}^{\alpha_1} ln(t_{r:n})\right]^2 \\
&= \sum_{i=1}^{r} t_{i:n}^{\alpha_1} \sum_{i=1}^{r} t_{i:n}^{\alpha_1}(ln(t_{i:n}))^2 - \left(\sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n})\right)^2 \\
&\quad +(n-r)t_{r:n}^{\alpha_1}\left[(ln(t_{r:n}))^2 \sum_{i=1}^{r} t_{i:n}^{\alpha_1} - 2ln(t_{r:n}) \sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n}) + \sum_{i=1}^{r} t_{i:n}^{\alpha_1}(ln(t_{i:n}))^2\right] \\
&= \sum_{i=1}^{r} t_{i:n}^{\alpha_1} \sum_{i=1}^{r} t_{i:n}^{\alpha_1}(ln(t_{i:n}))^2 - \left(\sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n})\right)^2 + (n-r)t_{r:n}^{\alpha_1} \sum_{i=1}^{r-1} t_{i:n}^{\alpha_1}\left(ln(t_{r:n}) - ln(t_{i:n})\right)^2 \\
&\geq 0.
\end{aligned}$$

Since by Cauchy-Schwartz inequality, $\sum_{i=1}^{r} t_{i:n}^{\alpha_1} \sum_{i=1}^{r} t_{i:n}^{\alpha_1}(ln(t_{i:n}))^2 - \left(\sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n})\right)^2 \geq 0.$

Therefore

$$\frac{\delta^2 l_1(Data; \alpha_1, \alpha_2)}{\delta \alpha_1^2} \leq 0.$$

Hence unique solution of equation (7) exists which maximize (6) with respect to $\alpha_1$.

## A.2 Proof of lemma 2

$$\begin{aligned}
\frac{\delta^2 l_1(Data; \alpha_1, \alpha_2)}{\delta \alpha_2^2} &= -\frac{n-r}{\alpha_2^2} - \frac{(n-r)A_2(\alpha_2)}{[D_2(\alpha_2)]^2} \\
&= -(n-r)u(\alpha_2)
\end{aligned}$$

Now if $u(\alpha_2) \geq 0$ then

$$\frac{\delta^2 l_1(Data; \alpha_1, \alpha_2)}{\delta \alpha_2^2} \leq 0.$$

Hence for all $u(\alpha_2) > 0$ unique solution of equation (8) exists which maximize (6) with respect to $\alpha_2$.

## A.3 Elements of Fisher information Matrix

$$\begin{aligned}
\frac{\delta^2 l}{\delta\theta_{ij}^2} &= -\frac{r_{ij}}{\theta_{ij}^2}, \qquad i=1,2; j=1,2,\\
\frac{\delta^2 l}{\delta\alpha_1^2} &= -\frac{r}{\alpha_1^2} - (\theta_{11}+\theta_{12})\Bigg[\sum_{i=1}^{r} t_{i:n}^{\alpha_1}(ln(t_{i:n}))^2 + (n-r)t_{r:n}^{\alpha_1}(ln(t_{r:n}))^2\Bigg],\\
\frac{\delta^2 l}{\delta\alpha_2^2} &= -\frac{n-r}{\alpha_2^2} - (\theta_{21}+\theta_{22})\Bigg[\sum_{i=r+1}^{n} t_{i:n}^{\alpha_2}(ln(t_{i:n}))^2 - (n-r)t_{r:n}^{\alpha_2}(ln(t_{r:n}))^2\Bigg],\\
\frac{\delta^2 l}{\delta\theta_{ij}\delta\theta_{kl}} &= 0, \qquad (ij)\neq(kl); i=1,2; j=1,2,\\
\frac{\delta^2 l}{\delta\theta_{11}\delta\alpha_1} &= \frac{\delta^2 l}{\delta\theta_{12}\delta\alpha_1} = -\sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n}) - (n-r)t_{r:n}^{\alpha_1} ln(t_{r:n}),\\
\frac{\delta^2 l}{\delta\theta_{21}\delta\alpha_2} &= \frac{\delta^2 l}{\delta\theta_{22}\delta\alpha_2} = -\sum_{i=r+1}^{n} t_{i:n}^{\alpha_2} ln(t_{i:n}) + (n-r)t_{r:n}^{\alpha_2} ln(t_{r:n}),\\
\frac{\delta^2 l}{\delta\theta_{11}\delta\alpha_2} &= \frac{\delta^2 l}{\delta\theta_{12}\delta\alpha_2} = \frac{\delta^2 l}{\delta\theta_{21}\delta\alpha_1} = \frac{\delta^2 l}{\delta\theta_{22}\delta\alpha_1} = \frac{\delta^2 l}{\delta\alpha_1\delta\alpha_2} = 0.
\end{aligned}$$

