Order Restricted Classical Inference of A Weibull Multiple Step-stress Model

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Abstract

In this paper, a multiple **step-stress** model is designed and analyzed when the data are Type-I censored. Lifetime distributions of the experimental units at each stress level are assumed to follow a two-parameter Weibull distribution. Further, distributions under each of the stress levels are connected through a tampered failure rate based model. In a **step-stress** experiment, as the stress level increases, the load on the experimental units increases and hence the mean lifetime is expected to be shortened. Taking this into account, the aim of this paper is to develop the **order restricted** inference of the model parameters of a multiple **step-stress** model based on the frequentist approach. An extensive simulation study has been carried out and two real data sets have been analyzed for illustrative purposes.

Key Words: **Step-stress** model; tampered failure rate based model; maximum likelihood estimator; isotonic regression; bootstrap confidence interval.

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1 Introduction

In a life-testing experiment, experimenters often come across units with high reliability. Owing to huge advancement in science and technology, this problem is very common nowadays in industries like automobile, aerospace, medical and so on. Thus, within a given time constraint and fixed cost, it is very difficult to obtain sufficient number of failure time data. Accelerated life testing(ALT) experiment is a very useful approach in reliability analysis to overcome this problem. Under an ALT experiment, the units are exposed to an external stress factor, for example high temperature, voltage, pressure in electrical appliances or toxicological effects in a medical drug study etc., in order to obtain early failures within a stipulated time period.

Real life applications of ALT is inevitable in today's world of consumerism, where the consumers are aware of their rights and insist on quality assurance, especially for the high-priced durable products. One such application is the prediction of warranty cost, since in the competitive business environment, warranty is an important sales feature, especially for consumer products. Attractive warranty packages are offered by manufacturers to customers with the hope of increased market share and high revenues. Often in such pursuits, manufacturers suffer drastic warranty burdens. For example, the automotive industry need to pay some $15 billion annually in the United States only for warranty repairs (Yang. [31]). Again, in some products, attainment of a prespecified design is a mandatory requirement. Thus, an early and accurate prediction of warranty cost turns out to be a natural choice. Such prediction can be carried out using the ALT conducted in the different phases of life cycle (Yang. [30]). In fact, ALT can be traced back to the inception of statistical quality control methods, which became popular as an effective tool of reducing cost and time of experiments without compromising on the quality of statistical inference drawn from the data. For some real life applications, see Willihnganz [29], Tian [27], Choi et al. [8] and see the references cited therein.

To carry out the ALT, test units are divided into several groups, each being put to an elevated stress level thus yielding early failures. The life data thus obtained from elevated stress levels are extrapolated to estimate the product life distribution at a design stress level (See Yang. [30]) or to estimate the field reliability (See Meeker et al. [19]). A special case of the ALT experiment is the **step-stress** accelerated life testing (SSALT) experiment, where the stress increases at given time points or after a specified number of failures. A general set up of a fixed time point multiple **step-stress** experiment can be described as follows. Suppose n items are exposed to an initial stress level s_1 and at fixed time points $\tau_1, \tau_2, \ldots, \tau_m$, the stress levels are increased to s_2, $s_3, \ldots, s_{m+1}$, respectively. In this process, distributional assumptions are made for the lifetime of units corresponding to each stress level and the failure times are recorded in an ascending order. Inference on the lifetime distributions under each stress level is performed, first under the accelerated set up, and then the results are translated back to predict the lifetime under the used (normal) conditions. Some of the key references on different ALT models are Nelson [21], Bagdonavicius and Nikulin [2] and the references cited therein.

Though, the duration of a life-testing experiment under a SSALT set up is guaranteed to shorten, yet it is not controlled. The control of the total experimental duration is equally crucial and can be achieved by imposing different censoring schemes. The experimenter can fix either (a) the total duration $\tau > \tau_m$ or (b) the observed value $r < n$ of the total number of failures (R). In case (a), the corresponding sample is Type-I censored and the number of failures $R(\geq 1)$ upto time τ is random. In case (b), the corresponding sample is Type-II censored and the experiment stops as soon as the r^{th} failure occurs.

To analyze data obtained from a SSALT experiment, one needs a model which relates the cumulative distribution functions (CDFs) of lifetimes under different stress levels to the CDF of the lifetime under the used conditions. Several models are available in the literature to describe this relationship. The most popular one is known as the cumulative exposure model (CEM) originally proposed by Sedyakin [26] and later quite extensively studied by Bagdonavicius [1] and Nelson [21]. This model assumes that at any time point, the remaining lifetime of an experimental unit depends only on the cumulative exposure accumulated and the current stress level, irrespective of how the exposure is actually accumulated. In a review article, Balakrishnan [4] has extensively discussed different inferential issues of a **step-stress** model under the CEM assumptions when the lifetime distributions of the experimental units follow exponential distribution. The recent monograph by Kundu and Ganguly [16] is another exhaustive collection of applications on different **step-stress** models. Another widely used model is the proportional hazards model (PHM), introduced by Cox [9]. It describes the influence of covariates on the lifetime distribution. Using the concept of PHM, Bhattacharyya and Soejoeti [6] introduced the tampered failure rate model (TFRM) for the simple SSALT and Madi [18] provided its generalization in case of multiple SSALT. This same model was also proposed by Khamis and Higgins [14] when lifetime at the different stress levels follow Weibull distribution with a common shape parameter and different scale parameters. It is popularly known as the Khamis-Higgins model (KHM). Analysis of an SSALT model under the frequentist set up is widely studied assuming the lifetime distribution to be Weibull. Komori [15] studied different properties of the CEM under Weibull distribution. Bai and Kim [3] and Kateri and Balakrishnan [11] discussed the inferential aspects of Weibull SSALT under Type-I censoring and Type-II censoring, respectively.

The main intent of an SSALT experiment is to reduce the lifetime of experimental units by increasing the stress level in order to obtain early failures. The experiments are so designed, that it is only logical to assume that the mean lifetime of the experimental units is lower at the higher stress level. Balakrishnan et al. [5] first incorporated this information and considered estimation of the model parameters based on the assumption that the lifetime distribution of the experimental units follow an exponential distribution. They obtained the **order restricted** maximum likelihood estimators (MLEs) using the isotonic regression technique in case of Type-I and Type-II censored data. Recently Samanta and Kundu [25] and Samanta et al. [23] also considered the same problem in case of generalized exponential and exponential distributions respectively and obtained the **order restricted** MLEs of the model parameters using a reparametrization algorithm which stops in a finite number of steps.

Often, in the context of multiple **step-stress** models, unrestricted MLEs of the parameters corresponding to the lifetime distributions at all the stress levels may not exist owing to non-availability of failures in some of the stress levels. **Order restricted** inference is a natural choice here and under this assumption, the restricted MLEs of the model parameters exist provided there is at least one failure at each of the two extreme stress levels (s_1 and s_{m+1}). In other words, MLEs always exist for the internal stress levels and thus the number of cases of non existence of the unrestricted MLEs, is significantly reduced.

A two-parameter Weibull (WE) distribution with the shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$ has the following CDF, probability density function (PDF) and hazard function (HF), respectively,

$$F(t;\alpha,\lambda) = 1 - e^{-\lambda t^{\alpha}}, \qquad t > 0, \tag{1}$$

$$f(t;\alpha,\lambda) = \alpha\lambda t^{\alpha-1} e^{-\lambda t^{\alpha}}, \qquad t > 0, \tag{2}$$

$$h(t;\alpha,\lambda) = \alpha\lambda t^{\alpha-1}, \qquad t > 0. \tag{3}$$

From now on, a WE distribution with the shape parameter α and the scale parameter λ will be denoted by WE(α, λ).

In this paper, the analysis of data obtained from a multiple SSALT experiment is considered when the data are Type-I censored. The lifetime distribution of the experimental units under each stress level is assumed to follow two parameter WE distribution with a common shape parameter but different scale parameters, which satisfies the TFRM assumptions. Assumption of the common shape parameter in Weibull distribution is quite common (See, for example, Mondal and Kundu [20], Samanta et al. [24], Wang and Fei [28]) and it makes the problem mathematically more tractable. However, different shape parameters may also be incorporated at the different stress levels. Based on the assumption that the mean lifetime of the experimental units decreases with the increase in stress level, we provide the **order restricted** classical inference of the model parameters. We obtain the estimators of the unknown parameters and the associated confidence intervals. Extensive simulation experiments have been performed to validate the effectiveness of the **order restricted** inference. We provide the analysis of two real data sets for illustrative purposes.

Rest of the paper is organized as follows. In Section 2, we provide the model assumptions, the structure of the available data and the likelihood function. In Section 3, we obtain the **order restricted** MLEs of the model parameters. In Section 4, we discuss the interval estimation of the model parameters. Extensive simulation study and analyses of two real life data are carried out in Section 5. Finally we conclude the paper in Section 6.

