On Weighted Bivariate Geometric Distribution

H. Najarzadegan [*], M. H. Alamatsaz [†], I. Kazemi [‡] Debasis Kundu [§]

Abstract

Discrete bivariate distributions play important roles for modeling bivariate lifetime count data. Recently a weighted geometric distribution has been introduced in the literature, and it is observed that it is more flexible than the classical geometric distribution. In this article we develop a new weighted bivariate geometric distribution with univariate weighted geometric distribution as the marginals. It is observed that the proposed weighted bivariate geometric distribution is more flexible than the classical bivariate geometric distribution and the later can be obtained as a special case of the proposed model. We develop different properties of the proposed distribution. An interesting characterization also has been provided. We further develop both the classical and Bayesian inference of the unknown parameters. Extensive Monte Carlo simulations have been conducted to see the performances of the different methods. Finally the analysis of a data set has been presented for illustrative purposes.

Keywords: Bivariate probability mass function, Estimation, Simulation, Moment, Moment generating function, Renyi entropy.

AMS 2000 Subject Classification: Primary: 62F10, Secondary: 62H10.

1 Introduction

Azzalini [4] introduced a very powerful tool to incorporate an extra parameter to a normal distribution and named the new distribution as the skew-normal distribution. Since then, several new weighted distributions have been introduced in

[*]Department of Statistics, University of Isfahan, Isfahan, Iran, e-mail: hnajarzadegan@yahoo.com

[†]Department of Statistics, University of Isfahan, Isfahan, Iran, Corresponding author: e-mail: alamatho@sci.ui.ac.ir (mh_alamatsaz@yahoo.com)

[‡]Department of Statistics, University of Isfahan, Isfahan, Iran, e-mail: ikazemi@sci.ui.ac.ir

[§]Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India, e-mail: kundu@iitk.ac.in

the literature based on Azzalini [4]'s method, see for example the monograph on the skew-elliptical distributions by Genton [8], weighted exponential distribution by Gupta and Kundu [9], two-parameter weighted exponential (TWE) distribution by Shakhatreh [19] and generalized weighted exponential (GWE) distribution by Kharazmi et al. [11], and the references cited therein. Although an extensive amount of work has been done on continuous univariate distributions, the amount of work on the discrete distribution is quite insignificant. Recently, Bhati and Joshi [6] introduced a weighted geometric (WGE) distribution and Najarzadegan and Alamatsaz [16] provided a generalization of the weighted geometric distribution (GWG) based on the method of Azzalini [4].

A random variable X is said to have a WGE distribution with parameters α and λ, denoted by $WGE(\alpha,\lambda)$, if it has the following probability mass function (PMF)

$$P(X=x)=\frac{(1-p)(1-p^{\lambda+1})}{(1-p^{\lambda})}p^{x-1}(1-p^{\lambda x}),\quad x\in\mathcal{N}_1=\{1,2,\ldots\},$$

where $\lambda>0$ and $0<p<1$. It is observed that the WGE provides a better fit than the classic geometric distribution and it has certain advantages over a discrete analogue of the generalized exponential distribution proposed by Nekoukhou et al. [17]. It is further observed that when $\lambda\in\mathcal{N}_1$, then the WGE distribution can be constructed by using Azzalini [4]'s method on geometric (GE) distributions as follows.

Suppose that X_1 and X_2 are two independent and identically distributed geometric random variabes with mean $\frac{1}{1-p}$, then for $\lambda\in\mathcal{N}_1=\{1,2,...\}$ we have

$$X\overset{d}{=}X_1\big|\big(X_2<\lambda X_1\big).$$

where $\overset{d}{=}$ means equality in distribution. It can be easily shown that the PMF of the WGE distribution is unimodal for all values $0<\lambda<\infty$ and $0<p<1$. The corresponding cumulative distribution function (CDF) and the hazard rate function (HRF) of the WGE distribution are

$$F(x)=1-\frac{\big\{1-p^{\lambda+1}-p^{\lambda([x]+1)}(1-p)\big\}}{1-p^{\lambda}}p^{[x]}\qquad x\geq 1,$$

where [.] is the integer function and

$$h(x)=\frac{(1-p)(1-p^{\lambda+1})p^{x-1}(1-p^{\lambda x})}{\big\{1-p^{\lambda+1}-p^{\lambda(x+1)}(1-p)\big\}p^{x}},\qquad x=1,2,\ldots,$$

respectively. As we can see, the CDF and HRF have explicit forms, hence it can be used quite conveniently to analyze discrete univariate data. The discrete WGE distribution has a non-decreasing HRF for all λ and p. It has also been observed that the shapes of the PMF of the WGE distribution are very similar to those of the PMFs of the discrete generalized exponential, discrete Weibull and discrete gamma distributions. The moment generating function (MGF) of $WG(\lambda, p)$ can be obtained as

$$M_X(t) = \frac{(1-p)(1-p^{\lambda+1})e^t}{(1-pe^t)(1-p^{\lambda+1}e^t)}, \qquad t \in (-\infty, \infty). \tag{1}$$

The mean and variance of a WGE distribution are given by

$$E(X) = \frac{1-p^{\lambda+2}}{(1-p)(1-p^{\lambda+1})} \quad \text{and} \quad Var(X) = \frac{p}{(1-p)^2} + \frac{p^{\lambda+1}}{(1-p^{\lambda+1})^2}.$$

Moreover, suppose that Z and W are two independent geometric random variables with means $\frac{1}{1-p}$ and $\frac{p^{\lambda+1}}{1-p^{\lambda+1}}$, respectively, then from the MGF (1), it can be easily obtained that

$$Z + W \sim WG(\lambda, p). \tag{2}$$

The above characterization (2) can be used quite conveniently to generate random sample from a WGE distribution. It can be used for other purposes also. For example, from (2) it easily follows that if $X_1, \ldots, X_n$ are independent identically distributed (i.i.d.) WGE random variables then $\sum_{i=1}^{n} X_i$ can be written as the sum of two independent negative binomial random variables. For other mathematical properties and for different estimation procedures, one is referred to Bhati and Joshi [6].

Although, an extensive amount of work has been done in generalizing Azzalini's method for different univariate distributions, and for multivariate symmetric distributions, not that much of work has been done for multivariate non-negative random variables. Recently, Al-Mutairi et al. [1, 2] developed bivariate/ multivariate distributions with weighted exponential and weighted Weibull marginals, respectively. Jamalizadeh and Kundu [10] developed new weighted MarshallOlkin bivariate exponential (WMOBE) distribution using the idea of Azzalini [4]. Interestingly, no work has been done in developing bivariate/ multivariate discrete models along this line.

In real situations, sometimes we cannot measure the life length of a device on a continuous scale. It has been observed that Azzalini's method provides a powerful tool in developing very flexible univariate and multivariate continuous distributions. However, there seems to have been not any attempt to construct discrete bivariate distributions using the idea of Azzalini [4]. This paper is an attempt towards that direction. The aim of this article is to propose a new discrete weighted bivariate geometric (WBGE) distribution with WGE marginals, by implementing an idea that is similar to Azzalini's method which was introduced by Jamalizadeh and Kundu [10]. The WBGE distribution can also be constructed as a hidden truncation model which was proposed by Arnold and Beaver [3]. In fact, this interpretation is one of the basic motivations of our proposed model. It is observed that the proposed WBGE has four parameters, and due to presence of one extra parameter, the joint PMF of a WBGE distribution is more flexible than the joint PMF of a bivariate geometric (BGE) distribution. Moreover, the BGE can be obtained as a limiting case of the WBGE distribution. We have derived several properties of the proposed model and provided several characteristics. We have also developed both classical and Bayesian inference of the unknown parameters. Extensive simulation experiments have been performed to see the performances of the different methods and one data analysis has been performed for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we define the proposed WBGE distribution. The PMF, CDF and some basic statistical properties of the WBGE distribution are provided in this section. In Section 3, we drive some other important properties and characteristics of a WBGE distribution. The maximum likelihood estimators (MLEs) and the Bayes estimators of the model parameters are discussed in Section 4. In Section 5, we compare the performances of the different estimators based on extensive Monte Carlo simulations. In Section 6, we provide the analysis of a data set for illustrative purposes, and finally we conclude the paper in Section 7.

2 Definition and Basic Properties

The BGE distribution was originally introduced by Basu and Dhar [5]. A bivariate random vector (Y_1, Y_2) is said to have a BGE distribution, denoted by $BGE(p_1, p_2, p_3)$, if it has the joint PMF

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} (1-p_1)(1-p_2p_3)(p_2p_3)^{y_2-1}p_1^{y_1-1}, & \text{if } y_2 > y_1, \\ (1-p_2)(1-p_1p_3)(p_1p_3)^{y_1-1}p_2^{y_2-1}, & \text{if } y_1 > y_2, \\ (1-p_1p_3-p_2p_3+p_1p_2p_3)(p_1p_2p_3)^{y-1}, & \text{if } y_1 = y_2 = y \end{cases} \quad (3)$$

where $y_1, y_2 \in \mathcal{N}_1$ and $0 < p_1, p_2, p_3 < 1$. It may be mentioned that Basu and Dhar [5] obtained the BGE distribution as follows:

$$(Y_1, Y_2) \stackrel{d}{=} (min(V_1, V_3), min(V_2, V_3))$$

where V_1, V_2 and V_3 are independent GE random variables with means $\frac{1}{1-p_1}$, $\frac{1}{1-p_2}$ and $\frac{1}{1-p_3}$, respectively. This is popularly known as the trivariate reduction technique and it was used by Marshall and Olkin [15] to construct the MOBE distribution.

Now, following the idea proposed by Jamalizadeh and Kundu [10], we would like to define the WBGE distribution as follows. [13]

Definition 2.1. *Suppose (Y_1, Y_2) is a BGE random vector with joint PMF (3) and Y is a $GE(1-p_4)$ random variable independent of (Y_1, Y_2) with the PMF*

$$P(Y = y) = (1-p_4)p_4^{y-1}, \qquad y = 1, 2, 3, \ldots \tag{4}$$

where $0 < p_4 < 1$. Then, we say that a random vector (X_1, X_2) has a WBGE distribution if

$$(X_1, X_2) \stackrel{d}{=} (Y_1, Y_2)\big| Y < \min\{Y_1, Y_2\}. \tag{5}$$

Hereafter, this is denoted by $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$.

Now, we obtain the joint PMF of a WBGE distribution in the following theorem.

Theorem 2.1. *Suppose $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$, then the PMF of (X_1, X_2) is given by*

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} K(1-p_1)(1-p_2p_3)(p_2p_3)^{x_2-1}p_1^{x_1-1}(1-p_4^{x_1}), & if\ x_2 > x_1, \\ K(1-p_2)(1-p_1p_3)(p_1p_3)^{x_1-1}p_2^{x_2-1}(1-p_4^{x_2}), & if\ x_1 > x_2, \\ K(1-p_1p_3-p_2p_3+p_1p_2p_3)(p_1p_2p_3)^{x-1}(1-p_4^{x}), & if\ x_1 = x_2 = x \end{cases} \tag{6}$$

where $K^{-1} = \frac{1-p_4}{1-p_1p_2p_3p_4}$ and $x_1, x_2 \in N_1$.

Proof: See in the Appendix.

Here in following theorem we shall establish that the WBGE distribution can be constructed as a hidden truncation model.