## A.4 Proof of lemma 3

$$\begin{aligned}
ln(\pi_1(\alpha_1)) &= -\alpha_1 a_0 + (\alpha_1 - 1)\big(\sum_{i=1}^{r} ln(t_{i:n})\big) + (r+b_0-1)ln(\alpha_1)\\
&\quad -(a_1 + r_{11} + r_{12}) ln\big(b_1 + D_1(\alpha_1)\big),\\
\frac{\delta^2 ln(\pi_1(\alpha_1))}{\delta\alpha_1^2} &= -\frac{r+b_0-1}{\alpha_1^2} - \frac{(a_1+r_{11}+r_{12})\Big[g_1(\alpha_1)g_1''(\alpha_1) - [g_1'(\alpha_1)]^2\Big]}{[g_1(\alpha_1)]^2}, \qquad \text{where}\\
g_1(\alpha_1) &= b_1 + \sum_{i=1}^{r} t_{i:n}^{\alpha_1} + (n-r)t_{r:n}^{\alpha_1},\\
g_1'(\alpha_1) &= \sum_{i=1}^{r} t_{i:n}^{\alpha_1} ln(t_{i:n}) + (n-r)t_{r:n}^{\alpha_1} ln(t_{r:n}),\\
g_1''(\alpha_1) &= \sum_{i=1}^{r} t_{i:n}^{\alpha_1}(ln(t_{i:n}))^2 + (n-r)t_{r:n}^{\alpha_1}(ln(t_{r:n}))^2.
\end{aligned}$$

Now

$$g_1(\alpha_1)g_1''(\alpha_1) - [g_1'(\alpha_1)]^2 = b_1 g''(\alpha_1) + A_1(\alpha_1) \geq 0.$$

Since $g_1''(\alpha_1) \geq 0$ and in appendix (A.1) we have shown that $A_1(\alpha_1) \geq 0$. Therefore $\frac{\delta^2 ln(\pi_1(\alpha_1))}{\delta\alpha_1^2} \leq 0$. Hence $\pi_1(\alpha_1)$ is log-concave.

## A.5 Proof of lemma 4

$$\begin{aligned}
ln(\pi_3(\alpha_2)) &= -\alpha_2 a_3 + (\alpha_2 - 1)\Bigg( \sum_{i=r+1}^{n} ln(t_{i:n})\Bigg) + (n - r + b_3 - 1) ln(\alpha_2) \\
&\quad -(a_4 + r_{21} + r_{22}) ln\Bigg( b_4 + D_2(\alpha_2)\Bigg), \\
\frac{\delta^2 ln(\pi_3(\alpha_2))}{\delta\alpha_2^2} &= -u_2(\alpha_2).
\end{aligned}$$

Therefore if $u_2(\alpha_2) \geq 0$ then $\frac{\delta^2 ln(\pi_3(\alpha_2))}{\delta\alpha_2^2} \leq 0$. Hence $\pi_3(\alpha_2)$ is log-concave.

## A.6 Determinant of Fisher information matrix

Consider the partition of Fisher information matrix as

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F'_{12} & F_{22} \end{bmatrix},$$

where $F_{11} = Diag(f_{11}, f_{22}, f_{33}, f_{44}, )$, $F_{22} = Diag(f_{55}, f_{66})$,

$$F_{12} = \begin{bmatrix} f_{15} & 0 \\ f_{25} & 0 \\ 0 & f_{36} \\ 0 & f_{46} \end{bmatrix} = \begin{bmatrix} x & 0 \\ x & 0 \\ 0 & y \\ 0 & y \end{bmatrix} (say)[\text{ Since} f_{15} = f_{25} \text{ and } f_{36} = f_{46}].$$

Now $Det(F) = Det(F_{11})Det(F_{22} - F'_{12}F_{11}^{-1}F_{12})$.

$$F_{22} - F'_{12}F_{11}^{-1}F_{12} = \begin{bmatrix} f_{55} - x^2(\frac{1}{f_{11}} + \frac{1}{f_{22}}) & 0 \\ 0 & f_{66} - y^2(\frac{1}{f_{33}} + \frac{1}{f_{44}}) \end{bmatrix}.$$

Therefore, $Det(F) = f_{11} \times f_{22} \times f_{33} \times f_{44} \times [f_{55} - x^2(\frac{1}{f_{11}} + \frac{1}{f_{22}})] \times [f_{66} - y^2(\frac{1}{f_{33}} + \frac{1}{f_{44}})]$.

