Nonlinear System Analysis
Lyapunov Based Approach

Lecture 4 Module 1

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Overview

- Non-linear Systems: An Introduction
- Linearization
- Lyapunov Stability Theory
- Examples
- Summary
Nonlinear Systems: An Introduction

Model:

\[
\begin{align*}
\dot{x} &= f(t, x, u) \\
y &= h(t, x, u)
\end{align*}
\]

Properties:

- Don’t follow the principle of superposition, i.e.,

\[
f(t, x, au_1 + bu_2) \neq af(t, x, u_1) + bf(t, x, u_2)
\]
- Multiple equilibrium points
- Limit cycles: oscillations of constant amplitude and frequency
- Subharmonic, harmonic oscillations for constant frequency inputs
- Chaos: randomness, complicated steady state behaviours
- Multiple modes of behaviour
Autonomous Systems: the nonlinear function does not explicitly depend on time $t$.

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u)
\end{align*}
\]

Affine System:

\[
\dot{x} = f(x) + g(x)u
\]

Unforced System: input $u(t) = 0$, \[ \dot{x} = f(x) \]
Example:

Pendulum:

\[ ml^2 \ddot{\theta} = -mglsin(\theta) - k\dot{\theta} \]

The state equations are

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l}sin(x_1) - \frac{k}{ml^2}x_2
\end{align*}
\]
Exercise

Identify the category to which the following differential equations belong to? Why?

1. $\dot{x} = tx$

2. $\dot{x} = -x + x^2$

3. $\dot{x} = ux$ where $u$ is an external input.

4. $\dot{x} = -2x + t^2$

5. $\dot{x} = -x$
Linearization

- **Concept of Equilibrium Point:** Consider a system

\[ \dot{x} = f(x, u) \]

where functions \( f_i(.) \) are continuously differentiable. The equilibrium point \((x_e, u_e)\) for this system is defined as

\[ f(x_e, u_e) = 0 \]
What is linearization?
Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point.

Why do we need it?
We have well-established tools to analyze and stabilize linear systems.
The method:
Let us write the general form of nonlinear system \( \dot{x} = f(x, u) \) as:

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{align*}
\]
Let $u_e = [u_{1e} \ u_{2e} \ \ldots \ u_{me}]^T$ be a constant input that forces the system $\dot{x} = f(x, u)$ to settle into a constant equilibrium state $x_e = [x_{1e} \ x_{2e} \ \ldots \ x_{ne}]^T$ such that $f(x_e, u_e) = 0$ holds true.
Linearization

We now perturb the equilibrium state by allowing:
\( x = x_e + \Delta x \) and \( u = u_e + \Delta u \). Taylor’s expansion yields

\[
\frac{dx}{dt} = f(x_e + \Delta x, u_e + \Delta u) \\
= f(x_e, u_e) + \frac{\partial f}{\partial x}(x_e, u_e) \Delta x + \frac{\partial f}{\partial u}(x_e, u_e) \Delta u + \ldots
\]
where

\[
\frac{\partial f}{\partial x}(x_e, u_e) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}_{x_e, u_e}
\]

\[
\frac{\partial f}{\partial u}(x_e, u_e) = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m}
\end{bmatrix}_{x_e, u_e}
\]

are the **Jacobian** matrices of $F$
Linearization

Note that
\[
\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}_e}{dt} + \frac{d(\Delta \mathbf{x})}{dt} = \frac{d(\Delta \mathbf{x})}{dt}
\]
because \(\mathbf{x}_e\) is constant. Furthermore, \(f(\mathbf{x}_e, \mathbf{u}_e) = 0\).

Let
\[
A = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}_e, \mathbf{u}_e) \quad \text{and} \quad B = \frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}_e, \mathbf{u}_e)
\]

Neglecting higher order terms, we arrive at the

*linear approximation*

\[
\frac{d(\Delta \mathbf{x})}{dt} = A \Delta \mathbf{x} + B \Delta \mathbf{u}
\]
Similarly, if the outputs of the nonlinear system model are of the form

\[
\begin{align*}
  y_1 &= h_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
  y_2 &= h_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
  &\vdots \\
  y_p &= h_p(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{align*}
\]

or in vector notation

\[
y = h(x, u)
\]
Taylor’s series expansion can again be used to yield the linear approximation of the above output equations. Indeed, if we let

\[ y = y_e + \Delta y \]

then we obtain

\[ \Delta y = C \Delta x + D \Delta u \]
Linearization

Example

Consider a first order system:

\[ \dot{x} = -x + x^2 \quad \text{where} \quad x(0) = x_0 \]

Linearize it about origin

\[ \dot{x} = -x \]

Its solution is: \( x(t) = x_0 e^{-t} \). Whatever may be the initial state \( x_0 \), the state will settle at \( x(t) = 0 \) as \( t \to \infty \), which is the only equilibrium point that this linearized system has.
However, the solution of actual nonlinear system is

\[
x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}
\]