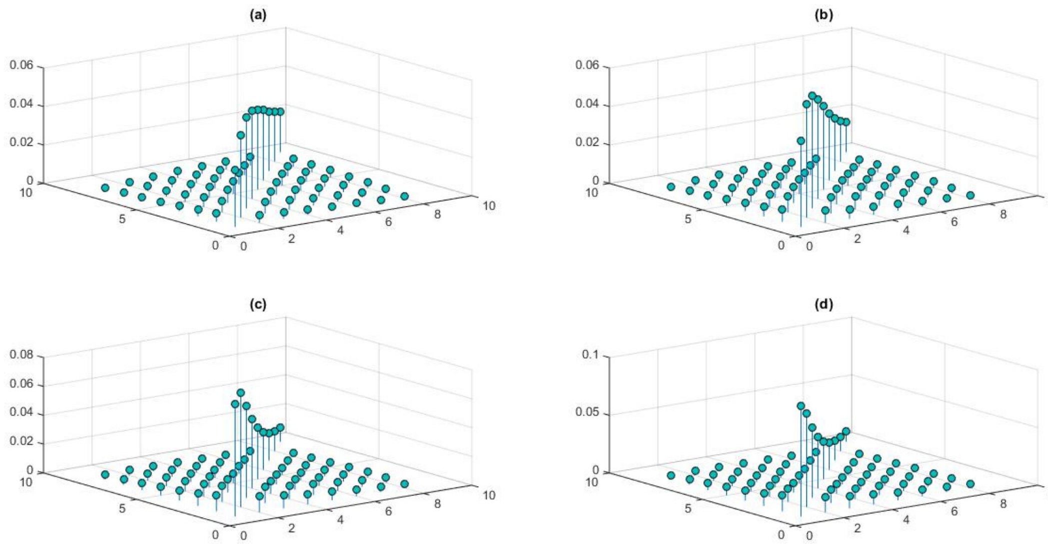

Figure 1: The plot of the joint PMF (6) when $p_1 = p_2 = p_3 = 0.9$ and (a) $p_4 = 0.9$ (b) $p_4 = 0.9^4$ (c) $p_4 = 0.9^9$ (d) $p_4 = 0.9^{17}$

Theorem 2.2. *Let V_1, V_2 and V_3 be three dependent random variables with the following joint pmf*

$$f_{V_1,V_2,V_3}(v_1, v_2, v_3) = \begin{cases} (1-p_4)(1-p_1)(1-p_2p_3)(p_2p_3)^{v_2-1}p_1^{v_1-1}p_4^{v_3v_1-1}, & \text{if } v_2 > v_1, \\ (1-p_4)(1-p_2)(1-p_1p_3)(p_1p_3)^{v_1-1}p_2^{v_2-1}p_4^{v_3v_2-1}, & \text{if } v_1 > v_2, \\ (1-p_4)(1-p_1p_3-p_2p_3+p_1p_2p_3)(p_1p_2p_3)^{v-1}p_4^{v_3v-1}, & \text{if } v_1 = v_2 = v \end{cases} \tag{7}$$

where $v_1, v_2, v_3 \in N_1$. Then the conditional random variables $\left((V_1, V_2)\middle|V_3 < 1\right)$ has WBGE distribution.

Proof: The proof follows simply using conditional probability for $\left((V_1, V_2)\middle|V_3 < 1\right)$. □

In Figure 1 we provide the plots of the joint PMF (6) for different values of p_4, for fixed $p_1 = p_2 = p_3 = .9$. It is observed that as p_4 decreases, the mode of the distribution moves towards the origin. In particular, when $p_4 \to 0$, the joint PMF (6) reduces to the joint PMF (3). Indeed, we have the following relation between the joint PMF (6) and the joint PMF (3)

$$f_{X_1,X_2}(x_1, x_2) = K(1 - p_4^{\min(x_1,x_2)})f_{Y_1,Y_2}(x_1, x_2). \tag{8}$$

In fact, $K(1-p_4^{\min(x_1,x_2)})$ is the weight function to construct a WBGE distribution from a BGE distribution. Now, in the next two theorems we obtain the CDF of a WBGE distribution.

Theorem 2.3. *If $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$, then the joint CDF of (X_1, X_2) is given by*

$$F(x_1,x_2) = P(X_1 \le x_1, X_2 \le x_2) = \begin{cases} KF_1(x_1,x_2), & \text{if} \quad x_1 > x_2, \\ KF_2(x_1,x_2), & \text{if} \quad x_2 > x_1, \\ KF_0(x_1,x_2), & \text{if} \quad x_1 = x_2 = x \end{cases} \tag{9}$$

where $x_1, x_2 \in \mathcal{N}_2 = \{2,3,4,\ldots\}$ and

$$\begin{aligned} F_1(x_1,x_2) = (1-p_1)p_2p_3\Big[&\frac{1-(p_1p_2p_3)^{x_2-1}}{1-p_1p_2p_3} - \frac{(1-p_1^{x_2-1})(p_2p_3)^{x_2-1}}{1-p_1}\Big] \\ &- (1-p_1)p_2p_3p_4^{\alpha}\Big[\frac{1-(p_1p_2p_3p_4)^{x_2-1}}{1-p_1p_2p_3p_4} - \frac{\{1-(p_1p_4)^{x_2-1}\}(p_2p_3)^{x_2-1}}{1-p_1p_4}\Big] \\ &+ (1-p_2)p_1p_3\Big[\frac{1-(p_1p_2p_3)^{x_2}}{1-p_1p_2p_3} - \frac{(1-p_2^{x_2})(p_1p_3)^{x_1-1}}{1-p_2}\Big] \\ &- (1-p_2)p_1p_3p_4\Big[\frac{1-(p_1p_2p_3p_4)^{x_2}}{1-p_1p_2p_3p_4} - \frac{\{1-(p_2p_4)^{x_2}\}(p_1p_3)^{x_1-1}}{1-p_2p_4}\Big] \\ &+ (1-p_1p_3-p_2p_3+p_1p_2p_3)\Big[\frac{1-(p_1p_2p_3)^{x_2}}{1-p_1p_2p_3}\Big], \\ &- (1-p_1p_3-p_2p_3+p_1p_2p_3)p_4\Big[\frac{1-(p_1p_2p_3p_4)^{x_2}}{1-p_1p_2p_3p_4}\Big], \end{aligned}$$

$$\begin{aligned} F_2(x_1,x_2) =&(1-p_2)p_1p_3\Big[\frac{1-(p_1p_2p_3)^{x_1-1}}{1-p_1p_2p_3} - \frac{(1-p_2^{x_1-1})(p_1p_3)^{x_1-1}}{1-p_2}\Big] \\ &- (1-p_2)p_1p_3p_4\Big[\frac{1-(p_1p_2p_3p_4)^{x_1-1}}{1-p_1p_2p_3p_4} - \frac{\{1-(p_2p_4)^{x_1-1}\}(p_1p_3)^{x_1-1}}{1-p_2p_4}\Big] \\ &+ (1-p_1)p_2p_3\Big[\frac{1-(p_1p_2p_3)^{x_1}}{1-p_1p_2p_3} - \frac{(1-p_1^{x_1})(p_2p_3)^{x_2-1}}{1-p_1}\Big] \\ &- (1-p_2)p_1p_3p_4\Big[\frac{1-(p_1p_2p_3p_4)^{x_1}}{1-p_1p_2p_3p_4} - \frac{\{1-(p_1p_4)^{x_1}\}(p_2p_3)^{x_2-1}}{1-p_1p_4}\Big] \\ &+ (1-p_1p_3-p_2p_3+p_1p_2p_3)\Big[\frac{1-(p_1p_2p_3)^{x_1}}{1-p_1p_2p_3}\Big], \\ &- (1-p_1p_3-p_2p_3+p_1p_2p_3)p_4\Big[\frac{1-(p_1p_2p_3p_4)^{x_1}}{1-p_1p_2p_3p_4}\Big], \end{aligned}$$

$$
\begin{aligned}
F_0(x_1,x_2) =&(1-p_1)p_2p_3\big[\frac{1-(p_1p_2p_3)^{x-1}}{1-p_1p_2p_3} - \frac{(1-p_1^{x-1})(p_2p_3)^{x-1}}{1-p_1}\big]\\
&-(1-p_1)p_2p_3p_4\big[\frac{1-(p_1p_2p_3p_4)^{x-1}}{1-p_1p_2p_3p_4} - \frac{\{1-(p_1p_4^{\alpha})^{x-1}\}(p_2p_3)^{x-1}}{1-p_1p_4}\big]\\
&+(1-p_2)p_1p_3\big[\frac{1-(p_1p_2p_3)^{x-1}}{1-p_1p_2p_3} - \frac{(1-p_2^{x-1})(p_1p_3)^{x-1}}{1-p_2}\big]\\
&-(1-p_2)p_1p_3p_4\big[\frac{1-(p_1p_2p_3)^{x-1}}{1-p_1p_2p_3} - \frac{\{1-(p_2p_4)^{x-1}\}(p_1p_3)^{x-1}}{1-p_2p_4}\big]\\
&+(1-p_1p_3-p_2p_3+p_1p_2p_3)\big[\frac{1-(p_1p_2p_3)^{x}}{1-p_1p_2p_3}\big],\\
&-(1-p_1p_3-p_2p_3+p_1p_2p_3)p_4\big[\frac{1-(p_1p_2p_3p_4)^{x}}{1-p_1p_2p_3p_4}\big].
\end{aligned}
$$

Proof: See in the Appendix.

Similarly, when $x_1 = 1$ and $x_2 \in \{1, 2, 3, ...\}$ or $x_2 = 1$ and $x_1 \in \{1, 2, 3, ...\}$, we have

Theorem 2.4. *If* $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$ *with joint distribution function* F*, then we have*

$$F(1, x_2) = \big\{1 - p_1p_3 - (1 - p_1)(p_2p_3)^{x_2}\big\}(1 - p_1p_2p_3p_4),$$

where $x_2 \in \{1, 2, 3, ...\}$ *and*

$$F(x_1, 1) = \big\{1 - p_2p_3 - (1 - p_2)(p_1p_3)^{x_1}\big\}(1 - p_1p_2p_3p_4),$$

where $x_1 \in \{1, 2, 3, ...\}$.

Proof: It can be obtained easily, hence it is avoided. □

Now we consider the MGF of a WBGE distribution.

Theorem 2.5. *The MGF of a* $WBGE(p_1, p_2, p_3, p_4)$ *is given by*

$$
\begin{aligned}
M_{X_1,X_2}(t_1,t_2) = E(e^{t_1X_1+t_2X_2}) =& \frac{(1-p_1)(1-p_2p_3)p_2p_3e^{t_1+2t_2}(1-p_1p_2p_3p_4)}{(1-p_2p_3e^{t_2})(1-p_1p_2p_3e^{t_1+t_2})(1-p_1p_2p_3p_4e^{t_1+t_2})}\\
&+\frac{(1-p_2)(1-p_1p_3)p_1p_3e^{2t_1+t_2}(1-p_1p_2p_3p_4)}{(1-p_1p_3e^{t_1})(1-p_1p_2p_3e^{t_1+t_2})(1-p_1p_2p_3p_4e^{t_1+t_2})}\\
&+\frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)e^{t_1+t_2}(1-p_1p_2p_3p_4)}{(1-p_1p_2p_3e^{t_1+t_2})(1-p_1p_2p_3p_4e^{t_1+t_2})},
\end{aligned}
$$

where $t_1 < -\ln(p_1p_3)$, $t_2 < -\ln(p_2p_3)$ *and* $t_1+t_2 < \min\{-\ln(p_1p_2p_3), -\ln(p_1p_2p_3p_4)\}$.

Proof: See in the Appendix. □

In the next theorem, using the MGF as provided in Theorem 2.5, we obtain marginal distributions of a WBGE distribution.