2 Model Assumptions and the likelihood function

We assume that n identical units are placed on a life testing experiment, and subjected to an initial stress level s_1 at the time point 0. At pre- fixed time points $\tau_1, \tau_2, \ldots, \tau_m$, the stress level increases to $s_2, s_3, \ldots, s_{m+1}$, respectively. Finally, the experiment terminates at a fixed time point τ. The observed Type-I censored failure time data thus obtained from a multiple SSALT experiment is given by

$$\mathcal{D} = \{t_{1:n} < \ldots < t_{\bar{n}_1:n} < \tau_1 < t_{\bar{n}_1+1:n} < \ldots < t_{\bar{n}_2:n} < \tau_2 < \ldots < \tau_m < t_{\bar{n}_m+1:n} < \ldots < \mathbf{t_{\bar{n}_{m+1}:n}} < \tau\}. \tag{4}$$

Here n_k is the number of failures under stress level s_k $(k = 1, \ldots, m+1)$ and $\bar{n}_j = \sum_{i=1}^{j} n_i$ is the total number of failures upto the jth stress level. Clearly, $n_1 = \bar{n}_1$. Further, it is assumed that the lifetime distribution of the experimental units under the stress level s_k follows WE(α, λ_k). Hence, for $\alpha > 0$, $\lambda_k > 0$ and $t > 0$, the CDF and the hazard function (HF) of the lifetime distribution at the k^{th} stress level is given by

$$F_k(t) = \left(1 - e^{-\lambda_k t^\alpha}\right), \; h_k(t) = \alpha\lambda_k t^{\alpha-1}. \quad k = 1, \ldots, m+1.$$

If s_j, $j = 1, 2, \ldots, m+1$, are the $m+1$ stress levels and τ_i, $i = 1, 2, \ldots, m$, are the pre determined time points at which the stress level changes from s_i to s_{i+1}, it is assumed that the hazard function of the distribution under the **step-stress** pattern is as follows:

$$h^*(t) = \begin{cases} \lambda_1 h_o(t) & \text{if} \quad 0 < t \leq \tau_1 \\ \lambda_k h_o(t) & \text{if} \quad \tau_{k-1} < t < \tau_k, ; \; k = 2, 3, \ldots, m \\ \lambda_{m+1} h_o(t) & \text{if} \quad \tau_m < t < \infty, \end{cases} \tag{5}$$

where $h_o(t)$ is the hazard function corresponding to the baseline distribution function $F_o(t) = 1 - e^{-t^\alpha}$. The flexibility of this model can be very useful in SSALT experiments. Some recent references for more insight and detailed interpretations are Kateri and Kamps [12] and Kateri and Kamps [13]. We denote $\Theta = (\alpha,\ \lambda_1,\ \lambda_2,\ \ldots, \lambda_{m+1})$, the set of model parameters to be estimated. Using the one-to-one correspondence between the HF and the CDF, the overall CDF and the associated probability density function (PDF) are given by

$$F(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-\lambda_1 t^\alpha} & \text{if } 0 < t \leq \tau_1 \\ 1 - e^{-\sum\limits_{j=1}^{k-1}(\lambda_j - \lambda_{j+1})\tau_j^\alpha - \lambda_k t^\alpha} & \text{if } \tau_{k-1} < t \leq \tau_k \ ; \ k = 2, 3, \ldots, m \\ 1 - e^{-\sum\limits_{j=1}^{m}(\lambda_j - \lambda_{j+1})\tau_j^\alpha - \lambda_{m+1} t^\alpha} & \text{if } \tau_m < t < \infty, \end{cases} \tag{6}$$

$$f(t) = \begin{cases} \alpha\lambda_1 t^{\alpha-1} e^{-\lambda_1 t^\alpha} & \text{if } 0 < t \leq \tau_1 \\ e^{-\sum\limits_{j=1}^{k-1}(\lambda_j - \lambda_{j+1})\tau_j^\alpha} \alpha\lambda_k t^{\alpha-1} e^{-\lambda_k t^\alpha} & \text{if } \tau_{k-1} < t \leq \tau_k \ ; \ k = 2, 3, \ldots, m \\ e^{-\sum\limits_{j=1}^{m}(\lambda_j - \lambda_{j+1})\tau_j^\alpha} \alpha\lambda_{m+1} t^{\alpha-1} e^{-\lambda_{m+1} t^\alpha} & \text{if } \tau_m < t < \infty \\ 0 & \text{otherwise} . \end{cases} \tag{7}$$

Therefore, based on the Type-I censored data (4), the likelihood function then can be written as

$$L(\Theta|\mathcal{D}) \ \propto \ \alpha^{\bar{n}_{m+1}} \lambda_1^{n_1} \lambda_2^{n_2} \ldots \lambda_{m+1}^{n_{m+1}} \left(\prod_{i=1}^{\bar{n}_{m+1}} t_{i:n}^{\alpha-1} \right) e^{-(\lambda_1 D_1(\alpha) + \lambda_2 D_2(\alpha) + \ldots + \lambda_{m+1} D_{m+1}(\alpha))}, \tag{8}$$

$$\text{where} \quad D_j(\alpha) \;=\; \sum_{i=\overline{n}_{j-1}+1}^{\overline{n}_j} t_{i:n}^{\alpha} + (n-\overline{n}_j)\tau_j^{\alpha} - (n-\overline{n}_{j-1})\tau_{j-1}^{\alpha}, \;\; j = 1,2,\ldots,m+1 \qquad (9)$$

$$\text{and} \;\; \overline{n}_0 = \tau_0 = 0\,.$$

3 Order restricted Maximum Likelihood Estimators

In a **step-stress** experiment, it is assumed that as the stress level increases, the load on the units increases, and hence the experimental lifetimes of the units get shortened. In this section this additional information is used to compute the MLEs of the unknown parameters. Therefore, we impose the natural restriction on the scale parameters as

$$\lambda_1 \leq \ldots \leq \lambda_{m+1}. \qquad (10)$$

Hence the **order restricted** MLEs of the unknown parameters can be obtained by maximizing (8) with respect to α, $\lambda_1, \ldots, \lambda_{m+1}$ based on the order restriction (10).

First, we introduce here some new notations. Let $I = \{1, 2, \ldots, m+1\}$ be the set of natural numbers upto $m+1$. In addition, we introduce the random set of indices corresponding to the stress levels at which items fail as $\tilde{I} = \{i \in I | n_i \geq 1\}$. The elements of $\tilde{I}$ are denoted as $i_1, i_2, \ldots, i_{|\tilde{I}|}$. Also, let $\tilde{J} = \tilde{I} \backslash \{i_1, i_{|\tilde{I}|}\}$. Note that the likelihood function in (8) can equivalently be written, up to a constant, as

$$\alpha^{\bar{n}_{m+1}} \prod_{j \in \tilde{I}} \lambda_j^{n_j} \exp\Big\{ -\sum_{j\in\tilde{I}} \lambda_j D_j(\alpha) \Big\} \Big\{ \prod_{i=1}^{\bar{n}_{m+1}} t_{i:n}^{\alpha-1} \Big\} \qquad (11)$$

However, for fixed shape parameter α, in the unrestricted case, one can only maximize (11) with respect to $\lambda_i (\, i \in \tilde{I})$. The unrestricted MLEs for Type-I censored data is then given by

$$\widehat{\lambda}_i^{(\alpha)} = \frac{n_i}{D_i(\alpha)}, \quad i \in \tilde{I}.$$

If $i \in \tilde{I}^c$, MLE of λ_i will not exist. However, consideration of **order restricted** inference greatly reduces the probability of non-existence of the unrestricted MLEs. To address this, we consider two scenarios of a multiple **step-stress** data and provide some results.

Case 1: At least one failure is present at every stress level.

When at least one failure is present at every stress level, the set $\tilde{I}$ is simply $\{1, 2, \ldots, m+1\}$. The following lemma provides the MLEs of $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}$ for a fixed shape parameter α, under the natural order restriction on the scale parameters $(\lambda_1 \leq \lambda_2 \ldots \leq \lambda_{m+1})$. It is to note that the **order restricted** MLEs considered in Case 1 are conditional on the event that at least one failure is present at every stress level. For the sake of brevity, we do not make it explicit in the notations. Similarly, in the next scenario (Case 2), the **order restricted** MLEs are conditional on the event that there is no failure in at least one of the internal stress levels.

Lemma 1. *For the fixed shape parameter α, if $L(\Theta \mid \mathcal{D})$ is maximized under the restriction on the scale parameters $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_{m+1}$, the MLE $\widetilde{\lambda}_k^{(\alpha)}$ of λ_k, for $k = 1, 2, \ldots, m+1$ is given by*

$$\widetilde{\lambda}_k^{(\alpha)} = \left[\min_{s \leq k} \max_{t \geq k} \frac{\sum_{r=s}^{t} D_r(\alpha)}{\sum_{r=s}^{t} n_r}\right]^{-1}, \quad s, t \in \tilde{\mathbf{I}} = \mathbf{I}.$$

Proof. See Appendix. $\square$

To obtain the MLE of α, we need to maximize the profile log likelihood function of α, say $m_1(\alpha)$ where

$$m_1(\alpha) = \alpha^{\bar{n}_{m+1}} \prod_{j \in \tilde{I}} \left[\widetilde{\lambda}_j^{(\alpha)}\right]^{n_j} \exp\Big\{-\sum_{j \in \tilde{I}} \widetilde{\lambda}_j^{(\alpha)} D_j(\alpha)\Big\}\Big\{\prod_{i=1}^{\bar{n}_{m+1}} t_{i:n}^{\alpha-1}\Big\}.$$

It cannot be obtained in explicit form. We need to use some numerical method to maximize $m_1(\alpha)$.

Case 2: There is no failure in at least one of the internal stress levels

Often, in a multiple **step-stress** data, there may be no failures in one or more stress levels. We are assuming here that there is at least one failure in the extreme stress levels (s_1 and s_{m+1}). Now, for fixed shape parameter α, let us first consider the problem of maximizing (11) with respect to $\lambda_i \quad i \in \tilde{I}$, under the natural order, i.e, $\lambda_{i_1} \leq \ldots \leq \lambda_{i_{|\tilde{I}|}}$ and then the problem of maximizing the likelihood function (11) with respect to $\lambda_1 \leq \ldots \leq \lambda_{m+1}$. We now state the following two lemmas for fixed shape parameter α.