Putting the values of $f_{ij}, x$ and $y$ we have the expression for $\phi_D(r)$.

## A.7 Diagonals of inverse of Fisher information matrix

$$F^{-1} = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}' & F_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (F_{11} - F_{12}F_{22}^{-1}F_{12}')^{-1} & -F_{11}^{-1}F_{12}(F_{22} - F_{12}'F_{11}^{-1}F_{12})^{-1} \\ -F_{22}^{-1}F_{12}'(F_{11} - F_{12}F_{22}^{-1}F_{12}')^{-1} & (F_{22} - F_{12}'F_{11}^{-1}F_{12})^{-1} \end{bmatrix}.$$

Here we are concerned only with the diagonals of $F^{-1}$. Hence consider two block diagonal matrices, namely, $(F_{11} - F_{12}F_{22}^{-1}F_{12}')^{-1}$ and $(F_{22} - F_{12}'F_{11}^{-1}F_{12})^{-1}$,

$$(F_{11} - F_{12}F_{22}^{-1}F_{12}')^{-1} = \begin{bmatrix} f_{11} - \frac{x^2}{f_{55}} & -\frac{x^2}{f_{55}} & 0 & 0 \\ -\frac{x^2}{f_{55}} & f_{22} - \frac{x^2}{f_{55}} & 0 & 0 \\ 0 & 0 & f_{33} - \frac{y^2}{f_{66}} & -\frac{y^2}{f_{66}} \\ 0 & 0 & -\frac{y^2}{f_{66}} & f_{44} - \frac{y^2}{f_{66}} \end{bmatrix}^{-1} = \begin{bmatrix} F_1^{-1} & 0 \\ 0 & F_2^{-1} \end{bmatrix},$$

where

$$F_1^{-1} = \begin{bmatrix} f_{11} - \frac{x^2}{f_{55}} & -\frac{x^2}{f_{55}} \\ -\frac{x^2}{f_{55}} & f_{22} - \frac{x^2}{f_{55}} \end{bmatrix}^{-1} = \frac{1}{(f_{11} - \frac{x^2}{f_{55}})(f_{22} - \frac{x^2}{f_{55}}) - \frac{x^4}{f_{55}^2}} \begin{bmatrix} f_{22} - \frac{x^2}{f_{55}} & \frac{x^2}{f_{55}} \\ \frac{x^2}{f_{55}} & f_{11} - \frac{x^2}{f_{55}} \end{bmatrix},$$

$$F_2^{-1} = \begin{bmatrix} f_{33} - \frac{y^2}{f_{66}} & -\frac{y^2}{f_{66}} \\ -\frac{y^2}{f_{66}} & f_{44} - \frac{y^2}{f_{66}} \end{bmatrix}^{-1} = \frac{1}{(f_{33} - \frac{y^2}{f_{66}})(f_{44} - \frac{y^2}{f_{66}}) - \frac{y^4}{f_{66}^2}} \begin{bmatrix} f_{44} - \frac{y^2}{f_{66}} & \frac{y^2}{f_{66}} \\ \frac{y^2}{f_{66}} & f_{33} - \frac{y^2}{f_{66}} \end{bmatrix},$$

$$(F_{22} - F_{12}'F_{11}^{-1}F_{12})^{-1} = \begin{bmatrix} \left(f_{55} - x^2(\frac{1}{f_{11}} + \frac{1}{f_{22}})\right)^{-1} & 0 \\ 0 & \left(f_{66} - y^2(\frac{1}{f_{33}} + \frac{1}{f_{44}})\right)^{-1} \end{bmatrix}.$$

Diagonals of $F_1^{-1}$, $F_2^{-1}$ and $(F_{22} - F_{12}'F_{11}^{-1}F_{12})^{-1}$ are the diagonals of inverse of Fisher information matrix.