Theorem 2.6. *If* $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$*, then* $X_1 \sim WGE(\frac{\ln p_2 p_4}{\ln p_1 p_3}, p_1 p_3)$ *and* $X_2 \sim WG(\frac{\ln p_2 p_4}{\ln p_2 p_3}, p_2 p_3)$.

Proof.

$$M_{X_1}(t_1) = M_{X_1,X_2}(t_1, 0) = \frac{(1-p_1p_3)e^{t_1}(1-p_1p_2p_3p_4)}{(1-p_1p_3e^{t_1)}(1-p_1p_2p_3p_4e^{t_1})}$$

and

$$M_{X_2}(t_2) = M_{X_1,X_2}(0, t_2) == \frac{(1-p_2p_3)e^{t_2}(1-p_1p_2p_3p_4)}{(1-p_2p_3e^{t_2)}(1-p_1p_2p_3p_4e^{t_2})}$$

which are MGFs of $X_1 \sim WGE(\frac{\ln p_2 p_4}{\ln p_1 p_3}, p_1 p_3)$ and $X_2 \sim WGE(\frac{\ln p_1 p_4}{\ln p_2 p_3}, p_2 p_3)$, respectively. Hence, the results follow. □

We can easily obtain the means, variances and the product moments of the marginals as follows.

$$E(X_1) = \frac{(1-p_1^2p_2p_3^2p_4)}{(1-p_1p_3)(1-p_1p_2p_3p_4)}, \qquad E(X_2) = \frac{(1-p_1p_2^2p_3^2p_4)}{(1-p_2p_3)(1-p_1p_2p_3p_4)},$$

$$Var(X_1) = \frac{p_1p_3}{(1-p_1p_3)^2} - \frac{p_1p_2p_3p_4}{(1-p_1p_2p_3p_4)^2}, \quad Var(X_2) = \frac{p_2p_3}{(1-p_2p_3)^2} - \frac{p_1p_2p_3p_4}{(1-p_1p_2p_3p_4)^2}.$$

$$\begin{aligned} E(X_1X_2) =& \frac{(1-p_1p_2p_3^2)(1-p_1p_2p_3p_4)}{(1-p_1p_2)(1-p_2p_3)(1-p_1p_2p_3)(1-p_4)} \\ &- \frac{(1-p_1p_2p_3)(1+p_1p_2p_3p_4)p_4}{(1-p_1p_2p_3p_4)^2(1-p_4)} \\ &- \frac{(p_2p_3-p_1p_2p_3)p_4}{(1-p_1p_2p_3p_4)(1-p_2p_3)(1-p_4)} \\ &- \frac{(p_1p_3-p_1p_2p_3)p_4}{(1-p_1p_2p_3p_4)(1-p_1p_3)(1-p_4)}. \end{aligned} \tag{10}$$

Hence, the correlation coefficient also can be obtained explcitly. Since when $p_4 \to 0$, the WBGE distribution weakly converges to a BGE distribution it follows that the correlation coefficient of a BGE distribution is given by

$$Corr(X_1, X_2) = \frac{(1-p_3)(p_1p_2)^{\frac{1}{2}}}{(1-p_1p_2p_3)}. \tag{11}$$

Hence, it can be observed that, in such a case, as $p_3 \to 1$ and $p_1p_2 \neq 1$, the correlation coefficient (11) tends to zero. Furthermore, when $p_3 \to 0$, the correlation coefficient (11) tends to $(p_1p_2)^{\frac{1}{2}}$.

The following theorem provides the conditional PMF and the conditional expected value of a WBGE distribution.

Theorem 2.7. *If $(X_1, X_2) \sim WBG(p_1, p_2, p_3, p_4)$, then*
(a) the conditional PMF of X_2 given X_1, is given by

$$f_{X_2|X_1=x_1}(x_2) = \begin{cases} \frac{(1-p_2p_4)(1-p_1)(1-p_2p_3)(1-p_4^{x_1})}{(1-p_4)(1-p_1p_3)\{1-(p_2p_4)^{x_1}\}} p_2^{x_2-1} p_3^{x_2-x_1}, & \text{if} \quad x_2 > x_1, \\ \frac{(1-p_2p_4)(1-p_2)}{(1-p_4)\{1-(p_2p_4)^{x_1}\}} p_2^{x_2-1}(1-p_4^{x_2}), & \text{if} \quad x_1 > x_2, \\ \frac{(1-p_2p_4)(1-p_1p_3-p_2p_3+p_1p_2p_3)}{(1-p_4)(1-p_1p_3)} \times \frac{(1-p_4^{x_2})p_2^{x_2-1}}{\{1-(p_2p_4)^{x_2}\}}, & \text{if} \quad x_2 = x_1 \end{cases}$$

(b) the conditional expectation of $(X_2|X_1 = x_1)$ is given by

$$\begin{aligned} E(X_2|X_1 = x_1) =& \frac{(1-p_2p_4)(1-p_2)}{(1-p_4)\{1-(p_2p_4)^{x_1}\}} \Big[\frac{1 - x_1 p_2^{x_1-1} + p_2^{x_1}(x_1 - 1)}{(1-p_2)^2}, \\ & - \frac{1 - x_1(p_2p_4)^{x_1-1} + (p_2p_4)^{x_1}(x_1 - 1)}{(1-p_2p_4)^2} p_4 \Big], \\ & + \frac{(1-p_2p_4)(1-p_1)(1-p_4^{x_1})}{(1-p_4)(1-p_1p_3)(1-p_2p_3)\{1-(p_2p_4)^{x_1}\}} \Big[(x_1+1)p_2^{x_1-1} - x_1 p_2^{x_1} p_3 \Big], \\ & + \frac{(1-p_2p_4)(1-p_1p_3-p_2p_3+p_1p_2p_3)x_1(1-p_4^{x_1})p_2^{x_1-1}}{(1-p_4)(1-p_1p_3)\{1-(p_2p_4)^{x_1}\}}. \qquad (12) \end{aligned}$$

Theorem 2.8. *Suppose $(X_1, X_2) \sim WBG(p_1, p_2, p_3, p_4)$. If $p_4 \longrightarrow 0$ and $p_3 \longrightarrow 1$, then X_1 and X_2 are independent GE random variables with means $\frac{1}{1-p_1}$ and $\frac{1}{1-p_2}$, respectively.*

Proof. The proof is straightforward by using Theorem 2.5. □

3 Other properties

In this section we consider some other important properties and characteristics of the WBG distribution. Suppose $GE(p)$ is a geometric distributions with the PMF

$$f_X(x) = p(1-p)^x, \qquad x = 0, 1, 2, \ldots,$$

then we have the following useful results. The first result can be used for generating samples from a WBG distribution.

Theorem 3.1. *Let $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$. Then, if $U_1 \sim GE(1 - p_1p_3)$, $U_2 \sim GE(1 - p_2p_3)$, $U_3 \sim GE(1 - p_1p_2p_3)$ and $U_4 \sim G_0(1 - p_1p_2p_3p_4)$ are independently distributed and $U = U_3 + U_4$, we have*

$$(X_1, X_2) \stackrel{d}{=} \begin{cases} (U, U_2 + U), & \text{with probability} \quad \frac{p_2p_3 - p_1p_2p_3}{1-p_1p_2p_3}, \\ (U_1 + U, U), & \text{with probability} \quad \frac{p_1p_3 - p_1p_2p_3}{1-p_1p_2p_3}, \\ (U, U), & \text{with probability} \quad \frac{1-p_1p_3-p_2p_3+p_1p_2p_3}{1-p_1p_2p_3}. \end{cases}$$

Proof. The joint MGF of (X_1, X_2) can be written as

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) =& \frac{p_2p_3 - p_1p_2p_3}{1-p_1p_2p_3}\Big\{\frac{(1-p_2p_3)e^{t_2}}{1-p_2p_3e^{t_2}}\Big\}\Big\{\frac{(1-p_1p_2p_3)e^{t_1+t_2}}{1-p_1p_2p_3e^{t_1+t_2}}\Big\}\Big\{\frac{1-p_1p_2p_3p_4}{1-p_1p_2p_3p_4e^{t_1+t_2}}\Big\} \\ &+ \frac{p_1p_3 - p_1p_2p_3}{1-p_1p_2p_3}\Big\{\frac{(1-p_1p_3)e^{t_1}}{1-p_1p_3e^{t_1}}\Big\}\Big\{\frac{(1-p_1p_2p_3)e^{t_1+t_2}}{1-p_1p_2p_3e^{t_1+t_2}}\Big\}\Big\{\frac{1-p_1p_2p_3p_4}{1-p_1p_2p_3p_4e^{t_1+t_2}}\Big\} \\ &+ \frac{1-p_1p_3-p_2p_3+p_1p_2p_3}{1-p_1p_2p_3}\Big\{\frac{(1-p_1p_2p_3)e^{t_1+t_2}}{1-p_1p_2p_3e^{t_1+t_2}}\Big\}\Big\{\frac{1-p_1p_2p_3p_4}{1-p_1p_2p_3p_4e^{t_1+t_2}}\Big\}, \end{aligned}$$

which implies the result. □

Theorem 3.2. *Let* $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$. *Then,*

$$X_{(1)} \sim WG(\frac{\ln p_4}{\ln p_1p_2p_3}, p_1p_2p_3),$$

where $X_{(1)} = min\{X_1, X_2\}$.

Proof: See in the Appendix.

An important measure of reliability in Engineering Sciences is the stress-strength reliability of a system. In the next theorem we present this measure for our model.

Theorem 3.3. *If* $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$, *then stress-strength reliability of the model is given by*

$$P(X_2 < X_1) = \frac{p_1p_3(1-p_2)}{(1-p_1p_2p_3)} \tag{13}$$

Proof: See in the Appendix.

A total positivity of order two (TP_2) property is another important property of a multivariate PMF which has considerable attention in the statistical literature. Let (T_1, T_2) be a discrete bivarita random variables on $\mathcal{N}_1 \times \mathcal{N}_1$ with PMF $f(\cdot, \cdot)$, then (T_1, T_2) has a TP_2 property if

$$\frac{f(t_{11}, t_{21})f(t_{12}, t_{22})}{f(t_{12}, t_{21})f(t_{11}, t_{22})} \geq 1 \tag{14}$$

for any $t_{11}, t_{12}, t_{21}, t_{22} \in N_1$ when $t_{11} < t_{12}$ and $t_{21} < t_{22}$. We have the following result.

Theorem 3.4. *Suppose* $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, p_4)$. *If* $p_1 = p_2$, *then* (X_1, X_2) *has the* TP_2 *property.*

Proof. For any $t_{11}, t_{12}, t_{21}, t_{22} \in N_1$ if $t_{11} < t_{21} < t_{12} < t_{22}$, clearly we have

$$\frac{f(t_{11}, t_{21})f(t_{12}, t_{22})}{f(t_{12}, t_{21})f(t_{11}, t_{22})} = \frac{p_3^{t_{21}}(1-p_4^{t_{12}})}{p_3^{t_{12}}(1-p_4^{t_{21}})} \geq 1.$$

Similarly, for other rearrangements of $t_{11}, t_{12}, t_{21}, t_{22}$ such as $t_{11} < t_{12} < t_{21} < t_{22}$, $t_{11} < t_{21} < t_{12} = t_{22}$, (14) holds. Hence, the result follows. $\square$

Entropy of a random vector (X_1, X_2) is a measure of unpredictability or uncertainty. This measure is very important in Engineering Sciences. There are numerous entropy and information indices from which the most popular is Renyi entropy. For a random vector (X_1, X_2) with joint PDF $f(x_1, x_2)$, Renyi entropy is defined as

$$I_F(\delta) = \frac{1}{1-\delta} \log \Big[H(\delta)\Big],$$

for $\delta > 0$ and $\delta \neq 1$, where

$$H(\delta) = \sum_{x_1=-\infty}^{\infty} \sum_{x_2=-\infty}^{\infty} f^{\delta}(x_1, x_2).$$

Here, we find a close form, although lengthy, expression for the Renyi entropy of a WBG distribution. To do this, it is sufficient to find $H(\delta)$.