Lemma 2. *If the maximization of the likelihood function under the natural order restriction on the scale parameters is restricted to $\tilde{I}$, then for fixed α, the MLE $\widetilde{\lambda}_k^{(\alpha)}$ of λ_k for $k \in \tilde{I}$ is given by*

$$\widetilde{\lambda}_k^{(\alpha)} = \left[\min_{s \leq k} \max_{t \geq k} \frac{\sum_{r=s}^{t} D_r(\alpha)}{\sum_{r=s}^{t} n_r} \right]^{-1}, \quad s, t \in \tilde{I}.$$

Proof. See Appendix. □

Lemma 3. *Under the natural order restriction on the scale parameters, for fixed α, there exists a solution for the maximization of the likelihood function* (11) *iff the maximization is restricted to $\lambda_{i_1}, \lambda_{i_1+1}, \ldots, \lambda_{i_{|\tilde{I}|}}$. The solution $(\lambda_{i_1}^{*(\alpha)}, \lambda_{i_1+1}^{*(\alpha)}, \ldots, \lambda_{i_{|\tilde{I}|}}^{*(\alpha)})$ is then given by*

$$\lambda_k^{*(\alpha)} = \widetilde{\lambda}_{i(k)}^{(\alpha)}, \quad i_1 \leq k \leq i_{|\tilde{I}|}$$

where $i(k)$ is defined by $i(k) = min\{i \in \tilde{I} | i \geq k\}$.

Proof. See Appendix. □

We denote

$$\widetilde{\lambda}_i^{(\alpha)} = \begin{cases} \widetilde{\lambda}_i^{(\alpha)}, & \text{if } i \in \tilde{I} \\ \lambda_i^{*(\alpha)}, & \text{if } i \in \tilde{J} \end{cases} \tag{12}$$

Plugging in the order restricted estimates $\widetilde{\lambda}_{i_1}^{(\alpha)}, \widetilde{\lambda}_{i_1+1}^{(\alpha)}, \ldots, \widetilde{\lambda}_{i_{|\tilde{I}|}}^{(\alpha)}$ in (11), we obtain the profile log likelihood function of α, say $m_2(\alpha)$ where

$$m_2(\alpha) = \alpha^{\bar{n}_{m+1}} \prod_{j \in \tilde{I}} \big[\widetilde{\lambda}_j^{(\alpha)}\big]^{n_j} \exp\Big\{ - \sum_{j \in \tilde{I} \cup \tilde{J}} \widetilde{\lambda}_j^{(\alpha)} D_j(\alpha) \Big\} \Big\{ \prod_{j=1}^{\bar{n}_{m+1}} t_{j:n}^{\alpha-1} \Big\}.$$

Now, $m_2(\alpha)$ needs to be maximized to obtain $\widehat{\alpha}$, the MLE of α. It cannot be obtained in explicit form. We need to use some numerical method to maximize $m_2(\alpha)$. Since the estimation method is computational, the pseudocode is presented in the following Algorithm 1 to obtain the order restricted MLEs of the model parameters using the method of generalized isotonic regression.

Algorithm 1: Pseudocode to obtain the order restricted MLEs

Data: $\{t_{1:n} < \ldots < t_{\bar{n}_1:n} < \tau_1 < t_{\bar{n}_1+1:n} < \ldots < t_{\bar{n}_2:n} < \tau_2 < \ldots < \tau_m < t_{\bar{n}_m+1:n} < \ldots < \mathbf{t_{\bar{n}_{m+1}:n}} < \tau\}$

Input: $n, \tau_1, \tau_2, \ldots, \tau_m, \tau$; and $n_1, n_2, \ldots, n_{m+1}$.

if $n_i > 0 \; \forall \, i = 1, 2, \ldots, m+1$ **then**

(i) set $\tilde{I} = \{1, 2, \ldots, m+1\}$, calculate $\widehat{\alpha}$, the MLE of α by maximizing $m_1(\alpha)$.

`/* Profile likelihood maximization */`

(ii) the MLE of λ_k, $k \in \tilde{I}$ is then $\widetilde{\lambda}_k^{(\widehat{\alpha})} = \left[\min_{s \le k} \max_{t \ge k} \frac{\sum_{r=s}^{t} D_r(\widehat{\alpha})}{\sum_{r=s}^{t} n_r} \right]^{-1}, \quad s, t \in \tilde{I}.$

`/* Order restricted MLEs of the scale parameters */`

else if $n_i = 0$ *for some* $i = 2, 3, \ldots, m$ **then**

(i) set $\tilde{I} = \{i : n_i \ge 1\} \subset \{1, 2, \ldots, m+1\}$, and $\tilde{J} = \tilde{I} \backslash \{i_1, i_{|\tilde{I}|}\}$.

(ii) calculate $\widehat{\alpha}$, the MLE of α by maximizing $m_2(\alpha)$.

`/* Profile likelihood maximization */`

(ii) the MLE of λ_k, $k \in \tilde{I}$ is then $\widetilde{\lambda}_k^{(\widehat{\alpha})} = \left[\min_{s \le k} \max_{t \ge k} \frac{\sum_{r=s}^{t} D_r(\widehat{\alpha})}{\sum_{r=s}^{t} n_r} \right]^{-1}, \quad s, t \in \tilde{I}.$

(iv)In addition , we are able to find the MLEs for $\lambda_k \; \forall \, k \in \tilde{J}$. The MLE of λ_k, $k \in \tilde{J}$ is then given by

$$\lambda_k^{*(\widehat{\alpha})} = \widetilde{\lambda}_{i(k)}^{(\widehat{\alpha})}, \quad i_1 \le k \le i_{|\tilde{I}|}$$

where $i(k)$ is defined by $i(k) = min\{i \in \tilde{I} | i \ge k\}$.

`/* Order restricted MLEs of the scale parameters */`

else

Not possible to obtain the **order restricted** MLEs of the model parameters.

4 Confidence intervals

In this section we provide two different methods for construction of the confidence intervals (CIs). First, we provide asymptotic CIs assuming the asymptotic normality of the MLEs and then, the parametric bootstrap CIs.

4.1 Asymptotic Confidence Intervals

For illustration, we have considered three stress levels ($m = 3$). Assuming the asymptotic normality of the MLEs, we obtain the CIs for $\alpha, \lambda_1, \lambda_2$ and λ_3 using the observed Fisher information matrix I. This method is useful for its computational flexibility and provides good coverage probabilities (close to the nominal value) for large sample sizes. At first, explicit expressions for elements of the Fisher information matrices I are obtained. Details are provided in the Appendix (Section 7.4). The $100(1-\gamma)\%$ asymptotic confidence intervals for α, λ_1, λ_2, and λ_3 are, respectively

$$(\widehat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}),\ \ (\widehat{\lambda}_1 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}),\ \ (\widehat{\lambda}_2 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}),\ \ (\widehat{\lambda}_3 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{44}}),$$

where z_q is the upper $q-$ point of a standard normal distribution, V_{ij} is the $(i,j)th$ element of the inverse of the Fisher information matrix I. For simulation purposes, we have assumed that there is at least one failure in every stress level. However, non - availability of failures at the internal stress levels will indicate that the observed Fisher information matrix is singular. The method to construct the asymptotic CIs under such a scenario is discussed in analyzing the Fish data set 2 (See section 5.2.2).

4.2 Parametric Bootstrap confidence intervals

In this subsection, we consider the parametric bootstrap CIs of the unknown parameters. We discuss the algorithm in the following steps to construct Bias adjusted percentile (BCa) bootstrap CIs of $\alpha, \lambda_1, \lambda_2, \ldots, \lambda_{m+1}$.

1. For given $n,\ \tau,\ \tau_1, \tau_2, \ldots, \tau_m$, the **order restricted** MLEs of $(\alpha, \lambda_1, \lambda_2, \ldots, \lambda_{m+1})$ are computed based on the original sample $t = (t_{1:n}, \ldots, t_{n_1:n}, t_{(n_1+1):n}, \ldots, t_{\bar{n}_{m+1}:n})$, say $\widehat{\boldsymbol{\delta}} = (\widehat{\delta}_1, \widehat{\delta}_2, \ldots, \widehat{\delta}_{m+2}) = (\widehat{\alpha}, \widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_{m+1})$.

2. Generate a bootstrap Type-I censored sample say $t^* = (t^*_{1:n}, \ldots, t^*_{n_1:n}, t^*_{(n_1+1):n}, \ldots, t^*_{\bar{n}_{m+1}:n})$, from (7) with the parameters $(\widehat{\alpha}, \widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_{m+1})$.

3. Based on $n,\ \tau,\ \tau_1, \tau_2, \ldots, \tau_m$ and the bootstrap sample t^*, the **order restricted** MLEs of $\alpha, \lambda_1, \lambda_2, \ldots, \lambda_{m+1}$ are computed, say $(\widehat{\alpha}^{(1)}, \widehat{\lambda}_1^{(1)}, \widehat{\lambda}_2^{(1)}, \ldots, \widehat{\lambda}_{m+1}^{(1)})$.

4. Suppose $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{m+2}) = (\alpha, \lambda_1, \ldots, \lambda_{m+1})$ and $\widehat{\boldsymbol{\delta}}^{(i)} = (\widehat{\delta}_1^{(i)}, \widehat{\delta}_2^{(i)}, \ldots, \widehat{\delta}_{m+2}^{(i)})$ $= (\widehat{\alpha}^{(i)}, \widehat{\lambda}_1^{(i)}, \ldots, \widehat{\lambda}_{m+1}^{(i)})$. Repeat steps $2-3$, B times to obtain B sets of MLE of $\boldsymbol{\delta}$, say $\widehat{\boldsymbol{\delta}}^{(i)}$; $i = 1, 2, \ldots, B$.