For various initial conditions, the system has two equilibrium points: \( x = 0 \) and \( x = 1 \) as can be seen in the following figure.
$x = 0$ is a Stable while $x = 1$ is an unstable equilibrium point.
Lyapunov Stability Theory

The concept of stability: Consider the nonlinear system

\[ \dot{x} = f(x) \]

Let an equilibrium point of the system be \( \bar{x} \),

\[ f(\bar{x}) = 0 \]
We say that $\bar{x}$ is \textit{stable in the sense of Lyapunov} if there exists positive quantity $\epsilon$ such that for every $\delta = \delta(\epsilon)$ we have

$$|x(t_0) - \bar{x}| < \delta \iff |x(t) - \bar{x}| < \epsilon$$

for all $t > t_0$. We say that $\bar{x}$ is \textit{asymptotically stable} if it is stable and

$$|x(t) - \bar{x}| \to 0 \text{ as } t \to \infty$$

We call $\bar{x}$ unstable if it is not stable.
How to determine the stability or instability of \( \bar{x} \) without explicitly solving the dynamic equations?

- **Lyapunov’s first or indirect method**: Start with a nonlinear system

\[
\dot{x} = f(x)
\]

Expand in Taylor series around \( \bar{x} \) (we also redefine \( x \rightarrow x - \bar{x} \))

\[
\dot{x} = Ax + g(x)
\]
where

$$A = \left. \frac{\partial f}{\partial x} \right|_{\bar{x}}$$

is the Jacobian matrix of $f(x)$ evaluated at $\bar{x}$ and $g(x)$ contains the higher order terms, i.e.,

$$\lim_{|x| \to 0} \frac{|g(x)|}{|x|} = 0$$

Then the nonlinear system $\dot{x} = f(x)$ is asymptotically stable if and only if the linear system $\dot{x} = Ax$ is stable, i.e., if all eigenvalues of $A$ have negative real parts.
Advantage: Easy to apply

Disadvantages:

– If some eigenvalues of $A$ are zero, then we cannot draw any conclusion about stability of the nonlinear system.

– It is valid only if initial conditions are “close” to the equilibrium $\bar{x}$. This method provides no indication as to how close is “close”.

Lyapunov’s second or direct method: Consider the nonlinear system

\[ \dot{x} = f(x) \]

Suppose that there exists a function, called Lyapunov function, \( V(x) \) with following properties:

1. \( V(\bar{x}) = 0 \)

2. \( V(x) > 0 \), for \( x \neq \bar{x} \): Positive definite

3. \( \dot{V}(x) < 0 \) along trajectories of \( \dot{x} = f(x) \): Negative
Definite

Then, \( \bar{x} \) is \textit{asymptotically stable}. The method hinges on the \textbf{existence} of a Lyapunov function, which is an energy-like function.

\[
\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial V}{\partial x} f(x) = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \cdots + \frac{\partial V}{\partial x_n} f_n
\]
Advantages:

- answers stability of nonlinear systems without explicitly solving dynamic equations
- can easily handle time varying systems
  \[ \dot{x} = f(x, t) \]
- can determine asymptotic stability as well as plain stability
- can determine the \textit{region} of asymptotic stability or the \textit{domain of attraction} of an equilibrium
Example

Oscillator with a nonlinear spring:

$$\ddot{y} + 3\dot{y} + y^3 = 0$$

Linearize this system,

$$\ddot{y} + 3\dot{y} = 0$$

The characteristic equation of linearized system is

$$s(s + 3) = 0.$$
The $-3$ characteristic root corresponds to the damping term but notice the existence of a 0 root from the lack of a linear term in the spring restoring force. The linearized version of the system cannot recognize the existence of a nonlinear spring term and it fails to produce a non-zero characteristic root related to the restoring force.
Let’s look at Lyapunov based approach. Consider the state space model

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -3x_2 - x_1^3
\end{align*}

with equilibrium \( \bar{x}_1 = \bar{x}_2 = 0 \). Let’s try for a Lyapunov function

\[ V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \]

We can see that \( V(x) > 0 \) for all \( x_1, x_2 \).
The time derivative of $V$ is

$$\dot{V}(x) = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2$$

$$= x_1^3 x_2 + x_2(-3x_2 - x_1^3)$$

$$= -3x_2^2$$

$$< 0$$

It follows then that $\bar{x}$ is asymptotically stable.
Disadvantage of Lyapunov based Approach

- There is no systematic way of obtaining Lyapunov functions
- Lyapunov stability criterion provides only sufficient condition for stability.
References