Theorem 3.5. *Let* $(X_1, X_2) \sim WBGE(p_1, p_2, p_3, , p_4)$*, then*

$$\begin{aligned} H(\delta) =& \frac{\big[K(p_2p_3 - p_1p_2p_3)(1-p_2p_3)\big]^{\delta}}{\{1-(p_2p_3)^{\delta}\}} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \frac{p_4^j}{\{1-(p_1p_2p_3)^{\delta} p_4^j\}}, \\ &+ \frac{\big[K(p_1p_3 - p_1p_2p_3)(1-p_1p_3)\big]^{\delta}}{\{1-(p_1p_3)^{\delta}\}} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \frac{p_4^j}{\{1-(p_1p_2p_3)^{\delta} p_4^j\}}, \\ &+ \big[K(1-p_1p_3 - p_2p_3 + p_1p_2p_3)\big]^{\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \frac{p_4^j}{\{1-(p_1p_2p_3)^{\delta} p_4^j\}}. \end{aligned}$$

Proof: See in the Appendix.

4 Maximum Likelihood (ML) and Bayes Estimations

4.1 Maximum Likelihood Estimation

In this section, we discuss maximum likelihood estimation of the parameters of a $WBGE(p_1, p_2, p_3, p_4)$ distribution based on a random sample of size n from that distribution. Suppose that $\{(x_{1i}, x_{2i}), i = 1, ..., n\}$ is a random sample of size n

from a $WBGE(p_1, p_2, p_3, p_4)$ distribution and we denote $\Theta = (p_1, p_2, p_3, p_4)$. Also let

$$
\begin{aligned}
K_1 &= \{i \in 1, 2, \ldots, n; x_{1i} < x_{2i}\},\\
K_2 &= \{i \in 1, 2, \ldots, n; x_{1i} > x_{2i}\},\\
K_3 &= \{i \in 1, 2, \ldots, n; x_{1i} = x_{2i} = x_i\},
\end{aligned}
$$

and n_1, n_2 and n_3 denote the number of elements in K_1, K_2 and K_3, respectively. Then, the log-likelihood function for $\Theta = (p_1, p_2, p_3, p_4)$ becomes

$$
\begin{aligned}
l(\Theta) = &\sum_{i\in K_1} \ln\Big[\frac{(1-p_1p_2p_3p_4)(1-p_1)(1-p_2p_3)}{(1-p_4)(p_1p_2p_3)} p_1^{x_{1i}} (p_2p_3)^{x_{2i}} (1-p_4^{x_{1i}})\Big],\\
&+ \sum_{i\in K_2} \ln\Big[\frac{(1-p_1p_2p_3p_4)(1-p_2)(1-p_1p_3)}{(1-p_4)(p_1p_2p_3)} (p_1p_3)^{x_{1i}} p_2^{x_{2i}} (1-p_4^{x_{2i}})\Big],\\
&+ \sum_{i\in K_3} \ln\Big[\frac{(1-p_1p_2p_3p_4)(1-p_1p_3-p_2p_3+p_2p_3)}{(1-p_4)(p_1p_2p_3)} (p_1p_2p_3)^{x_i} (1-p_4^{x_i})\Big],\\
=&n_1\ln(1-p_1) + n_1\ln(1-p_2p_3) + n_2\ln(1-p_2) + n_2\ln(1-p_1p_3),\\
&+ (n_1+n_2+n_3)\ln(1-p_1p_2p_3p_4) - (n_1+n_2+n_3)\ln(1-p_4) - (n_1+n_2+n_3)\ln(p_1p_2p_3),\\
&+ n_3\ln(1-p_1p_3-p_2p_3+p_1p_2p_3) + (\sum_{i\in K_1\bigcup K_2} x_{1i} + \sum_{i\in K_3} x_i)\ln p_1,\\
&+ (\sum_{i\in K_1\bigcup K_2} x_{2i} + \sum_{i\in K_3} x_i)\ln p_2 + (\sum_{i\in K_1} x_{1i} + \sum_{i\in K_2} x_{2i} + \sum_{i\in K_3} x_i)\ln p_3 + \sum_{i\in K_1}\ln(1-p_4^{x_{1i}}),\\
&+ \sum_{i\in K_2}\ln(1-p_4^{x_{2i}}) + \sum_{i\in K_3}\ln(1-p_4^{x_i}).
\end{aligned}
$$

The first derivatives of $l(\Theta)$ with respect to the parameters p_1, p_2,p_3 and p_4 are:

$$
\begin{aligned}
\frac{\partial l(\Theta)}{\partial p_1} =& -\frac{n_1}{1-p_1} - \frac{n_2p_3}{1-p_1p_3} - \frac{(n_1+n_2+n_3)p_2p_3p_4}{(1-p_1p_2p_3p_4)} - \frac{(n_1+n_2+n_3)}{p_1},\\
&- \frac{n_3(p_3-p_2p_3)}{1-p_1p_3-p_2p_3+p_1p_2p_3} + \frac{T_1}{p_1} = 0,\\
\frac{\partial l(\Theta)}{\partial p_2} =& -\frac{n_2}{1-p_2} - \frac{n_1p_3}{1-p_2p_3} - \frac{(n_1+n_2+n_3)p_1p_3p_4}{(1-p_1p_2p_3p_4)} - \frac{(n_1+n_2+n_3)}{p_2},\\
&- \frac{n_3(p_3-p_1p_3)}{1-p_1p_3-p_2p_3+p_1p_2p_3} + \frac{T_2}{p_2} = 0,\\
\frac{\partial l(\Theta)}{\partial p_3} =& -\frac{n_1p_2}{1-p_2p_3} - \frac{n_2p_1}{1-p_1p_3} - \frac{(n_1+n_2+n_3)p_1p_2p_4}{(1-p_1p_2p_3p_4)} - \frac{(n_1+n_2+n_3)}{p_3},\\
&- \frac{n_3(p_1+p_2-p_1p_2)}{1-p_1p_3-p_2p_3+p_1p_2p_3} + \frac{T_3}{p_3} = 0,\\
\frac{\partial l(\Theta)}{\partial p_4} =& -\frac{(n_1+n_2+n_3)p_1p_2p_3p_4}{1-p_1p_2p_3p_4} + \frac{(n_1+n_2+n_3)}{1-p_4} - \frac{T_4}{p_4} = 0,
\end{aligned}
$$

where

$$T_1 = \Big(\sum_{i \in K_1 \bigcup K_2} x_{1i} + \sum_{i \in K_3} x_i \Big),$$
$$T_2 = \Big(\sum_{i \in K_1 \bigcup K_2} x_{2i} + \sum_{i \in K_3} x_i \Big),$$
$$T_3 = \Big(\sum_{i \in K_1} x_{1i} + \sum_{i \in K_2} x_{2i} + \sum_{i \in K_3} x_i \Big).$$
$$T_4 = \Big(\sum_{i \in K_1} \frac{x_{1i} p_4^{x_{1i}}}{1 - p_4^{x_{1i}}} + \sum_{i \in K_2} \frac{x_{2i} p_4^{x_{2i}}}{1 - p_4^{x_{2i}}} + \sum_{i \in K_3} \frac{x_i p_4^{x_i}}{1 - p_4^{x_i}} \Big).$$

The solutions of above equations yield the MLE of $\Theta = (p_1, p_2, p_3, p_4)$, which can be obtained by using function "optim" in the statistical software R. Based on the observed Fisher information matrix we have obtained the approximate confidence intervals of the unknown parameters. Let $\hat{\Theta} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4)$ be MLE of $\Theta = (p_1, p_2, p_3, p_4)$, then by assuming the regularity conditions, as $n \longrightarrow \infty$ we have

$$\sqrt{n}\{(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) - (p_1, p_2, p_3, p_4)\} \sim N(0, I^{-1}(\theta)), \tag{15}$$

where $I(\theta) = -E[\frac{\partial^2 \ell(\Theta)}{\partial \Theta^2}]$ is the information matrix with

$$\frac{\partial^2 \ell(\Theta)}{\partial \Theta^2} = \begin{bmatrix} \frac{\partial^2 \ell(\Theta)}{\partial p_1^2} & \frac{\partial^2 \ell(\Theta)}{\partial p_1 \partial p_2} & \frac{\partial^2 \ell(\Theta)}{\partial p_1 \partial p_3} & \frac{\partial^2 \ell(\Theta)}{\partial p_1 \partial p_4} \\ \frac{\partial^2 \ell(\Theta)}{\partial p_2 \partial p_1} & \frac{\partial^2 \ell(\Theta)}{\partial p_2^2} & \frac{\partial^2 \ell(\Theta)}{\partial p_2 \partial p_3} & \frac{\partial^2 \ell(\Theta)}{\partial p_2 \partial p_4} \\ \frac{\partial^2 \ell(\Theta)}{\partial p_3 \partial p_1} & \frac{\partial^2 \ell(\Theta)}{\partial p_3 \partial p_2} & \frac{\partial^2 \ell(\Theta)}{\partial p_3^2} & \frac{\partial^2 \ell(\Theta)}{\partial p_3 \partial p_4} \\ \frac{\partial^2 \ell(\Theta)}{\partial p_4 \partial p_1} & \frac{\partial^2 \ell(\Theta)}{\partial p_4 \partial p_2} & \frac{\partial^2 \ell(\Theta)}{\partial p_4 \partial p_3} & \frac{\partial^2 \ell(\Theta)}{\partial p_4^2} \end{bmatrix}.$$

Here, we have

$$\begin{aligned} \frac{\partial^2 l(\Theta)}{\partial p_k^2} &= -\frac{n_k}{(1-p_k)^2} - \frac{n_j p_3^2}{(1-p_j p_3)^2} - \frac{n p_j^2 p_3^2 p_4^2}{(1-p_1 p_2 p_3 p_4)^2} + \frac{n}{p_k^2}, \\ &- \frac{n_3 (p_3 - p_j p_3)^2}{(1 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3)^2} - \frac{T_k}{p_k^2}, \quad j,k = 1,2 \quad and \quad j \neq k, \\ \frac{\partial^2 l(\Theta)}{\partial p_3^2} &= -\frac{n_1 p_2^2}{(1-p_2 p_3)^2} - \frac{n_2 p_1^2}{(1-p_1 p_3)^2} - \frac{n p_1^2 p_2^2 p_4^2}{(1-p_1 p_2 p_3 p_4)^2} + \frac{n}{p_3^2}, \\ &- \frac{n_3 (p_1 + p_2 - p_1 p_2)^2}{(1 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3)^2} - \frac{T_3}{p_3^2}, \\ \frac{\partial^2 l(\Theta)}{\partial p_4^2} &= -\frac{n p_1 p_2 p_3}{(1-p_1 p_2 p_3 p_4)^2} + \frac{n}{(1-p_4)^2} - \frac{T - T_4}{p_4^2}, \\ \frac{\partial^2 l(\Theta)}{\partial p_1 p_2} &= -\frac{n p_3 p_4}{(1-p_1 p_2 p_3 p_4)^2} + \frac{n_3 p_3 (1-p_3)}{(1 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3)^2}, \end{aligned}$$

$$\frac{\partial^2 l(\Theta)}{\partial p_k p_3} = -\frac{n_j}{(1-p_k p_3)^2} - \frac{n p_j p_4}{(1-p_1 p_2 p_3 p_4)^2},$$

$$- \frac{n_3(1-p_j)}{(1-p_1 p_3 - p_2 p_3 + p_1 p_2 p_3)^2}, \quad j,k = 1,2 \quad - and \quad j \neq k,$$

$$\frac{\partial^2 l(\Theta)}{\partial p_j p_4} = -\frac{n p_l p_k}{(1-p_1 p_2 p_3 p_4)^2}, \qquad j,k,l = 1,2,3 \quad and \quad j \neq k \neq l,$$

and

$$T = \Big(\sum_{i \in K_1} \frac{x_{1i}^2 p_4^{x_{1i}}}{(1-p_4^{x_{1i}})^2} + \sum_{i \in K_2} \frac{x_{2i}^2 p_4^{x_{2i}}}{(1-p_4^{x_{2i}})^2} + \sum_{i \in K_3} \frac{x_i^2 p_4^{x_i}}{(1-p_4^{x_i})^2} \Big).$$

Therefore, we can obtain $I(\theta)$ as given in Appendix.