5. Arrange $\widehat{\delta}_j^{(1)}, \widehat{\delta}_j^{(2)}, \ldots, \widehat{\delta}_j^{(B)}$ in ascending order and denote the ordered MLEs as $\widehat{\delta}_j^{[1]}, \widehat{\delta}_j^{[2]}, \ldots, \widehat{\delta}_j^{[B]}$; $j = 1, 2, \ldots, (m+2)$.

A two sided $100(1-\gamma)\%$ bootstrap confidence interval of δ_j is then given by

$$(\widehat{\delta}_j^{[\gamma_1 B]}, \widehat{\delta}_j^{[\gamma_2 B]})$$

where $\gamma_1 = \Phi\Big\{\widehat{z}_o + \frac{\widehat{z}_o + z_{1-\frac{\gamma}{2}}}{1 - a(\widehat{z}_o + z_{1-\frac{\gamma}{2}})}\Big\}$, $\gamma_2 = \Phi\Big\{\widehat{z}_o + \frac{\widehat{z}_o + z_{\frac{\gamma}{2}}}{1 - a(\widehat{z}_o + z_{\frac{\gamma}{2}})}\Big\}$, and $[x]$ denotes the largest integer less than or equal to x. Here, $\Phi(.)$ denotes the CDF of the standard normal distribution, z_γ is the upper $\gamma-$ point of the standard normal distribution, and

$$\widehat{z}_o = \Phi^{-1}\left\{\frac{\#\widehat{\delta}_j^{[k]} < \widehat{\delta}_j}{B}\right\}\ j = 1, 2, \ldots, (m+2),\ \ k = 1, 2, \ldots, B.$$

An estimate of the acceleration factor a is

$$\widehat{a} = \frac{\sum_{i=1}^{\bar{n}_{m+1}} [\widehat{\delta_j^{(.)}} - \widehat{\delta}_j^i]^3}{6\{\sum_{i=1}^{\bar{n}_{m+1}} [\widehat{\delta_j^{(.)}} - \widehat{\delta}_j^i]^2\}^{\frac{3}{2}}},$$

where $\widehat{\delta_j^i}$ is the MLE of δ_j based on the original sample with the $i-th$ observation deleted, and

$$\widehat{\delta_j^{(.)}} = \frac{\sum_{i=1}^{\bar{n}_{m+1}} \widehat{\delta_j^i}}{\bar{n}_{m+1}}$$

The performance of all these confidence intervals are evaluated through an extensive simulation study in Section 5.

5 Simulation studies and data analysis :

5.1 Simulation studies

In this section, we have performed an extensive simulation study to see the effectiveness of the proposed methods. For simulation purpose, a multiple **step-stress** test consisting of three stress levels is considered and the model parameters are set as $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$. Further, different sample sizes and time points for step acceleration $(n = 30,\ 40,\ 50,\ 60)$; $(\tau_1, \tau_2) = (0.4, 0.6),\ (0.4, 0.7),\ (0.5, 0.7)$ are considered for illustration. For each n, with these choices of (τ_1, τ_2), we have assumed that the experiment terminates at time points $\tau = 1.0$.

Table 1: MLEs and MSEs of model parameters (Order restricted case).
Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	τ	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE
30	1.0	0.4	0.6	2.4527	0.4943	1.2629	0.6803	2.1167	0.5750	3.3206	0.8557
		0.4	0.7	2.4555	0.4421	1.2457	0.5647	2.1009	0.3881	3.3671	1.1703
		0.5	0.7	2.5828	0.6111	1.1855	0.5142	2.1339	0.4651	3.3265	1.1099
40	1.0	0.4	0.6	2.5077	0.4769	1.2069	0.6116	2.1190	0.4737	3.2731	0.6734
		0.4	0.7	2.4806	0.4454	1.1736	0.5304	2.1044	0.3084	3.2970	0.9158
		0.5	0.7	2.5836	0.5312	1.1532	0.4466	2.1341	0.3511	3.2409	0.8422
50	1.0	0.4	0.6	2.5271	0.4255	1.1842	0.5838	2.1179	0.4183	3.2196	0.5261
		0.4	0.7	2.5257	0.4076	1.1756	0.5149	2.0944	0.2559	3.2472	0.7444
		0.5	0.7	2.5910	0.4504	1.1540	0.3866	2.1071	0.2793	3.1894	0.6833
60	1.0	0.4	0.6	2.5378	0.3744	1.1785	0.5392	2.1123	0.3497	3.1758	0.4014
		0.4	0.7	2.5191	0.3749	1.1297	0.4597	2.0862	0.2170	3.2012	0.6266
		0.5	0.7	2.5888	0.4014	1.1533	0.3647	2.0929	0.2376	3.1477	0.5245

Table 2: MLEs and MSEs of model parameters (Unrestricted case)
with proportion of cases where $(\widehat{\lambda}_1 < \widehat{\lambda}_2 < \widehat{\lambda}_3)$ will not hold.
Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3		
n	τ	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE	Proportion
30	1.0	0.4	0.6	2.6299	0.7365	1.7429	2.6454	2.3200	1.2109	3.2635	0.8985	0.4258
		0.4	0.7	2.6391	0.7107	1.7412	2.6292	2.2483	0.7132	3.2588	1.2624	0.4036
		0.5	0.7	2.7730	0.9993	1.5466	1.8871	2.2664	0.8286	3.2231	1.1925	0.4082
40	1.0	0.4	0.6	2.6518	0.7047	1.5914	2.2046	2.3000	1.0200	3.2302	0.7027	0.3694
		0.4	0.7	2.6270	0.6719	1.5606	2.1002	2.2241	0.5485	3.2144	0.9878	0.3560
		0.5	0.7	2.7302	0.8394	1.4247	1.4482	2.2418	0.6151	3.1632	0.9163	0.3390
50	1.0	0.4	0.6	2.6435	0.6120	1.4885	1.8114	2.2653	0.8430	3.1867	0.5509	0.3314
		0.4	0.7	2.6462	0.6056	1.4974	1.8136	2.1921	0.4370	3.1846	0.8036	0.3114
		0.5	0.7	2.7019	0.6772	1.3602	1.1378	2.1887	0.4604	3.1307	0.7410	0.2978
60	1.0	0.4	0.6	2.6351	0.5287	1.4362	1.5705	2.2340	0.6795	3.1481	0.4210	0.2946
		0.4	0.7	2.6161	0.5456	1.3858	1.5020	2.1700	0.3736	3.1510	0.6743	0.2772
		0.5	0.7	2.6729	0.5667	1.3096	0.9012	2.1513	0.3551	3.1036	0.5686	0.2602

We have computed both **order restricted** and unrestricted MLEs of the unknown parameters and the associated mean squared errors (MSEs). The results are reported in Table 1 and Table 2. It is to note that when the unrestricted MLEs are actually ordered, they exactly coincide with the **order restricted** estimates. From Table 1 and Table 2, it can be observed in all the cases that the **order restricted** estimates perform better compared to the unrestricted case with respect to MSEs. Again, in small sample scenario, the **order restricted** estimates are close to the true value giving an extra edge over the unrestricted estimation.

Table 3: CP and AL of 95% asymptotic CI based on 5000 simulations with $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	τ	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL
30	1.0	0.4	0.6	0.9632	3.7501	0.9296	5.5900	0.9482	4.0862	0.9800	3.8650
		0.4	0.7	0.9646	3.7789	0.9320	5.4255	0.9616	3.0552	0.9882	4.9194
		0.5	0.7	0.9612	3.8856	0.9162	4.0799	0.9654	3.1855	0.9834	4.6150
40	1.0	0.4	0.6	0.9608	3.3794	0.9114	4.6403	0.9516	3.4954	0.9746	3.2247
		0.4	0.7	0.9654	3.4101	0.9150	4.5331	0.9660	2.6315	0.9874	4.1516
		0.5	0.7	0.9622	3.4211	0.9116	3.4701	0.9644	2.7094	0.9832	3.8721
50	1.0	0.4	0.6	0.9600	3.0792	0.9046	4.1027	0.9550	3.1194	0.9724	2.8137
		0.4	0.7	0.9696	3.1208	0.9108	4.0398	0.9622	2.3561	0.9838	3.6103
		0.5	0.7	0.9632	3.0629	0.9132	3.0386	0.9674	2.3698	0.9774	3.3818
60	1.0	0.4	0.6	0.9612	2.8394	0.9032	3.6565	0.9524	2.8099	0.9682	2.5241
		0.4	0.7	0.9626	2.8542	0.9048	3.5977	0.9570	2.1205	0.9846	3.3022
		0.5	0.7	0.9624	2.8102	0.9202	2.7996	0.9646	2.1396	0.9758	3.0141

Table 4: CP and AL of 95% BCa bootstrap CI based on 5000 simulations with $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	τ	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL
30	1.0	0.4	0.6	0.9220	2.5332	0.8675	2.8542	0.9170	2.9538	0.9665	3.2136
		0.4	0.7	0.9215	2.3902	0.8850	2.4788	0.9320	2.4729	0.9520	3.8101
		0.5	0.7	0.9295	2.9521	0.8685	2.4624	0.9435	2.6563	0.9545	3.6663
40	1.0	0.4	0.6	0.9480	2.4625	0.8950	2.4941	0.9140	2.6010	0.9540	2.9207
		0.4	0.7	0.9350	2.3773	0.8910	2.3192	0.9280	2.1797	0.9540	3.3924
		0.5	0.7	0.9340	2.7619	0.8775	2.2469	0.9265	2.2696	0.9470	3.2526
50	1.0	0.4	0.6	0.9540	2.3752	0.9105	2.3364	0.9225	2.3659	0.9490	2.6229
		0.4	0.7	0.9290	2.3113	0.8960	2.2069	0.9265	1.9602	0.9520	3.1153
		0.5	0.7	0.9445	2.5858	0.9125	2.0922	0.9330	2.0261	0.9480	2.9784
60	1.0	0.4	0.6	0.9425	2.2742	0.9030	2.2863	0.9260	2.2050	0.9440	2.3468
		0.4	0.7	0.9295	2.2177	0.9005	2.1206	0.9365	1.7804	0.9380	2.9253
		0.5	0.7	0.9345	2.4039	0.9195	1.9858	0.9295	1.8649	0.9290	2.7588

For interval estimation purpose, we have focussed mainly on constructing the asymptotic and the parametric bootstrap CIs. The average lengths (ALs) and the associated **coverage probabilities**(CPs) for 95% level of confidence are reported in Table 3 and Table 4. It can be observed that both the confidence intervals perform quite well. As the sample size increases, the average lengths decrease. The average lengths in asymptotic CIs are slightly longer compared to that in parametric bootstrap CIs. In all the above cases, the results are based on 5000 replications.