2. Applied Nonlinear Control, *J. J. E. Slotine* and *W. Li*, Prentice Hall

Nonlinear System Analysis
Lyapunov Based Approach

Lecture 5 Module 1

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Stability of Linear Systems

Consider a linear system in the form

\[ \dot{x} = Ax \]

Choose as Lyapunov function the quadratic form

\[ V(x) = x^T Px \]

where \( P \) is a symmetric positive definite matrix.
Then we have

\[
\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}
\]
\[
= (Ax)^T P x + x^T P Ax
\]
\[
= x^T A^T P x + x^T P A x
\]
\[
= x^T (A^T P + PA) x
\]
\[
= -x^T Q x
\]

where

\[
A^T P + PA = -Q \quad \text{Lyapunov Matrix Equation}
\]
If the matrix $Q$ is positive definite, then the system is asymptotically stable. Therefore, we could pick $Q = I$, the identity matrix and solve

$$A^T P + PA = -I$$

for $P$ and see if $P$ is positive definite.
Note: The usefulness of Lyapunov’s matrix equation for linear systems is that it can provide an initial estimate for a Lyapunov function for a nonlinear system in cases where this is done computationally. Furthermore, it can be used to show stability of the linear quadratic regulator design.
Lyapunov Stability Theory

LaSalle-Yoshizawa Theorem

Let \( x = 0 \) be an equilibrium point of \( \dot{x} = f(x, u) \). Let \( V(x) \) be a continuously differentiable, positive definite and radially unbounded function such that

\[
\dot{V} = \frac{\partial V}{\partial x} f(x, t) \leq -W(x) \leq 0
\]

where \( W \) is a continuous function. Then, all solutions of \( \dot{x} = f(x, u) \) are globally uniformly bounded and satisfy

\[
\lim_{t \to \infty} W(x(t)) = 0
\]
In addition, if $W(x)$ is positive definite, then the equilibrium $x = 0$ is **globally uniformly asymptotically stable**.
Lyapunov Stability Theory

More Examples

We will discuss three examples that demonstrate applications of Lyapunov’s method, namely

1. How to assess the importance of nonlinear terms in stability or instability.

2. How to estimate the domain of attraction of an equilibrium point.

3. How to design a control law that guarantees global asymptotic stability.
Example 1

Consider the system

\[
\begin{align*}
\dot{x}_1 &= -x_2 + ax_1 x_2^2 \\
\dot{x}_2 &= x_1 - bx_1^2 x_2
\end{align*}
\]

with \( a \neq b \). Find the equilibrium of the system by solving the following equations:

\[
\begin{align*}
-x_2 + a\bar{x}_1 \bar{x}_2^2 &= 0 \\
\bar{x}_1 - b\bar{x}_1^2 \bar{x}_2 &= 0
\end{align*}
\]
Lyapunov Stability Theory: Examples

Multiply the first equation by $\bar{x}_1$, the second by $\bar{x}_2$ and add them to get

$$\bar{x}_1^2 \bar{x}_2^2 (a - b) = 0$$

Hence, the equilibrium point is $\bar{x}_1 = \bar{x}_2 = 0$.

The linearized system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
Lyapunov Stability Theory: Examples

The characteristic equation is

\[
\det \begin{vmatrix} -s & -1 \\ 1 & -s \end{vmatrix} = 0 \implies s^2 + 1 = 0 \implies s = \pm \omega i
\]

Since the characteristic roots are purely imaginary, we can not draw any conclusion on the stability of the nonlinear system.

Now we resort to Lyapunov based approach. Choose the Lyapunov function \( V(x) \) to be the sum of the kinetic and potential energy of the linear system.
(this does not work always!):

\[ V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \]

We see that \( V(x) > 0 \) for all \( x_1, x_2 \). Then

\[
\begin{align*}
\dot{V}(x) &= x_1(-x_2 + ax_1 x_2^2) + x_2(x_1 - bx_1^2 x_2) \\
&= -x_1 x_2 + a x_1^2 x_2^2 + x_1 x_2 - b x_1^2 x_2^2 \\
&= (a - b) x_1^2 x_2^2
\end{align*}
\]
Therefore, we see that \( b \):

- if \( a < b \), the system is asymptotically stable
- if \( a > b \), the system is unstable

\[ b \text{this result can not be obtained obtained by linearization} \]
Example .2

Suppose we want to determine the stability of the origin \((0, 0)\) of the nonlinear system (Show that this is the equilibrium of the system.),

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 + x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= -x_1 - x_2 + x_2(x_1^2 + x_2^2)
\end{align*}
\]
The easiest way to show the stability is by linearization. The linearized form of the system is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

The characteristic equation is

\[s^2 + 2s + 2 = 0\]

We can see that the system is stable. *Wait!* since this result is based on linearization, it says that if initial condition is "close" to the equilibrium point (0, 0)
then the solution will tend to the equilibrium as $t \to \infty$.