4.2 Initial Estimate

To obtain the MLE of the unknown parameters we need some initial estimates. For this purpose, we shall use the marginals and their minimum distributions. Infact, As given in Theorems 2.6 and 3.2, we have $X_1 \sim WGE(\frac{\ln p_2 p_4}{\ln p_1 p_3}, p_1 p_3)$, $X_2 \sim WGE(\frac{\ln p_2 p_4}{\ln p_2 p_3}, p_2 p_3)$ and $X_{(1)} \sim WGE(\frac{\ln p_4}{\ln p_1 p_2 p_3}, p_1 p_2 p_3)$. Thus, we can obtain the initial estimates of $\frac{\ln p_2 p_4}{\ln p_1 p_3}$ and $p_1 p_3$ by fitting WGE distribution to X_1. Similarly, we obtain the initial estimates of $\frac{\ln p_2 p_4}{\ln p_2 p_3}$, $p_2 p_3$, $\frac{\ln p_4}{\ln p_1 p_2 p_3}$ and $p_1 p_2 p_3$ by fitting WGE distribution to X_2 and $X_{(1)}$, respectively. Consequently, we shall obtain the initial estimates of p_1,p_2,p_3 and p_4.

4.3 Testing of Hypothesis

As we know $BGE(p_1, p_2, p_3)$ is a submodel of the family of $WBGE(p_1, p_2, p_3, p_4)$ distributions when $p_4 = 0$. Thus, we can compare these two models by making the following test of hypotheses:

$$H_0 : p_4 = 0 (BGE) \qquad v.s \qquad H_1 : p_4 > 0, (WBGE) \tag{16}$$

Under the null hypothesis, p_4 is a boundary value and , thus, standard results do not work. However, by Theorem 3 of Self and Liang [18], it follows that.

$$-2(-l_{WBG}(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) + l_{BG}(\hat{p}_1, \hat{p}_2, \hat{p}_3)) \sim \frac{1}{2} + \frac{1}{2}\chi_1^2, \tag{17}$$

where $l_{BGE}(.)$ and $l_{WBGE}(.)$ are the log-likelihood functions of BGE and WBGE distributions, respectively.

4.4 Bayes Estimation

Bayes estimation for bivariate geometric distribution have been proposed by Krishna and Pundir [12], with bivariate Dirichlet distribution (BDD) as the prior, and by Li and Dhar [14], with a uniform prior distribution. Now, for WBG distribution, suppose $\Theta = (p_1, p_2, p_3, p_4)$ has uniform distribution with probability density function:

$$f(p_1, p_2, p_3, p_4) = \begin{cases} 1 & \text{if} \quad 0 < p_1, p_2, p_3, p_4 < 1 \\ 0 & \quad o.w \end{cases}$$

Then, the joint posterior PDF of $\Theta = (p_1, p_2, p_3, p_4)$ is

$$\begin{aligned} f(p_1, p_2, p_3, p\big|x, y) =& A^{-1} k^n p_1^a p_2^b p_3^c (1-p_1)^d (1-p_2)^e (1-p_1p_3)^e (1-p_2p_3)^d, \\ & .(1 - p_1p_3 - p_2p_3 + p_1p_2p_3)^f (1-p_4^g) \end{aligned}$$

where $0 < p_1, p_2, p_3, p_4 < 1$ and $a = \sum_{i=1}^n x_{1i} - n$, $b = \sum_{i=1}^n x_{2i} - n$, $c = \sum_{i=1}^n x_{2i} I[x_{1i} < x_{2i}] + \sum_{i=1}^n x_{1i} I[x_{2i} \leq x_{1i}] - n$, $d = \sum_{i=1}^n I[x_{1i} < x_{2i}]$, $e = \sum_{i=1}^n I[x_{2i} < x_{1i}]$, $f = \sum_{i=1}^n I[x_{1i} = x_{2i}]$, $g = \sum_{i=1}^n x_{1i} I[x_{1i} < x_{2i}] + \sum_{i=1}^n x_{2i} I[x_{2i} \leq x_{1i}] - n$ and A is the constant

$$\begin{aligned} A = \int_0^1 \int_0^1 \int_0^1 \int_0^1 & k^n p_1^a p_2^b p_3^c (1-p_1)^d (1-p_2)^e (1-p_1p_3)^e (1-p_2p_3)^d, \\ & .(1 - p_1p_3 - p_2p_3 + p_1p_2p_3)^f (1-p_4^g) dp_1 dp_2 dp_3 dp_4. \end{aligned}$$

Then, the marginal posterior pdf of p_1, p_2, p_3 and p_4 are

$$f(p_i\big|x, y) = \int_0^1 \int_0^1 \int_0^1 f(p_1, p_2, p_3, p_4\big|x, y) dp_j dp_k dp_l$$

where $i, j, k, l = 1, 2, 3, 4$ and $i \neq j \neq k \neq l$.

Thus, under the mean squared error loss function, the Bayes estimator of the unknown parameter is

$$\tilde{p}_i = \int_0^1 p_i f(p_i\big|x, y) dp_i$$

where $p_i = p_1, p_2, p_3, p_4$. Analytical solutions for the above two integrals are not available. Therefore, we obtain them numerically by using the WinBUGS software. WinBUGS is a software for Bayesian inference analyzes which utilizes Gibbs sampling. Infact, in WinBUGS program we need to define three sections; i.e., Model, Data, and Initial Values to obtain the Bayes estimates of unknown prameters. Then, in this way, by loading our statistical model into a WinBUGS program with uniform prior distributions for p_1, p_2, p_3 and p_4, we have

```
model{for(i in 1:N){
dummy[i] <- 0
dummy[i] ~ dloglik(logLike[i])
id1[i] <- step(y[i] -x[i]) - equals(x[i], y[i])
id2[i] <- equals(y[i],x[i])
id3[i] <- step(x[i] -y[i]) - equals(x[i], y[i])

f1[i] <- (x[i]-1)*log(p1) +(y[i]-1)*log(p2*p3)+log(1-
p1)+log(1-p2*p3)+log(1-p1*p2*p3*p4)-log(1-p4)+log(1-
pow(p4,(x[i])))
f2[i] <-  (x[i]-1)*log(p1*p2*p3)  +log(1-p1*p3-
p2*p3+p1*p2*p3)+log(1-p1*p2*p3*p4)-log(1-p4)+log(1-
pow(p4,(x[i])))
f3[i] <-  (y[i]-1)*log(p2)  +(x[i]-1)*log(p1*p3)+log(1-
p2)+log(1-p1*p3)+log(1-p1*p2*p3*p4)-log(1-p4)+log(1-
pow(p4,(y[i])))
logLike[i] <-  f1[i]*id1[i]  + f2[i]*id2[i]  + f3[i]*id3[i]

}
p1~dunif(0,1)
p2~dunif(0,1)
p3~dunif(0,1)
p4~dunif(0,1)
}
```

Thus, also by loading data and initial values for parameters and doing several more operations, Bayes estimates of unknown parameters are obtained.

5 Simulation

Here, we perform a Monte Carlo simulation study by generating one hundred samples from $\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$ of sizes $n = 50$, $n = 75$ and $n = 100$ to show advantage of this model to fit the simulated data set and compare the performances of the maximum likelihood estimate and the Bayes estimate of parameters. One of the most popular methods which can be used to generate a bivariate distribution is using the marginal distribution of X_2 and the conditional distribution of X_1 given X_2. Also we can generate this distribution by using Theorem 3.1. Generating bivariate distribution based on PMF by using the WinBUGS software is another way which have been employed in this paper. It has the following scheme.

1. Define model based on PMF of WBG distribution (12) and then click check model.

2. Load data with different sample sizes of n and 500 iterations.

3. Load initial values of (X_1, X_2) with different sample sizes of n and 500 iterations.

We update this scheme 10000 times and discard them to increase the accuracy of sampling by using the UPDATE tool. Finally, by each update of the scheme we can get 500 samples of different sizes.

Tables 1 reveals the MLEs and Bayes estimates of parameters of the $WBGE(p_1, p_2, p_3, p_4)$ distribution along with their mean squared errors (MSEs) based on simulated data for fixed $p_1 = p_2 = p_3 = 0.9$ and different values of n and p_4.

It is clearly observed from this table that as the sample size increases, both MSE of MLEs (MSE_1) and MSE of Bayes estimates (MSE_2) decrease for all four parameters. Comparing MSE_1 and MSE_2, we conclude that the Bayes estimates provide more precise estimates for the parameters than MLE estimates.

Table 1. MLEs and Bayes estimates of parameters of the WBG distribution and mean squared error of MLEs (MSE_1) and Bayes estimates (MSE_2) for fixed $p_1 = p_2 = p_3 = 0.9$ and different values of n and p_4.

n	50							
p	0.5				0.9			
Estimator	MLE	Bayes	MSE_1	MSE_2	MLE	Bayes	MSE_1	MSE_2
p_1	0.8880	0.8822	0.00062	0.00045	0.8921	0.8975	0.00053	0.00036
p_2	0.8960	0.8916	0.00046	0.00033	0.9020	0.9064	0.00038	0.00034
p_3	0.8949	0.8956	0.00039	0.00028	0.8993	0.9012	0.00097	0.00038
p	0.4134	0.4009	0.00521	0.00358	0.8865	0.8127	0.00491	0.00264
n	75							
p	0.5				0.9			
Estimator	MLE	Bayes	MSE_1	MSE_2	MLE	Bayes	MSE_1	MSE_2
p_1	0.9000	0.8861	0.00037	0.00031	0.8940	0.8958	0.00049	0.00039
p_2	0.8994	0.8855	0.00038	0.00036	0.8937	0.8961	0.00036	0.00031
p_3	0.9081	0.9047	0.00036	0.00021	0.9105	0.9109	0.00053	0.00027
p	0.5867	0.5987	0.00325	0.00308	0.9047	0.8371	0.00262	0.00225
n	100							
p	0.5				0.9			
Estimator	MLE	Bayes	MSE_1	MSE_2	MLE	Bayes	MSE_1	MSE_2
p_1	0.9008	0.8938	0.00032	0.00016	0.9042	0.9103	0.00046	0.00028
p_2	0.8961	0.8894	0.00029	0.00018	0.9027	0.9072	0.00017	0.00010
p_3	0.9026	0.9019	0.00026	0.00011	0.8948	0.8974	0.00019	0.00012
p	0.5039	0.5304	0.00314	0.00146	0.8833	0.8684	0.00210	0.00201

6 Application

In this section, we apply a real data set (cf. Dhar [7]), scores given by seven judges from seven different countries recorded in a video recording, indicated in Table 2, to demonstrate the advantage of our proposed model. The score given by each judge is a discrete random variable taking positive integer values and also the midpoints of consecutive integers between zero and ten. Dhar [7] used the data set presented in Table 2 to reveal the applicability of the bivariate geometric model. In the following, we shall use several measures of goodness-of-fit to demonstrate that for this data set $WBGE(p_1, p_2, p_3, p_4)$ provides a better fit compared to $BGE(p_1, p_2, p_3)$ model. First of all, to estimate the unknown parameters by MLEs and Bayes methods, we need to provide initial guesses for $\theta = (p_1, p_2, p_3, p_4)$. As mentioned in Subsection 4.2, we have fitted WGE distribution to X_1, X_2 and $X_{(1)} = min\{X_1, X_2\}$, respectively. The initial MLE extimates are $p_1 = 0.9921$, $p_2 = 0.9908$, $p_3 = 0.8698$ and $p_4 = 0.9999$. Now we fit the WBGE distributions to the bivariate data set in Table 2 and estimate their parameters using both MLE and Bayes methods.