5.2 Data Analysis

In this section we provide analyses of two real life multiple **step-stress** fish data sets obtained from Greven et al. [10].

5.2.1 Fish data set 1: At least one failure is present at every stress level

A sample of 14 fishes were swum at initial flow rate 15 cm/sec. The time at which a fish could not maintain its position is recorded as the failure time. **To ensure the early failure, the stress level was increased (flow rate by 5 cm/sec) at time 110, 130, and 150 minutes. The observed failure time data is**

83.50, 91.00, 91.00, 97.00, 107.00, 109.50, 114.00, 115.41, 128.61, 133.53, 138.58, 140.00, 152.08, 155.10.

There are four stress levels and number of failures at each stress level is 6, 3, 3, and 2 respectively. For computational purpose, we have subtracted 80 from each data points, divide them by 100 and then analyze the data with $n = 14$, $\tau_1 = 0.3$, $\tau_2 = 0.5$, $\tau = 0.7$. We assume that the above failure data follow WE distribution at each stress level with common shape parameter but different scale parameters. Under the above model assumptions the **order restricted** MLEs are obtained as $\widehat{\alpha} = 1.3121$, $\widehat{\lambda_1} = 2.5699$, $\widehat{\lambda_2} = 2.5699$, $\widehat{\lambda_3} = 4.3691$. Although we could not prove it theoretically, but it is observed that the profile likelihood function is an unimodal function, see Figure 2. Next to check the performance of the fitted model we calculate the Kolmogorov-Smirnov (KS) distance between the empirical distribution function (EDF) and the fitted distribution function (FDF) and also obtain the associated p-value. The KS distance and the associated p-value based on the **order restricted** MLEs are 0.2208 and 0.5468 respectively which indicate good fit of the given data. **The plot of the empirical v/s the fitted CDFs is shown in Figure 1. In Figure 2, the profile likelihood function of** α **is plotted. Clearly, the function is unimodal and the maximum value occurs at** $\alpha = 1.3121$.

As $\widehat{\lambda_1} = \widehat{\lambda_2}$, it is adequate to provide the CIs based on the assumption $\lambda_1 = \lambda_2 = \lambda_1^*$ (say). In this case, the resulting likelihood is

$$L^{\text{Data-1}}(\alpha, \lambda_1^*, \lambda_3|\text{Data-I}) \;\propto\; \alpha^{n_1^*+n_3}\lambda_1^{*n_1^*}\lambda_3^{n_3}\Big(\prod_{i=1}^{n_1^*+n_3} t_{i:n}^{\alpha-1}\Big)e^{-(\lambda_1^* D_1^*(\alpha)+\lambda_3 D_3(\alpha))} \tag{13}$$

where $n_1^* = 9,\; n_3 = 3$, and

$$D_1^*(\alpha) \;= \sum_{i=1}^{n_1^*} t_{i:n}^{\alpha} + (n - n_1^*)\tau_2^{\alpha}, \qquad D_3(\alpha) \;= \sum_{i=n_1^*+1}^{n_1^*+n_3} t_{i:n}^{\alpha} + (n - n_1^* - n_3)\tau^{\alpha} - (n - n_1^*)\tau_2^{\alpha}. \tag{14}$$

The explicit expressions for elements of the associated Fisher information matrices $I^{\text{Data-I}}$ are provided in the Appendix (Section 7.1.2). The $100(1-\gamma)\%$ asymptotic confidence intervals for $\alpha,\ \lambda_1^*,\ \lambda_3$ are respectively

$$(\widehat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{W_{11}}),\;\; (\widehat{\lambda}_1^* \pm z_{1-\frac{\gamma}{2}}\sqrt{W_{22}}),\;\; (\widehat{\lambda}_3 \pm z_{1-\frac{\gamma}{2}}\sqrt{W_{33}}),$$

where z_q is the upper $q-$ point of a standard normal distribution, W_{ij} is the $(i,j)th$ element of the inverse of the Fisher information matrix $I^{\text{Data-I}}$. **Asymptotic CIs and bootstrap CIs of model parameters are given in Table (5) and Table (6) respectively.**

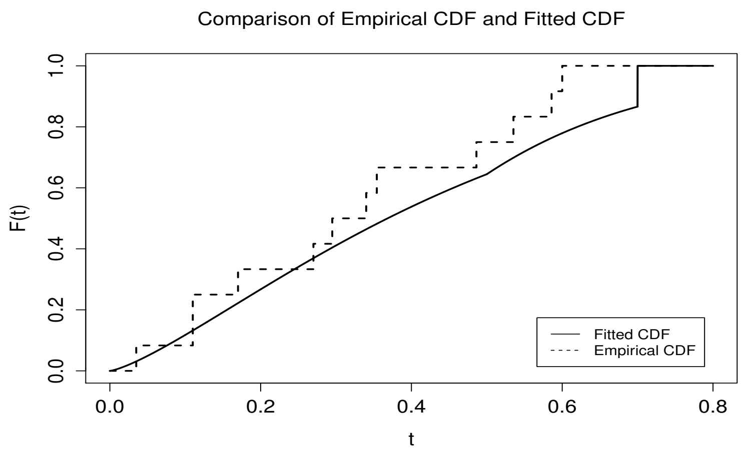

Figure 1: Plot of empirical and fitted CDFs of Fish data set 1.

In case the lower limit (LL) of the asymptotic CI of any model parameter

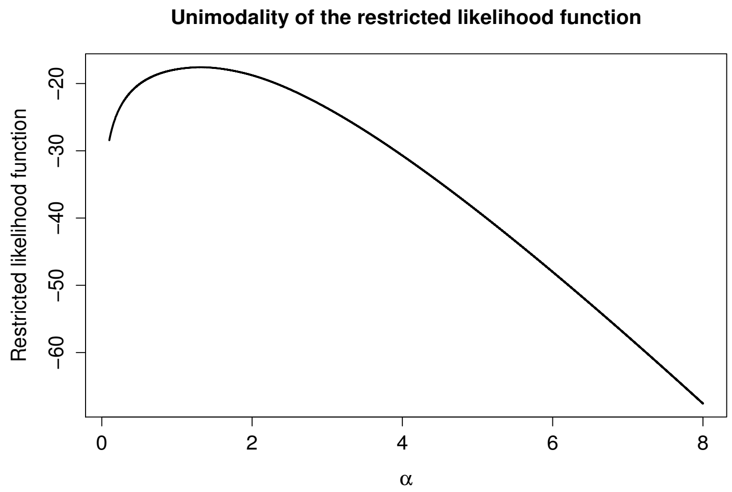

Figure 2: Unimodality of the restricted likelihood function in Fish data set 1

turns out to be negative, it is replaced by zero. It can be observed that in all the instances, the length of the asymptotic CIs is small compared to that of the bootstrap CIs.

Table 5: Asymptotic CI of parameters based on the Fish data set 1

	α		λ_1^*		λ_3	
Level	LL	UL	LL	UL	LL	UL
90%	0.6688	1.9554	0.5238	4.6160	0.1765	8.5618
95%	0.5455	2.0787	0.1317	5.0081	0	9.3652
99%	0.3046	2.3196	0	5.7744	0	10.9354

Table 6: Bootstrap CI of parameters based on the Fish data set 1

	α		λ_1^*		λ_3	
Level	LL	UL	LL	UL	LL	UL
90%	0.8407	2.2401	1.1921	5.5605	2.4146	12.4913
95%	0.7679	2.4859	1.0127	6.3831	2.1033	15.4114
99%	0.6489	3.0552	0.7344	8.6361	1.6602	21.6260

5.2.2 Fish data set 2: No failure in at least one of the internal stress levels

A sample of 15 fishes were swum at initial flow rate 15 cm/sec. The time at which a fish could not maintain its position is recorded as the failure time. To ensure the early failure, the stress level was increased (flow rate by 5 cm/sec) at time 110, 130, 150 and 170. The observed failure time data is

91.00, 93.00, 94.00, 98.20, 115.81, 116.00, 116.50, 117.25, 126.75, 127.50, 154.33, 159.50, 164.00, 184.14, 188.33.