To find how close is “close” we need to get an estimate of the *domain of attraction*. We can do this by using Lyapunov theory.

Let’s try a Lyapunov function candidate

$$V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$$
Lyapunov Stability Theory: Examples

Take its time derivative

\[ \dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 \]

\[ = x_1(-x_1 + x_2 + x_1^3 + x_1 x_2^2) + x_2(-x_1 - x_2 + x_2 x_1^2) \]

\[ = -x_1^2 + x_1 x_2 + x_1^4 + x_1^2 x_2^2 - x_1 x_2 - x_2^2 + x_1^2 x_2^2 + x_2^4 \]

\[ = x_1^4 + x_2^4 + 2x_1^2 x_2^2 - x_1^2 - x_2^2 \]

\[ = (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \]

We can see, therefore, that stability is guaranteed if

\[ \dot{V}(x) < 0 \text{ or } x_1^2 + x_2^2 < 1 \]
This means that the domain of attraction of the equilibrium is a circular disk of radius 1. As long as the initial conditions are inside the disk, it is guaranteed that the solution will end up at the stable equilibrium. In case, the initial conditions lie outside the disk then convergence is not guaranteed.
Note: It should be mentioned that the above disk is an estimate of the domain attraction based on the particular Lyapunov function we selected. A different Lyapunov function could have produced a different estimate of the domain of attraction.
Example 3

Trajectory Tracking

Consider a single link manipulator

\[ m l^2 \ddot{\theta} + K \dot{\theta} + mgl \cos(\theta) = \tau \]

Now we want to find a control so that \( \theta \) tracks a desired trajectory \( \theta_d \). Define \( e = \theta - \theta_d \). Then above equation may be written as

\[ ml^2 \ddot{e} + K \dot{e} + mgl \cos \theta = \tau \quad \text{error dynamic equation} \]
Choose a Lyapunov Function Candidate

\[ V = \frac{1}{2} ml^2 \dot{e}^2 + \frac{1}{2} K_p e^2 \]

Take its time derivative

\[ \dot{V} = ml^2 \ddot{e} \dot{e} + K_p e \dot{e} \]

\[ = \dot{e}(u - K \dot{e} - mgl\cos\theta) + K_p e \]
Lyapunov Stability Theory: Examples

We can choose our control input as
\[ u = mgl\cos \theta - K_p e - K_D \dot{e}. \]
Then
\[ \dot{V} = -(K_D + K) \dot{e}^2 \quad \text{assume} \ (K_D + K) > 0 \]
\[ < 0 \]
This implies that \( \dot{e} \to 0 \) and \( \ddot{e} \to 0 \). But, it does not imply that \( e \to 0 \). Substituting the control into error dynamic equation, we get
\[ ml^2 \ddot{e} + (K + K_D) \dot{e} + K_p e = 0 \]
This implies that \( K_p e = 0 \implies e = 0. \)
Example 4

Back-Stepping

Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\
\dot{x}_2 &= u
\end{align*}
\]

Start with the scalar system

\[
\dot{x}_1 = x_1^2 - x_1^3 + x_2
\]

Design the feedback control \( x_2 = \phi(x_1) \) to stabilize the origin \( x_1 = 0 \).
Lyapunov Stability Theory: Examples

We cancel the nonlinear term $x_1^2$ by taking

$$x_2 = \phi(x_1) = -x_1^2 - x_1$$

Thus we obtain

$$\dot{x}_1 = -x_1 - x_1^3$$

Taking Lyapunov function candidate as

$$V(x_1) = \frac{1}{2}x_1^2$$

its time derivative satisfies

$$\dot{V} = -x_1^2 - x_1^4 < 0 \quad \forall x_1 \in \mathbb{R}$$

Hence, the origin is globally asymptotically stable.
Lyapunov Stability Theory: Examples

Now to back step, we use the change of variable

\[ z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2 \]

So the transformed system

\[ \dot{x}_1 = -x_1 - x_1^3 + z_2 \]
\[ \dot{z}_2 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \]
Taking $V_c(x) = \frac{1}{2} x_1^2 + \frac{1}{2} z_2^2$ as a composite Lyapunov Function, we obtain

$$
\begin{align*}
\dot{V}_c &= x_1 (-x_1 - x_1^3 + z_2) \\
&\quad + z_2 [u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)] \\
&= -x_1^2 - x_1^4 \\
&\quad + z_2 [x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u]
\end{align*}
$$
Choosing control input as

\[
    u = -x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \\
    = -x_1 + (1 + 2x_1)(-x_1^2 - x_1^3 + x_2) \\
    - (x_2 + x_1 + x_1^2)
\]

we get

\[
    \dot{V}_c = -x_1^2 - x_1^4 - z_2^2
\]

Hence with this control law, the origin is globally asymptotically stable.
References