Tables 3 and 4 list the MLEs and Bayes estimates of the parameters, negative log-likelihood function ($-\log L$), Akaike Information Criterion, $AIC = -2\log L + 2k$, Bayesian information criterion, $BIC = -2\log L + k\log(n)$, and HannanQuinn information criterion (HQIC), $HQIC = -2L + 2k\log(\log(n))$, respectively, where n is the number of observations, k is number of parameters in the model and L is the maximized value of the likelihood function, for two models.

By considering AIC, BIC and HQIC indices for MLEs and Bayes estimates, as indicated in Tables 3 and 4, we can see that $WBGE(p_1, p_2, p_3, p_4)$ provides a better fit compared to $BGE(p_1, p_2, p_3)$ model based on both estimators. Also, by comparing Tables 3 and 4, we conclude that the maximum likelihood estimators have lower AIC, BIC and HQIC indices than the Bayes estimators of the parameters. Therefore, for this real data set we choose the maximum likelihood estimators as first candidates to estimate the parameters.

The 95 percent confidence interval of parameters p_1, p_2, p_3 and p_4 from WBGE distribution by MLE method are $(0.9265, 0.9505)$, $(0.9701, 0.9827)$, $(0.8973, 0.9213)$ and $(0.9615, 0.9999)$, respectively. The corresponding Bayes 95 percent confidence interval for parametars are $(0.9035, 0.9659)$, $(0.9427, 0.9932)$, $(0.8672, 0.9485)$ and $(0.9037, 0.9999)$, respectively.

Here we would like to test $WBGE(p_1, p_2, p_3, p_4)$ versus $BGE(p_1, p_2, p_3)$ as defined in (50). Using the negative log-likelihood values of WBGE and BGE distributions for MLE and Bayes estimates, as mentioned in Tables 3 and 4, we obtain the test statistic (51) as 22.94 and 20.58, respectively. Also, thier corresponding p-values are less than 0.0001. Therefore, the null hypothesis that the data come from a BGE distribution is rejected.

Table 2. Scores taken from a video recorded during the summer of 1995 relayed by NBC sports TV, IX World Cup diving competition, Atlanta, Georgia.

Item	Diver	X: max score, Asian and Caucasus	Y: max score, West
1	Sun Shuwei, China	19	19
2	David Pichler, USA	15	15
3	Jan Hempel, Germany	13	14
4	Roman Volodkuv, Ukraine	11	12
5	Sergei Kudrevich, Belarus	14	14
6	Patrick Jeffrey, USA	15	14
7	Valdimir Timoshinin, Russia	13	16
8	Dimitry Sautin, Russia	7	5
9	Xiao Hailiang, China	13	13
10	Sun Shuwei, China	15	16
11	David Pichler, USA	15	15
12	Jan Hempel, Germany	17	18
13	Roman Volodkuv, Ukraine	16	16
14	Sergei Kudrevich, Belarus	12	13
15	Patrick Jeffrey, USA	14	14
16	Valdimir Timoshinin, Russia	12	13
17	Dimitry Sautin, Russia, Russia	17	18
18	Xiao Hailiang, China, Russia	9	10
19	Sun Shuwei, China, Russia	18	18

Table 3. MLEs of the parameters, negative log-likelihood, AIC, BIC and HQIC of the BGE(p_1, p_2, p_3) and WBGE(p_1, p_2, p_3, p_4) models for data of Table 2.

Distribution	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$	$\hat{p}_4$	$-\log L$	AIC	BIC	HQIC
$BGE(p_1, p_2, p_3)$	0.9616	0.9854	0.9401	-	117.64	241.28	244.11	241.75
$WBGE(p_1, p_2, p_3, p_4)$	0.9385	0.9764	0.9093	0.9999	106.17	220.34	224.11	220.97

Table 4. Bayes estimation of the parameters, negative log-likelihood, AIC, BIC and HQIC of the BGE(p_1, p_2, p_3) and WBGE(p_1, p_2, p_3, p_4) models for data of Table 2.

Distribution	$\tilde{p}_1$	$\tilde{p}_2$	$\tilde{p}_3$	$\tilde{p}_4$	$-\log L$	AIC	BIC	HQIC
$BGE(p_1, p_2, p_3)$	0.9577	0.9798	0.9390	-	117.85	241.70	244.53	242.17
$WBGE(p_1, p_2, p_3, p_4)$	0.9382	0.9722	0.9109	0.9722	107.08	223.12	226.89	223.75

7 Conclusions

In this paper we have developed BWGE distributions and it has WGE as the marginals. It is observed that due to presence of one extra parameter it is more flexible than a classical BGE distribution. We have derived different properties and characteristics of the proposed bivariate distribution. It seems the method can be extended for other bivariate discrete distributions also. More work is needed along that direction.

Acknowledgments

The first author is grateful to the Graduate Office of the University of Isfahan for its support.

Appendix

PROOF OF THEOREM 2.1

By Equation (11), we have

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2) &= P(Y_1 = x_1, Y_2 = x_2 \big| Y < m \min\{Y_1, Y_2\}), \\ &= \frac{P(Y_1 = x_1, Y_2 = x_2) P(Y < m \min\{x_1, x_2\})}{P(Y < m \min\{Y_1, Y_2\})}. \end{aligned}$$

Now, let $K^{-1} = P(Y < m\min\{Y_1, Y_2\})$. Then, we obtain

$$
\begin{aligned}
K^{-1} =& P(Y < m\min\{Y_1, Y_2\}) = \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty} P(Y_1 = x_1, Y_2 = x_2) P(Y < m\min\{x_1, x_2\}) \\
=& \frac{(1-p_1)(1-p_2p_3)}{p_1p_2p_3} \sum_{x_1=1}^{\infty}\sum_{x_2=x_1+1}^{\infty} (p_2p_3)^{x_2} p_1^{x_1}(1-p_4^{x_1}) \\
&+ \frac{(1-p_2)(1-p_1p_3)}{p_1p_2p_3} \sum_{x_2=1}^{\infty}\sum_{x_1=x_2+1}^{\infty} (p_1p_3)^{x_1} p_2^{x_2}(1-p_4^{x_2}) \\
&+ \frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)}{p_1p_2p_3} \sum_{x=1}^{\infty}(p_1p_2p_3)^{x}(1-p_4^{x}) \\
=& 1 - \frac{(1-p_1)(1-p_2p_3)}{p_1p_2p_3} \sum_{x_1=1}^{\infty}(p_1p_4)^{x_1} \sum_{x_2=x_1+1}^{\infty} (p_2p_3)^{x_2} \\
&- \frac{(1-p_2)(1-p_1p_3)}{p_1p_2p_3} \sum_{x_2=1}^{\infty}(p_2p_4)^{x_2} \sum_{x_1=x_2+1}^{\infty} (p_1p_3)^{x_1} \\
&- \frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)}{p_1p_2p_3} \sum_{x=1}^{\infty}(p_1p_2p_3p_4)^{x} \\
=& 1 - \frac{(1-p_1)}{p_1} \sum_{x_1=1}^{\infty}(p_1p_2p_3p_4)^{x_1} - \frac{(1-p_2)}{p_2} \sum_{x_2=1}^{\infty}(p_1p_2p_3p_4)^{x_2} \\
&- \frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)}{p_1p_2p_3} \sum_{x=1}^{\infty}(p_1p_2p_3p_4)^{x} \\
=& 1 - \frac{(1-p_1p_2p_3)p_4}{(1-p_1p_2p_3p_4)} = \frac{(1-p_4)}{(1-p_1p_2p_3p_4)}
\end{aligned}
$$

which implies the result.

Proof of Theorem 2.2: Note that

$$
F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = \begin{cases} KF_1(x_1, x_2), & \text{if} \quad x_1 > x_2, \\ KF_2(x_1, x_2), & \text{if} \quad x_2 > x_1, \\ KF_0(x_1, x_2), & \text{if} \quad x_1 = x_2 = x, \end{cases}
$$

where

$$
\begin{aligned}
F_1(x_1, x_2) =& \frac{(1-p_1)(1-p_2p_3)}{p_1p_2p_3}\left[\sum_{t_1=1}^{x_2-1}\sum_{t_2=t_1+1}^{x_2} (p_2p_3)^{t_2} p_1^{t_1} - \sum_{t_1=1}^{x_2-1}\sum_{t_2=t_1+1}^{x_2} (p_2p_3)^{t_2}(p_1p_4)^{t_1}\right], \\
&+ \frac{(1-p_2)(1-p_1p_3)}{p_1p_2p_3}\left[\sum_{t_2=1}^{x_2}\sum_{t_1=t_2+1}^{x_1} (p_1p_3)^{t_1} p_2^{t_2} - \sum_{t_2=1}^{x_2}\sum_{t_1=t_2+1}^{x_1} (p_1p_3)^{t_1}(p_2p_4)^{t_2}\right], \\
&+ \frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)}{p_1p_2p_3}\left[\sum_{t=1}^{x_2}(p_1p_2p_3)^{t} - \sum_{t=1}^{x_2}(p_1p_2p_3p_4)^{t}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_2(x_1,x_2) =&\frac{(1-p_2)(1-p_1p_3)}{p_1p_2p_3}\Bigg[\sum_{t_2=1}^{x_1-1}\sum_{t_1=t_2+1}^{x_1}(p_1p_3)^{t_1}p_2^{t_2}-\sum_{t_2=1}^{x_1-1}\sum_{t_1=t_2+1}^{x_1}(p_1p_3)^{t_1}(p_2p_4)^{t_2}\Bigg],\\
&+\frac{(1-p_1)(1-p_2p_3)}{p_1p_2p_3}\Bigg[\sum_{t_1=1}^{x_1}\sum_{t_2=t_1+1}^{x_2}(p_2p_3)^{t_2}p_1^{t_1}-\sum_{t_1=1}^{x_1}\sum_{t_2=t_1+1}^{x_2}(p_2p_3)^{t_2}(p_1p_4)^{t_1}\Bigg],\\
&+\frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)}{p_1p_2p_3}\Bigg[\sum_{t=1}^{x_1}(p_1p_2p_3)^t-\sum_{t=1}^{x_1}(p_1p_2p_3p_4)^t\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
F_0(x_1,x_2) =&\frac{(1-p_1)(1-p_2p_3)}{p_1p_2p_3}\Bigg[\sum_{t_1=1}^{x-1}\sum_{t_2=t_1+1}^{x}(p_2p_3)^{t_2}p_1^{t_1}-\sum_{t_1=1}^{x-1}\sum_{t_2=t_1+1}^{x}(p_2p_3)^{t_2}(p_1p_4)^{t_1}\Bigg],\\
&+\frac{(1-p_2)(1-p_1p_3)}{p_1p_2p_3}\Bigg[\sum_{t_2=1}^{x-1}\sum_{t_1=t_2+1}^{x}(p_1p_3)^{t_1}p_2^{t_2}-\sum_{t_2=1}^{x-1}\sum_{t_1=t_2+1}^{x}(p_1p_3)^{t_1}(p_2p_4)^{t_2}\Bigg],\\
&+\frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)}{p_1p_2p_3}\Bigg[\sum_{t=1}^{x}(p_1p_2p_3)^t-\sum_{t=1}^{x}(p_1p_2p_3p_4)^t\Bigg].
\end{aligned}
$$

Therefore, the result follows by the geometric series expansions.