There are five stress levels and number of failures at each stress level is 4, 6, 0, 3 and 2 respectively. Here, we consider the first four stress levels and the observations in the fifth stress level are assumed to be censored. For computational purpose, we have subtracted 80 from each data points, divide them by 150 and then analyze the data with $n = 15$, $\tau_1 = 0.20$, $\tau_2 = 0.33$, $\tau_3 = 0.46$, $\tau = 0.6$. We assume that the above failure data follow WE distribution at each stress level with common shape parameter but different scale parameters. Under the above model assumptions the **order restricted** MLEs are obtained as $\widehat{\alpha} = 1.2302$, $\widehat{\lambda_1} = 2.3028$, $\widehat{\lambda_2} = 6.268$, $\widehat{\lambda_3} = 6.268$, $\widehat{\lambda_4} = 6.268$. In this case also the unimodality of the profile likelihood function can be observed in Figure 4. Next to check the performance of the fitted model we calculate the **KS** distance between the **EDF** and the **FDF** and also obtain the associated p-value. The **KS** distance and the associated p-value based on the **order restricted** MLEs are 0.1741 and 0.7856 respectively which indicates good fit of the given data. **The plot of the empirical v/s the fitted CDFs is shown in Figure 3. In Figure 4, the profile likelihood function of α is plotted. Clearly, the function is unimodal and the maximum value occurs at $\alpha = 1.2302$.**

As $\widehat{\lambda_2} = \widehat{\lambda_3} = \widehat{\lambda_4}$, it is adequate to provide the CIs based on the assumption $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_2^*$ (say). In this case, the resulting likelihood is

$$L^{\text{Data-2}}(\alpha, \lambda_1, \lambda_2^* | \text{Data-2}) \quad \propto \quad \alpha^{n_1+n_2^*} \lambda_1^{n_1} \lambda_2^{*n_2^*} \Big(\prod_{i=1}^{n_1+n_2^*} t_{i:n}^{\alpha-1} \Big) e^{-(\lambda_1 D_1(\alpha) + \lambda_2^* D_2^*(\alpha))} \qquad (15)$$

where $n_1 = 4,\ n_2^* = 9$ and

$$D_1(\alpha) \ = \sum_{i=1}^{n_1} t_{i:n}^{\alpha} + (n - n_1)\tau_1^{\alpha}, \qquad D_2^*(\alpha) \ = \sum_{i=n_1+1}^{n_1+n_2^*} t_{i:n}^{\alpha} + (n - n_1 - n_2^*)\tau^{\alpha} - (n - n_1)\tau_1^{\alpha}.$$

The explicit expressions for elements of the associated Fisher information matrices $I^{\text{Data-2}}$ are provided in the Appendix (Section 7.1.3). The $100(1-\gamma)\%$ asymptotic confidence intervals for $\alpha,\ \lambda_1,\ \lambda_2^*$, are, respectively

$$(\widehat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{X_{11}}),\ \ (\widehat{\lambda}_1 \pm z_{1-\frac{\gamma}{2}}\sqrt{X_{22}}),\ \ (\widehat{\lambda}_2^* \pm z_{1-\frac{\gamma}{2}}\sqrt{X_{33}}),$$

where z_q is the upper $q-$ point of a standard normal distribution, X_{ij} is the $(i,j)th$ element of the inverse of the Fisher information matrix $I^{\text{Data-2}}$. **Asymptotic CIs and bootstrap CIs of model parameters are given in Table (7) and Table (8) respectively.**

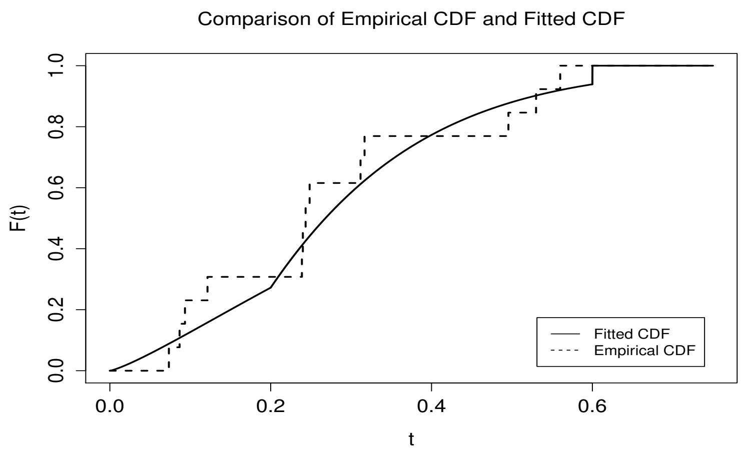

Figure 3: Plot of empirical and fitted CDFs of Fish data set 2.

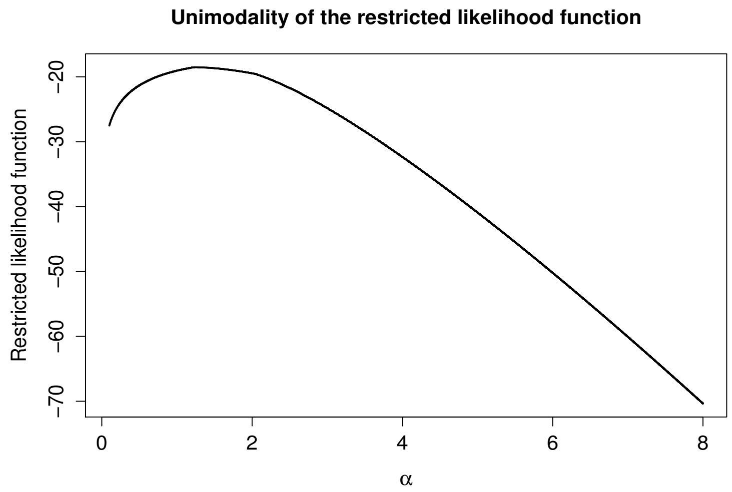

Figure 4: Unimodality of the restricted likelihood function in Fish data set 2

Table 7: Asymptotic CI of parameters based on the Fish data set 2

	α		λ_1		λ_2^*	
Level	LL	UL	LL	UL	LL	UL
90%	0	2.8077	0	8.7679	1.3017	11.2359
95%	0	3.1100	0	10.0069	0.3499	12.1878
99%	0	3.7008	0	12.4282	0	14.0481

Table 8: Bootstrap CI of parameters based on the Fish data set 2

	α		λ_1		λ_2^*	
Level	LL	UL	LL	UL	LL	UL
90%	0.6740	2.2648	0.6570	11.1307	3.9179	14.5997
95%	0.5912	2.4320	0.5035	13.5393	3.5025	16.6727
99%	0.4412	2.7301	0.3341	17.9613	2.7928	19.1963

In case the lower limit (LL) of the asymptotic CI of any model parameter turns out to be negative, it is replaced by zero. It can be observed that except for the shape parameter α, the length of the asymptotic CIs is small compared to that of the bootstrap CIs.

6 Conclusion:

In this paper, **order restricted** classical inference of the unknown model parameters is considered when the data come from a multiple **step-stress** model. The lifetime distribution at each stress level follows WE distribution and satisfies the TFRM assumptions. The method of generalized isotonic regression is applied to obtain the **order restricted** MLEs. For the fixed shape parameter, the **order restricted** MLEs of the scale parameters can be obtained in closed form. However, to obtain the MLE of the shape parameter, we need to solve a one dimensional optimization problem. Thus, the problem of solving the MLEs becomes simplified. We have performed some simulation studies and it is observed that the MLEs and the associated MSEs are satisfactory. For interval estimation purpose, the asymptotic CIs and the parametric bootstrap CIs are provided for the unknown parameters. Two real data sets have been analyzed and we have observed that the proposed model fits the data well. Our method can be extended to more general lifetime distributions also.

Another interesting work can be the consideration of a more general model referred to as accelerated degradation testing (ADT) framework discussed in Park and Padgett [22] and Liu et al. [17] which incorporates the observed measurements of degradation along with the actual failure times with multiple acceleration variables for inference. But it is not immediate how the order restricted inference can be incorporated in this set up. More work is needed along that direction.

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7 Appendix

7.1 Proof of Lemma 1

Before going to the main proof, we briefly introduce here the generalized isotonic regression problem, see Brunk et al. [7] in this respect.

Let A be a finite set, $g(.)$ be an arbitrary given function on A and $w(.)$ be a given positive weight function on A. Also, assume that $\Phi(.)$ is a convex function finite on an interval J, containing the range of the function $g(.)$ and infinite elsewhere. Further, $\phi(.)$ is the derivative of $\Phi(.)$ defined and finite on J. Clearly, $\phi(.)$ is non decreasing. Let us define for the numbers u and v, the function

$$\Delta(u,v) = \Delta_{\phi}(u,v) = \begin{cases} \Phi(u) - \Phi(v) - (u-v)\phi(v) & \text{if} \quad u, v \in J \\ \infty & \text{if} \quad v \in J,\ u \notin J. \end{cases} \tag{16}$$

Theorem 4. *If f(.) is isotonic on A and if the range of f(.) is in J, then g^*, the generalized isotonic regression of g with weights w minimizes $\sum_{x \in A} \Delta[g(x), f(x)]w(x)$ in the class of isotonic functions f(.) with range in J and is given by*

$$g^*(x_i) = \min_{t \geq i} \max_{s \leq i} \frac{\sum_{r=s}^{t} g(x_r)w(x_r)}{\sum_{r=s}^{t} w(x_r)}$$

Proof. See Chapter 1 of Brunk et al. [7] □

For fixed α, the problem here is to maximize the following likelihood function

$$L(\lambda_1, \lambda_2, \ldots, \lambda_{m+1} | \alpha, \mathcal{D}) = \alpha^{\bar{n}_{m+1}} \Big\{ \prod_{j=1}^{m+1} \lambda_j^{n_j} \Big\} \exp\Big\{ - \sum_{j=1}^{m+1} \lambda_j D_j(\alpha) \Big\} \Big\{ \prod_{i=1}^{\bar{n}_{m+1}} t_{i:n}^{\alpha-1} \Big\} \tag{17}$$

with respect to λ_i, $i \in I$ based on the order restriction $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1}$. Now

$$\max_{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1}} L(\lambda_1, \lambda_2, \ldots, \lambda_{m+1}|\alpha, \mathcal{D}) = \max_{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1}} \{\prod_{j \in I} \lambda_j^{n_j} e^{-\lambda_j D_j(\alpha)}\}$$