PROOF OF THEOREM 2.4:

Suppose that $t_1 < -\ln(p_1p_3)$, $t_2 < -\ln(p_2p_3)$ and $t_1+t_2 < \min\{-\ln(p_1p_2p_3), -\ln(p_1p_2p_3p_4)\}$, then we have

$$
\begin{aligned}
M_{X_1,X_2}(t_1,t_2) =&K\sum_{x_1=1}^{\infty}\sum_{x_2=x_1+1}^{\infty}e^{t_1x_1+t_2x_2}(1-p_1)(1-p_2p_3)(p_2p_3)^{x_2-1}p_1^{x_1-1}(1-p_4^{x_1})\\
&+K\sum_{x_2=1}^{\infty}\sum_{x_1=x_2+1}^{\infty}e^{t_1x_1+t_2x_2}(1-p_2)(1-p_1p_3)(p_1p_3)^{x_1-1}p_2^{x_2-1}(1-p_4^{x_2})\\
&+K\sum_{x=1}^{\infty}e^{t_1x+t_2x}(1-p_1p_3-p_2p_3+p_1p_2p_3)(p_1p_2p_3)^{x-1}(1-p_4^{x})\\
=&K\frac{(1-p_1)(1-p_2p_3)}{p_1p_2p_3}[\sum_{x_1=1}^{\infty}(p_1e^{t_1})^{x_1}\frac{(p_2p_3e^{t_2})^{x_1+1}}{(1-p_2p_3e^{t_2})}-\sum_{x_1=1}^{\infty}(p_1p_4e^{t_1})^{x_1}\frac{(p_2p_3e^{t_2})^{x_1+1}}{(1-p_2p_3e^{t_2})}]\\
&+K\frac{(1-p_2)(1-p_1p_3)}{p_1p_2p_3}[\sum_{x_2=1}^{\infty}(p_2e^{t_2})^{x_2}\frac{(p_1p_3e^{t_1})^{x_2+1}}{(1-p_1p_3e^{t_1})}-\sum_{x_2=1}^{\infty}(p_2p_4e^{t_2})^{x_2}\frac{(p_1p_3e^{t_1})^{x_2+1}}{(1-p_1p_3e^{t_1})}]\\
&+K\frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)}{p_1p_2p_3}[\sum_{x=1}^{\infty}(p_1p_2p_3e^{t_1+t_2})^{x}-\sum_{x=1}^{\infty}(p_1p_2p_3p_4e^{t_1+t_2})^{x}]\\
=&\frac{(1-p_1)(1-p_2p_3)p_2p_3e^{t_1+2t_2}(1-p_1p_2p_3p_4)}{(1-p_2p_3e^{t_2})(1-p_1p_2p_3e^{t_1+t_2})(1-p_1p_2p_3p_4e^{t_1+t_2})}\\
&+\frac{(1-p_2)(1-p_1p_3)p_1p_3e^{2t_1+t_2}(1-p_1p_2p_3p_4)}{(1-p_1p_3e^{t_1})(1-p_1p_2p_3e^{t_1+t_2})(1-p_1p_2p_3p_4e^{t_1+t_2})}\\
&+\frac{(1-p_1p_3-p_2p_3+p_1p_2p_3)e^{t_1+t_2}(1-p_1p_2p_3p_4)}{(1-p_1p_2p_3e^{t_1+t_2})(1-p_1p_2p_3p_4e^{t_1+t_2})}
\end{aligned}
$$

Proof of Theorem 2.6:

Since the marginal PMF of X_1 is

$$f_{X_1}(x_1) = \frac{(1-p_1p_3)(1-p_1p_2p_3p_4)}{(1-p_2p_4)}(p_1p_3)^{x_1-1}\{1-(p_2p_4)^{x_1}\},$$

the proof of (a) is straightforward. For part (b), we have

$$\begin{aligned}
E(X_2|X_1=x_1) = \sum_{x_2=1}^{\infty} x_2 f_{X_2|X_1=x_1}(x_2) &= \sum_{x_2=1}^{x_1-1} x_2 \frac{(1-p_2p_4)(1-p_2)}{(1-p_4)\{1-(p_2p_4)^{x_1}\}} p_2^{x_2-1}(1-p_4^{x_2}), \\
&+ \sum_{x_2=x_{1+1}}^{\infty} x_2 \frac{(1-p_2p_4)(1-p_1)(1-p_2p_3)(1-p_4^{x_1})}{(1-p_4)(1-p_1p_3)\{1-(p_2p_4)^{x_1}\}} p_2^{x_2-1}p_3^{x_2-x_1}, \\
&+ x_1 \frac{(1-p_2p_4)(1-p_1p_3-p_2p_3+p_1p_2p_3)}{(1-p_4)(1-p_1p_3)} \times \frac{(1-p_4^{x_1})p_2^{x_1-1}}{\{1-(p_2p_4)^{x_1}\}}, \\
&= \frac{(1-p_2p_4)(1-p_2)}{(1-p_4)\{1-(p_2p_4)^{x_1}\}} \Big[\frac{1-x_1p_2^{x_1-1}+p_2^{x_1}(x_1-1)}{(1-p_2)^2}, \\
&- \frac{1-x_1(p_2p_4)^{x_1-1}+(p_2p_4)^{x_1}(x_1-1)}{(1-p_2p_4)^2} p_4 \Big], \\
&+ \frac{(1-p_2p_4)(1-p_1)(1-p_4^{x_1})}{(1-p_4)(1-p_1p_3)(1-p_2p_3)\{1-(p_2p_4)^{x_1}\}} \Big[(x_1+1)p_2^{x_1-1} - x_1p_2^{x_1}p_3 \Big], \\
&+ \frac{(1-p_2p_4)(1-p_1p_3-p_2p_3+p_1p_2p_3)x_1(1-p_4^{x_1})p_2^{x_1-1}}{(1-p_4)(1-p_1p_3)\{1-(p_2p_4)^{x_1}\}}.
\end{aligned}$$

As required.

Proof of Theorem 3.2:

Using the law of total probability, we obtain

$$
\begin{aligned}
P(X_{(1)} = x) =& P(X_1 = x, X_2 > x) + P(X_2 = x, X_1 > x) + P(X_1 = X_2 = x),\\
=& \sum_{x_2=x+1}^{\infty} K(1-p_1)(1-p_2p_3)(p_2p_3)^{x_2-1}p_1^{x-1}(1-p_4^x),\\
&+ \sum_{x_1=x+1}^{\infty} K(1-p_2)(1-p_1p_3)(p_1p_3)^{x_1-1}p_2^{x-1}(1-p_4^x),\\
&+ K(1-p_1p_3-p_2p_3+p_1p_2p_3)(p_1p_2p_3)^{x-1}(1-p_4^x),\\
=& K(1-p_1)(1-p_2p_3)p_1^{x-1}(1-p_4^x)\sum_{x_2=x+1}^{\infty}(p_2p_3)^{x_2-1},\\
&+ K(1-p_2)(1-p_1p_3)p_2^{x-1}(1-p_4^x)\sum_{x_1=x+1}^{\infty}(p_1p_3)^{x_1-1},\\
&+ K(1-p_1p_3-p_2p_3+p_1p_2p_3)(p_1p_2p_3)^{x-1}(1-p_4^x),\\
=& K(p_2p_3-p_1p_2p_3)(p_1p_2p_3)^{x-1}(1-p_4^x),\\
&+ K(p_1p_3-p_1p_2p_3)(p_1p_2p_3)^{x-1}(1-p_4^x),\\
&+ K(1-p_1p_3-p_2p_3+p_1p_2p_3)(p_1p_2p_3)^{x-1}(1-p_4^x),\\
=& \frac{(1-p_1p_2p_3)(1-p_1p_2p_3p_4)}{1-p_4}(p_1p_2p_3)^{x-1}(1-p_4^x),
\end{aligned}
$$

which is the PMF of a $WGE(\frac{\ln p_4}{\ln p_1p_2p_3}, p_1p_2p_3)$.

Proof of Theorem 3.3:

From (12) we have

$$
\begin{aligned}
P(X_1 < X_2) &= \sum_{x_1=1}^{\infty}\sum_{x_2=x_1+1}^{\infty} K(1-p_1)(1-p_2p_3)(p_2p_3)^{x_2-1}p_1^{x_1-1}(1-p_4^{x_1}),\\
&= \frac{K(1-p_1)(1-p_2p_3)}{p_1p_2p_3}\sum_{x_1=1}^{\infty} p_1^{x_1}(1-p_4^{x_1})\sum_{x_2=x_1+1}^{\infty}(p_2p_3)^{x_2},\\
&= \frac{K(1-p_1)}{p_1}\sum_{x_1=1}^{\infty}(p_1p_2p_3)^{x_1}(1-p_4^{x_1}),\\
&= \frac{K(1-p_1)}{p_1}\Big[\sum_{x_1=1}^{\infty}(p_1p_2p_3)^{x_1} - \sum_{x_1=1}^{\infty}(p_1p_2p_3p_4)^{x_1}\Big],\\
&= \frac{K(1-p_1)}{p_1}\Big[\frac{p_1p_2p_3}{(1-p_1p_2p_3)} - \frac{p_1p_2p_3p_4}{(1-p_1p_2p_3p_4)}\Big],\\
&= \frac{p_2p_3(1-p_1)}{(1-p_1p_2p_3)},
\end{aligned}
\tag{18}
$$

as required.