As $\log(x)$ is a monotonically increasing function, this is equivalent to

$$\begin{aligned}
\max_{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1}} \{\sum_{j \in I} [n_j \log \lambda_j - \lambda_j D_j(\alpha)]\} &= \max_{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1}} \{\sum_{j \in I} [-n_j \log(\frac{1}{\lambda_j}) - \frac{D_j(\alpha)}{\frac{1}{\lambda_j}}]\} \\
&= \max_{\frac{1}{\lambda_1} \geq \frac{1}{\lambda_2} \geq \ldots \geq \frac{1}{\lambda_{m+1}}} \sum_{j \in I} \{[-\log(\frac{1}{\lambda_j}) - \frac{D_j(\alpha)}{\frac{n_j}{\lambda_j}}] n_j\} \\
&= \min_{\frac{1}{\lambda_1} \geq \frac{1}{\lambda_2} \geq \ldots \geq \frac{1}{\lambda_{m+1}}} \sum_{j \in I} \{[\log(\frac{1}{\lambda_j}) + \frac{D_j(\alpha)}{\frac{n_j}{\lambda_j}}] n_j\}.
\end{aligned}$$

Considering the convex function $\Phi(u) = -\log u$, $\Delta_\phi(g(x), f(x)) = -\log g(x) + \log f(x) + \frac{g(x)-f(x)}{f(x)}$. Now the problem

$$\text{Minimize } \sum_{x \in A} \Delta_\Phi[g(x), f(x)] w(x)$$

subject to f is isotonic, is equivalent to

$$\text{Minimize } \sum_{x \in A} [\log f(x) + \frac{g(x)}{f(x)}] w(x)$$

subject to f is isotonic. Let $A = \{1, 2, \ldots, m+1\}$, $g(j) = \frac{D_j(\alpha)}{n_j}$, $f(j) = \frac{1}{\lambda_j}$, $w(j) = n_j$, $j \in I$. By Theorem (4), g^*, the generalized isotonic regression of g solves the problem and is given by

$$\frac{1}{\widetilde{\lambda}_k^{(\alpha)}} = \Big[\min_{s \leq k} \max_{t \geq k} \frac{\sum_{r=s}^{t} D_r(\alpha)}{\sum_{r=s}^{t} n_r}\Big], \;\; s, k, t \in I.$$

Thus, using the invariance property of the MLEs, the result follows.

7.2 Proof of Lemma 2

For fixed α, the problem here is to maximize the following likelihood function

$$L(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{|\tilde{I}|}}|\alpha, \mathcal{D}) = \alpha^{\tilde{n}_{m+1}} \prod_{j\in\tilde{I}} \lambda_j^{n_j} \exp\Big\{ - \sum_{j\in\tilde{I}} \lambda_j D_j(\alpha)\Big\}\Big\{ \prod_{i=1}^{\tilde{n}_{m+1}} t_{i:n}^{\alpha-1}\Big\}. \quad (18)$$

with respect to λ_i, $i \in \tilde{I}$ based on the order restriction $\lambda_{i_1} \leq \lambda_{i_2} \leq \ldots \lambda_{i_{|\tilde{I}|}}$. Now

$$\max_{\lambda_{i_1}\leq\lambda_{i_2}\leq\ldots\leq\lambda_{i_{|\tilde{I}|}}} L(\lambda_1, \lambda_2, \ldots, \lambda_{m+1}|\alpha, \mathcal{D}) = \max_{\lambda_{i_1}\leq\lambda_{i_2}\leq\ldots\leq\lambda_{i_{|\tilde{I}|}}} \{\prod_{j\in\tilde{I}} \lambda_j^{n_j} e^{-\lambda_j D_j(\alpha)}\}$$

As $\log(x)$ is a monotonically increasing function, this is equivalent to

$$\begin{aligned}
\max_{\lambda_{i_1}\leq\lambda_{i_2}\leq\ldots\leq\lambda_{i_{|\tilde{I}|}}} \{\sum_{j\in\tilde{I}} [n_j \log\lambda_j - \lambda_j D_j(\alpha)]\} &= \max_{\lambda_{i_1}\leq\lambda_{i_2}\leq\ldots\leq\lambda_{i_{|\tilde{I}|}}} \{\sum_{j\in\tilde{I}} [-n_j \log(\frac{1}{\lambda_j}) - \frac{D_j(\alpha)}{\frac{1}{\lambda_j}}]\} \\
&= \max_{\frac{1}{\lambda_{i_1}}\geq\frac{1}{\lambda_{i_2}}\geq\ldots\geq\frac{1}{\lambda_{i_{|\tilde{I}|}}}} \sum_{j\in\tilde{I}} \{[-\log(\frac{1}{\lambda_j}) - \frac{D_j(\alpha)}{\frac{n_j}{\lambda_j}}]n_j\} \\
&= \min_{\frac{1}{\lambda_{i_1}}\geq\frac{1}{\lambda_{i_2}}\geq\ldots\geq\frac{1}{\lambda_{i_{|\tilde{I}|}}}} \sum_{j\in\tilde{I}} \{[\log(\frac{1}{\lambda_j}) + \frac{D_j(\alpha)}{\frac{n_j}{\lambda_j}}]n_j\}.
\end{aligned}$$

Considering the convex function $\Phi(u) = -\log u$, $\Delta_\phi(g(x), f(x)) = -\log g(x) + \log f(x) + \frac{g(x)-f(x)}{f(x)}$. Now the problem

$$\text{Minimize } \sum_{x\in A} \Delta_\Phi[g(x), f(x)]w(x)$$

subject to f is isotonic, is equivalent to

$$\text{Minimize } \sum_{x\in A} [\log f(x) + \frac{g(x)}{f(x)}]w(x)$$

subject to f is isotonic. Let $A = \{i_1, i_2, \ldots, i_{|\tilde{I}|}\}$, $g(j) = \frac{D_j(\alpha)}{n_j}$, $f(j) = \frac{1}{\lambda_j}$, $w(j) = n_j$, $j \in \tilde{I}$. By Theorem (4), g^*, the generalized isotonic regression of g solves the problem and is given by

$$\frac{1}{\widetilde{\lambda}_k^{(\alpha)}} = \Big[\min_{s\le k}\max_{t\ge k}\frac{\sum_{r=s}^{t} D_r(\alpha)}{\sum_{r=s}^{t} n_r}\Big],\ s,k,t\in\tilde{I}$$

Thus, using the invariance property of the MLEs, the result follows.

7.3 Proof of Lemma 3

For the fixed shape parameter α, the likelihood function $L(\lambda_1,\lambda_2\ldots,\lambda_{m+1}|\alpha,\mathcal{D})$ can be expressed as

$$\underbrace{\exp\Big\{-\sum_{j=1}^{i_1-1}\lambda_j D_j(\alpha)\Big\}}_{A_1}\times\underbrace{\Big\{\prod_{j=i_1}^{\max(\tilde{I})}\lambda_j^{n_j}\Big\}\times\exp\Big\{-\sum_{j=i_1}^{\max(\tilde{I})}\lambda_j D_j(\alpha)\Big\}}_{A_2}$$
$$\times\underbrace{\exp\Big\{-\sum_{i\notin\tilde{I},i_1\le i\le|\tilde{I}|}\lambda_i D_i(\alpha)\Big\}}_{A_3} \qquad (19)$$

For fixed $\lambda_i, i\notin\tilde{I}$, A_2 in (19) is maximized at $\lambda_j^{*(\alpha)}=\widetilde{\lambda}_j^{(\alpha)}, j\in\tilde{I}$. Note that, $\lambda_j^{*(\alpha)}, j\in\tilde{I}$ does not depend on $\lambda_i, i\in\tilde{I}$. Now, for the fixed shape parameter α, $\exp\{-\lambda_j D_j(\alpha)\}$ is a decreasing function of λ_j. Hence, given the natural order restriction on the scale parameters i.e $\lambda_1\le\ldots\le\lambda_{m+1}$, A_3 is maximized at $\lambda_k^{*(\alpha)}=\widetilde{\lambda}_{i(k)}^{(\alpha)}, k\notin\tilde{I}, i_1\le k\le|\tilde{I}|$.