Proof of Theorem 3.4:

$$
\begin{aligned}
H(\delta) = & \sum_{x_1=1}^{\infty} \sum_{x_2=x_1+1}^{\infty} \frac{K^\delta (1-p_1)^\delta (1-p_2p_3)^\delta}{(p_1p_2p_3)^\delta} (p_2p_3)^{\delta x_2} p_1^{\delta x_1} (1-p_4^{x_1})^\delta, \\
& + \sum_{x_2=1}^{\infty} \sum_{x_1=x_2+1}^{\infty} \frac{K^\delta (1-p_2)^\delta (1-p_1p_3)^\delta}{(p_1p_2p_3)^\delta} (p_1p_3)^{\delta x_1} p_2^{\delta x_2} (1-p_4^{x_2})^\delta, \\
& + \sum_{x=1}^{\infty} \frac{K^\delta (1-p_1p_3-p_2p_3+p_1p_2p_3)^\delta}{(p_1p_2p_3)^\delta} (p_1p_2p_3)^{\delta x} (1-p_4^{x})^\delta, \\
= & \frac{K^\delta (1-p_1)^\delta (1-p_2p_3)^\delta}{(p_1p_2p_3)^\delta} \sum_{x_1=1}^{\infty} \sum_{x_2=x_1+1}^{\infty} (p_2p_3)^{\delta x_2} p_1^{\delta x_1} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j p_4^{x_1 j}, \\
& + \frac{K^\delta (1-p_2)^\delta (1-p_1p_3)^\delta}{(p_1p_2p_3)^\delta} \sum_{x_2=1}^{\infty} \sum_{x_1=x_2+1}^{\infty} (p_1p_3)^{\delta x_1} p_2^{\delta x_2} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j p_4^{x_2 j}, \\
& + \frac{K^\delta \left(1-p_1p_3-p_2p_3+p_1p_2p_3\right)^\delta}{(p_1p_2p_3)^\delta} \sum_{x=1}^{\infty} (p_1p_2p_3)^{\delta x} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j p_4^{xj}, \\
= & \frac{K^\delta (1-p_1)^\delta (1-p_2p_3)^\delta}{(p_1p_2p_3)^\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \sum_{x_1=1}^{\infty} (p_1^\delta p_4^j)^{x_1} \sum_{x_2=x_1+1}^{\infty} (p_2p_3)^{\delta x_2}, \\
& + \frac{K^\delta (1-p_2)^\delta (1-p_1p_3)^\delta}{(p_1p_2p_3)^\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \sum_{x_2=1}^{\infty} (p_2^\delta p_4^j)^{x_2} \sum_{x_1=x_2+1}^{\infty} (p_1p_3)^{\delta x_1}, \\
& + \frac{K^\delta \left(1-p_1p_3-p_2p_3+p_1p_2p_3\right)^\delta}{(p_1p_2p_3)^\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \sum_{x=1}^{\infty} \{(p_1p_2p_3)^\delta p_4^j\}^x, \\
= & \frac{K^\delta (1-p_1)^\delta (1-p_2p_3)^\delta}{(p_1p_2p_3)^\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \sum_{x_1=1}^{\infty} (p_1^\delta p_4^j)^{x_1} \frac{(p_2p_3)^{\delta(x_1+1)}}{\{1-(p_2p_3)^\delta\}}, \\
& + \frac{K^\delta (1-p_2)^\delta (1-p_1p_3)^\delta}{(p_1p_2p_3)^\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \sum_{x_2=1}^{\infty} (p_2^\delta p_4^j)^{x_2} \frac{(p_1p_3)^{\delta(x_2+1)}}{\{1-(p_1p_3)^\delta\}}, \\
& + \frac{K^\delta \left(1-p_1p_3-p_2p_3+p_1p_2p_3\right)^\delta}{(p_1p_2p_3)^\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \sum_{x=1}^{\infty} \{(p_1p_2p_3)^\delta p_4^j\}^x, \\
= & \frac{K^\delta (p_2p_3-p_1p_2p_3)^\delta (1-p_2p_3)^\delta}{\{1-(p_2p_3)^\delta\}} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \frac{p_4^j}{\{1-(p_1p_2p_3)^\delta p_4^j\}}, \\
& + \frac{K^\delta (p_1p_3-p_1p_2p_3)^\delta (1-p_1p_3)^\delta}{\{1-(p_1p_3)^\delta\}} \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \frac{p_4^j}{\{1-(p_1p_2p_3)^\delta p_4^j\}}, \\
& + K^\delta \left(1-p_1p_3-p_2p_3+p_1p_2p_3\right)^\delta \sum_{j=0}^{\infty} \binom{\delta}{j} (-1)^j \frac{p_4^j}{\{1-(p_1p_2p_3)^\delta p_4^j\}},
\end{aligned}
$$

Therefore, by substituting (36) into Equation (35), we obtain Renyi entropy of a random vector (X_1, X_2).

FISHER INFORMATION MATRIX:

Since $I(\theta) = -E[\frac{\partial^2 \ell(\Theta)}{\partial \Theta^2}]$, then by using Theorem 3.3 and the following result

$$P(X_1 = X_2) = \frac{(1 - p_1p_3 - p_2p_3 + p_1p_2p_3)}{(1 - p_1p_2p_3)}$$

we have

$$\begin{aligned}
E\left[\frac{\partial^2 l(\Theta)}{\partial p_k^2}\right] &= -n\Bigg[\frac{p_jp_3}{(1-p_1p_2p_3)(1-p_k)} + \frac{p_kp_3^3(1-p_j)}{(1-p_1p_2p_3)(1-p_jp_3)^2} + \frac{p_j^2p_3^2p_4^2}{(1-p_1p_2p_3p_4)^2}, \\
&\quad - \frac{1}{p_k^2} + \frac{(p_3-p_jp_3)^2}{(1-p_1p_2p_3)(1-p_1p_3-p_2p_3+p_1p_2p_3)}, \\
&\quad + \frac{K(p_kp_3-p_1p_2p_3)}{(1-p_kp_3)p_k^2}\left\{\frac{(2-p_kp_3(1+p_j))}{(1-p_1p_2p_3)^2} - \frac{p_4(2-p_kp_3(1+p_jp_4))}{(1-p_1p_2p_3p_4)^2}\right\}\Bigg], \\
&\quad + \frac{K(1-p_kp_3)}{p_k^2}\left\{\frac{1}{(1-p_1p_2p_3)^2} - \frac{p_4}{(1-p_1p_2p_3p_4)^2}\right\}, \quad j,k=1,2 \quad and \quad j \neq k, \\
E\left[\frac{\partial^2 l(\Theta)}{\partial p_3^2}\right] &= -n\Bigg[\frac{p_2p_3(1-p_1)p_2^2}{(1-p_1p_2p_3)(1-p_2p_3)^2} + \frac{p_1p_3(1-p_2)p_1^2}{(1-p_1p_2p_3)(1-p_1p_3)^2} + \frac{p_1^2p_2^2p_4^2}{(1-p_1p_2p_3p_4)^2}, \\
&\quad - \frac{1}{p_3^2} + \frac{(p_1+p_2-p_1p_2)^2}{(1-p_1p_2p_3)(1-p_1p_3-p_2p_3+p_1p_2p_3)}, \\
&\quad + \frac{K(1-p_1p_2p_3)}{p_3^2}\left\{\frac{1}{(1-p_1p_2p_3)^2} - \frac{p_4}{(1-p_1p_2p_3p_4)^2}\right\}\Bigg], \\
E\left[\frac{\partial^2 l(\Theta)}{\partial p_4^2}\right] &= -n\Bigg[\frac{p_1p_2p_3}{(1-p_1p_2p_3p_4)^2} - \frac{1}{(1-p_4)^2} + \frac{K(1-p_1p_2p_3)A(p_1,p_2,p_3,p_4)}{(p_1p_2p_3p_4^2)}, \\
&\quad - \frac{Kp_4(1-p_1p_2p_3)}{p_4^2(1-p_1p_2p_3p_4)^2}\Bigg], \\
E\left[\frac{\partial^2 l(\Theta)}{\partial p_1p_2}\right] &= -n\left[\frac{p_3p_4}{(1-p_1p_2p_3p_4)^2} - \frac{p_3(1-p_3)}{(1-p_1p_2p_3)(1-p_1p_3-p_2p_3+p_1p_2p_3)}\right], \\
E\left[\frac{\partial^2 l(\Theta)}{\partial p_kp_3}\right] &= -n\Bigg[\frac{p_kp_3(1-p_j)}{(1-p_1p_2p_3)(1-p_kp_3)^2} + \frac{p_jp_4}{(1-p_1p_2p_3p_4)^2}, \\
&\quad + \frac{(1-p_j)}{(1-p_1p_2p_3)(1-p_1p_3-p_2p_3+p_1p_2p_3)}\Bigg], \quad j,k=1,2 \quad and \quad j \neq k, \\
E\left[\frac{\partial^2 l(\Theta)}{\partial p_jp_4}\right] &= -n\left[\frac{p_lp_k}{(1-p_1p_2p_3p_4)^2}\right], \quad j,k,l=1,2,3 \quad and \quad j \neq k \neq l,
\end{aligned}$$

where $A(p_1, p_2, p_3, p_4) = \sum_{t=1}^{\infty} \frac{t^2(p_1p_2p_3p_4)^t}{(1-p_4^t)}$.
For geting more detail to obtain above elements we refer to Kundu and Gupta [13].

References

[1] D. K. Al-Mutairi, M. E. Ghitany, and D. Kundu, *A new bivariate distribution with weighted exponential marginals and its multivariate generalization*, Statistical Papers. 52 (2011), 921-936.

[2] D. K. Al-Mutairi, M. E. Ghitany, and D. Kundu, *Weighted Weibull distribution: bivariate and multivariate cases*, Brazilian Journal of Probability and Statistics, 2017, to appear.

[3] B.C. Arnold, R.J. Beaver, *Skewed multivariate models related to hidden truncation and /or selective reporting (with discussion).* Test II. (2002), 754 .

[4] A. Azzalini, *A class of distributions which includes the normal ones.* Scandinavian Journal of Statistics. 12 (1985), 171-178

[5] A. P. Basu, and S. Dhar, *Bivariate Geometric Distribution*, Journal Applied Statistical Science. 2(1) (1995), 3344.

[6] D. Bhati, and S. Joshi, *Weighted geometric distribution with a new characterization of geometric distribution*, arXiv preprint arXiv:1512.07139. (2015).

[7] S. K. Dhar, *Modeling with a bivariate geometric distribution*, Advances on Methodological and Applied Aspects of Probability and Statistics. 1 (2003), 101-109.

[8] M.G. Genton, *Skew-Elliptical Distributions and their Applications: A Journey Beyond Normality*, Chapman and Hall/CRS, New York, 2004.

[9] R.D. Gupta, and D. Kundu, *A new class of weighted exponential distributions*, Statistics. 43 (2009), 621-634.

[10] A. Jamalizadeh, and D. Kundu, *Weighted MarshallOlkin bivariate exponential distribution*, Statistics. 47(5) (2013), 917-928.

[11] O. Kharazmi, A. Mahdavi, and M. Fathizadeh, *Generalized weighted exponential distribution*, Communications in Statistics-Simulation and Computation. 44 (6) (2015), 1557-1569.

[12] H. Krishna, and P. S. Pundir, *A bivariate geometric distribution with applications to reliability*, Communications in StatisticsTheory and Methods. 38(7) (2009),1079-1093.

[13] D. Kundu, and R.D. Gupta, *Bivariate generalized exponential distribution*, Journal of Multivariate Analysis. 100 (2009), 581 - 593.

[14] J. Li, and S. K. Dhar, *Modeling with bivariate geometric distributions*, Communications in Statistics-Theory and Methods. 42(2) (2013), 252-266.

[15] A.W. Marshall, and I. Olkin, *A generalized bivariate exponential distribution*, Journal of Applied Probability. 4(02) (1967), 291-302.

[16] H. Najarzadegan, and M. H. Alamatsaz, *A new generalization of weighted geometric distribution and its properties*, preprint. (2017).

[17] V. Nekoukhou, M. H. Alamatsaz, and H. Bidram, *A discrete analogue of the generalized exponential distribution*, Communications in Statistics Theory Methods. 41 (2011), 2000-2013.

[18] S. G. Self, and K-Y Liang, *Asymptotic properties of the maximum likeli- hood estimators and likelihood ratio test under non-standard conditions*, Journal of the American Statistical Association. 82 (1987), 605 - 610.

[19] M, K. Shakhatreh, *A two-parameter of weighted exponential distributions*, Statistics and Probability Letters. 82 (2012), 252-261.