7.4 Elements of Fisher Information Matrix

The elements of the Fisher information matrix

$$I(\alpha,\lambda_1,\lambda_2,\lambda_3)=\begin{bmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_1} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_2} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_3}\\ -\frac{\partial^2 l}{\partial\lambda_1\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_1^2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_3}\\ -\frac{\partial^2 l}{\partial\lambda_2\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_2^2} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_3}\\ -\frac{\partial^2 l}{\partial\lambda_3\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_3\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_3\partial\lambda_2} & -\frac{\partial^2 l}{\partial\lambda_3^2}\end{bmatrix}$$

can be expressed as follows.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \alpha^2} &= -\frac{n_1+n_2+n_3}{\alpha^2} - \lambda_1 D_1''(\alpha) - \lambda_2 D_2''(\alpha) - \lambda_3 D_3''(\alpha), \\
\frac{\partial^2 l}{\partial \lambda_1^2} &= -\frac{n_1}{\lambda_1^2}, \frac{\partial^2 l}{\partial \lambda_2^2} = -\frac{n_2}{\lambda_2^2}, \frac{\partial^2 l}{\partial \lambda_3^2} = -\frac{n_3}{\lambda_3^2}, \\
\frac{\partial^2 l}{\partial \alpha \partial \lambda_1} &= \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} = -D_1'(\alpha), \\
\frac{\partial^2 l}{\partial \alpha \partial \lambda_2} &= \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} = -D_2'(\alpha), \\
\frac{\partial^2 l}{\partial \alpha \partial \lambda_3} &= \frac{\partial^2 l}{\partial \lambda_3 \partial \alpha} = -D_3'(\alpha), \\
\frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} &= \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1} = \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_3} = \frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_2} = \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_3} = \frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_1} = 0.
\end{aligned}
$$

where

$$
\begin{aligned}
D_1'(\alpha) &= \sum_{i=1}^{\overline{n}_1} t_{i:n}^{\alpha} \ln t_{i:n} + (n-\overline{n}_1)\tau_1^{\alpha} \ln \tau_1, \\
D_2'(\alpha) &= \sum_{i=\overline{n}_1+1}^{\overline{n}_2} t_{i:n}^{\alpha} \ln t_{i:n} + (n-\overline{n}_2)\tau_2^{\alpha} \ln \tau_2 - (n-\overline{n}_1)\tau_1^{\alpha} \ln \tau_1, \\
D_3'(\alpha) &= \sum_{i=\overline{n}_2+1}^{\overline{n}_3} t_{i:n}^{\alpha} \ln t_{i:n} + (n-\overline{n}_3)\tau^{\alpha} \ln \tau - (n-\overline{n}_2)\tau_2^{\alpha} \ln \tau_2, \\
D_1''(\alpha) &= \sum_{i=1}^{\overline{n}_1} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n-\overline{n}_1)\tau_1^{\alpha} (\ln \tau_1)^2, \\
D_2''(\alpha) &= \sum_{i=\overline{n}_1+1}^{\overline{n}_2} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n-\overline{n}_2)\tau_2^{\alpha} (\ln \tau_2)^2 - (n-\overline{n}_1)\tau_1^{\alpha} (\ln \tau_1)^2, \\
D_3''(\alpha) &= \sum_{i=\overline{n}_2+1}^{\overline{n}_3} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n-\overline{n}_3)\tau^{\alpha} (\ln \tau)^2 - (n-\overline{n}_2)\tau_2^{\alpha} (\ln \tau_2)^2.
\end{aligned}
$$

7.5 Fish Data Set 1: Elements of Fisher Information Matrix

Let us denote $l^{\text{Data-1}} = \ln L^{\text{Data-1}}(\alpha, \lambda_1^*, \lambda_3 | \text{Data-I})$. The elements of the Fisher information matrix

$$I^{\text{Data-I}}(\alpha, \lambda_1^*, \lambda_3) = \begin{bmatrix} -\frac{\partial^2 l^{\text{Data-1}}}{\partial \alpha^2} & -\frac{\partial^2 l^{\text{Data-1}}}{\partial \alpha \partial \lambda_1^*} & -\frac{\partial^2 l^{\text{Data-1}}}{\partial \alpha \partial \lambda_3} \\ -\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_1^* \partial \alpha} & -\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_1^{*2}} & -\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_1^* \partial \lambda_3} \\ -\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_3 \partial \alpha} & -\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_3 \partial \lambda_1^*} & -\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_3^2} \end{bmatrix}$$

can be expressed as follows.

$$\begin{aligned}
\frac{\partial^2 l^{\text{Data-1}}}{\partial \alpha^2} &= -\frac{n}{\alpha^2} - \lambda_1^* D_1^{*\prime\prime}(\alpha) - \lambda_3 D_3''(\alpha), \\
\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_1^{*2}} &= -\frac{n_1^*}{\lambda_1^{*2}}, \ \frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_3^2} = -\frac{n_3}{\lambda_3^2}, \\
\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_1^* \partial \alpha} &= \frac{\partial^2 l^{\text{Data-1}}}{\partial \alpha \partial \lambda_1^*} = -D_1^{*\prime}(\alpha), \\
\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_3 \partial \alpha} &= \frac{\partial^2 l^{\text{Data-1}}}{\partial \alpha \partial \lambda_3} = -D_3'(\alpha), \\
\frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_1^* \partial \lambda_3} &= \frac{\partial^2 l^{\text{Data-1}}}{\partial \lambda_3 \partial \lambda_1^*} = 0.
\end{aligned}$$

where

$$\begin{aligned}
D_1^*(\alpha) &= \sum_{i=1}^{n_1^*} t_{i:n}^\alpha + (n - n_1^*)\tau_2^\alpha, \\
D_3(\alpha) &= \sum_{i=n_1^*+1}^{n_1^*+n_3} t_{i:n}^\alpha + (n - n_1^* - n_3)\tau^\alpha - (n - n_1^*)\tau_2^\alpha, \\
D_1^{*\prime}(\alpha) &= \sum_{i=1}^{n_1^*} t_{i:n}^\alpha \ln t_{i:n} + (n - n_1^*)\tau_2^\alpha \ln \tau_2, \\
D_3'(\alpha) &= \sum_{i=n_1^*+1}^{n_1^*+n_3} t_{i:n}^\alpha \ln t_{i:n} + (n - n_1^* - n_3)\tau^\alpha \ln \tau - (n - n_1^*)\tau_2^\alpha \ln \tau_2, \\
D_1^{*\prime\prime}(\alpha) &= \sum_{i=1}^{n_1^*} t_{i:n}^\alpha (\ln t_{i:n})^2 + (n - n_1^*)\tau_2^\alpha (\ln \tau_2)^2, \\
D_3''(\alpha) &= \sum_{i=n_1^*+1}^{n_1^*+n_3} t_{i:n}^\alpha (\ln t_{i:n})^2 + (n - n_1^* - n_3)\tau^\alpha (\ln \tau)^2 - (n - n_1^*)\tau_2^\alpha (\ln \tau_2)^2.
\end{aligned}$$

7.6 Fish Data Set 2: Elements of Fisher Information Matrix

Let us denote $l^{\text{Data-2}} = \ln L^{\text{Data-2}}(\alpha, \lambda_1, \lambda_2^*|\text{Data-2})$. The elements of the Fisher information matrix

$$I^{\text{Data-2}}(\alpha, \lambda_1, \lambda_2^*) = \begin{bmatrix} -\frac{\partial^2 l^{\text{Data-2}}}{\partial \alpha^2} & -\frac{\partial^2 l^{\text{Data-2}}}{\partial \alpha \partial \lambda_1} & -\frac{\partial^2 l^{\text{Data-2}}}{\partial \alpha \partial \lambda_2^*} \\ -\frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_1 \partial \alpha} & -\frac{\partial^2 l^{\text{Data-2}}}{\partial {\lambda_1}^2} & -\frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_1 \partial \lambda_2^*} \\ -\frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_2^* \partial \alpha} & -\frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_2^* \partial \lambda_1} & -\frac{\partial^2 l^{\text{Data-2}}}{\partial {\lambda_2^*}^2} \end{bmatrix}$$

can be expressed as follows.

$$\begin{aligned} \frac{\partial^2 l^{\text{Data-2}}}{\partial \alpha^2} &= -\frac{n}{\alpha^2} - \lambda_1 {D_1}''(\alpha) - \lambda_2^* D_2^{*\prime\prime}(\alpha), \\ \frac{\partial^2 l^{\text{Data-2}}}{\partial {\lambda_1}^2} &= -\frac{n_1}{{\lambda_1}^2}, \ \frac{\partial^2 l^{\text{Data-2}}}{\partial {\lambda_2^*}^2} = -\frac{n_2^*}{{\lambda_2^*}^2}, \\ \frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_1 \partial \alpha} &= \frac{\partial^2 l^{\text{Data-2}}}{\partial \alpha \partial \lambda_1} = -{D_1}'(\alpha), \\ \frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_2^* \partial \alpha} &= \frac{\partial^2 l^{\text{Data-2}}}{\partial \alpha \partial \lambda_2^*} = -D_2^{*\prime}(\alpha), \\ \frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_1 \partial \lambda_2^*} &= \frac{\partial^2 l^{\text{Data-2}}}{\partial \lambda_2^* \partial \lambda_1} = 0. \end{aligned}$$

where

$$\begin{aligned} D_1(\alpha) &= \sum_{i=1}^{n_1} t_{i:n}^{\alpha} + (n - n_1)\tau_1^{\alpha}, \\ D_2^*(\alpha) &= \sum_{i=n_1+1}^{n_1+n_2^*} t_{i:n}^{\alpha} + (n - n_1 - n_2^*)\tau^{\alpha} - (n - n_1)\tau_1^{\alpha}, \\ {D_1}'(\alpha) &= \sum_{i=1}^{n_1} t_{i:n}^{\alpha} \ln t_{i:n} + (n - n_1)\tau_1^{\alpha} \ln \tau_1, \\ D_2^{*\prime}(\alpha) &= \sum_{i=n_1+1}^{n_1+n_2^*} t_{i:n}^{\alpha} \ln t_{i:n} + (n - n_1 - n_2^*)\tau^{\alpha} \ln \tau - (n - n_1)\tau_1^{\alpha} \ln \tau_1, \\ {D_1}''(\alpha) &= \sum_{i=1}^{n_1} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n - n_1)\tau_1^{\alpha} (\ln \tau_1)^2, \\ D_2^{*\prime\prime}(\alpha) &= \sum_{i=n_1+1}^{n_1+n_2^*} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n - n_1 - n_2^*)\tau^{\alpha} (\ln \tau)^2 - (n - n_1)\tau_1^{\alpha} (\ln \tau_1)^2. \end{aligned}$$