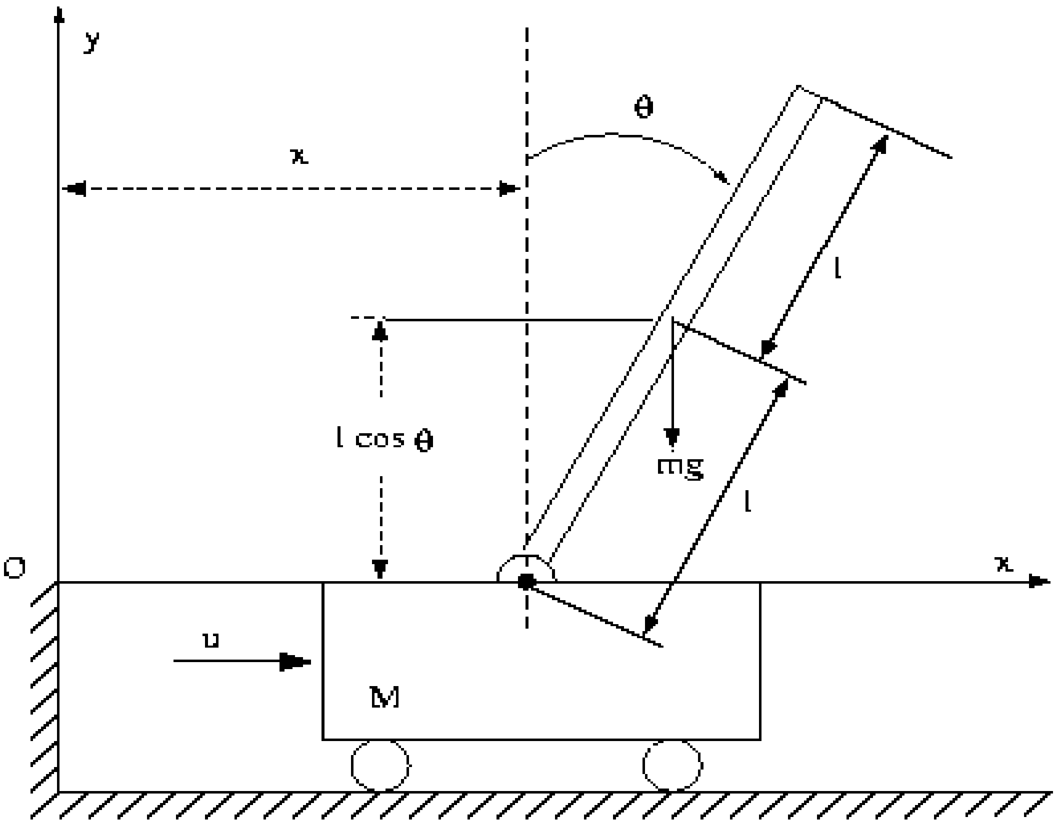
EE380: Control Systems Laboratory

Experiment No. 6
Balancing an inverted pendulum

An inverted pendulum mounted on a motor-driven cart is shown in the following figure. This is a model of the attitude control of a space booster on takeoff. (The objective of the attitude control problem is to keep the space booster in a vertical position.) The inverted pendulum is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider only a two-dimensional problem in which the pendulum moves only in the plane of page. The control force u is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center.



System Parameters:

Cart Mass M = 0.5 kg
Pendulum Mass m = 0.23 kg
Motor Torque Constant K_m = 0.00767 Nm/A
Motor gear ratio K_g = 3.71

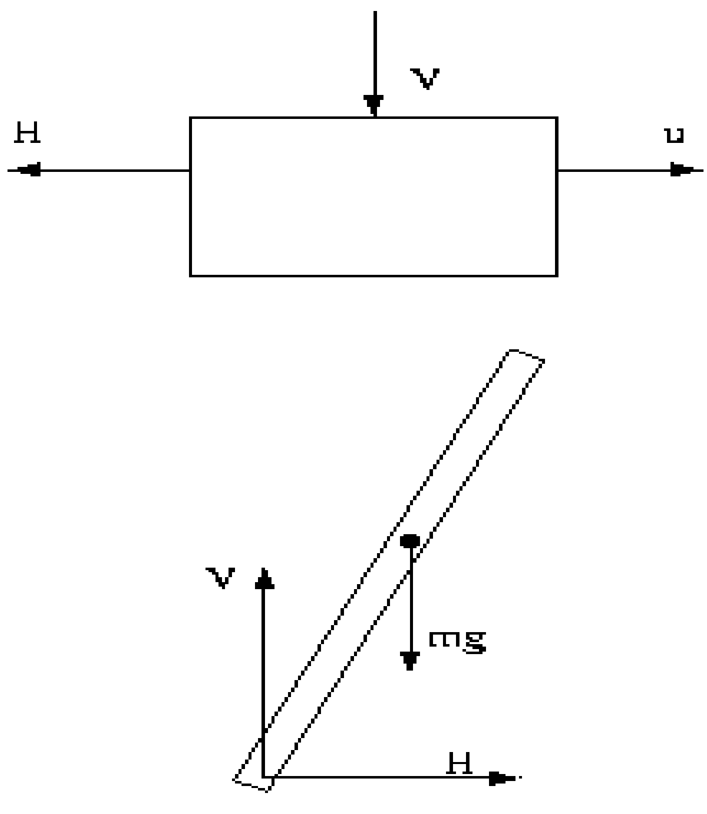
Pendulum length from pivot to the center of gravity l = 0.32065 m
Motor armature resistance R = 2.3585 Ω
radius of gear connected to motor shaft r = 0.00635 m

Mathematical Model of the system

Define the angle of the rod from the vertical line as θ . Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G, y_G) . Then

$$x_G = x + l\sin\theta$$
$$y_G = l\cos\theta$$

To derive the equations of motion for the system, consider the free-body diagram shown in the following figure.



The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta}=V\,l\sin\,\theta-H\,l\cos\,\theta \tag{1}$$

where I is the moment of inertia of the rod about its center of gravity. The horizontal motion of center of gravity of pendulum rod is given by

$$m\,\frac{d^2}{dt^2}(x+l\sin\theta)=H \tag{2}$$

The vertical motion of center of gravity of pendulum rod is

$$m\,\frac{d^2}{dt^2}(l\,\cos\,\theta)=V-mg \tag{3}$$

The horizontal motion of cart is described by

$$M\,\frac{d^2\,x}{dt^2}=u-H \tag{4}$$

Equations (1) – (4) describe the motion of inverted pendulum-on-the cart system. Because these equations involve $\sin\theta$ and $\cos\theta$, they are nonlinear equations.

Since, we are assuming that the mass of the pendulum rod is concentrated at its center of gravity, we take $I=0$. Since our objective is to balance the pole at its top vertical position, we linearize the system around $\theta=0$. For small θ, $\sin\theta\approx\theta$ and $\cos\theta\approx 1$. For small θ, we can linearize the equations (1) – (4) as follows:

$$I\,\ddot{\theta}=V\,l\,\theta-H\,l \tag{5}$$

$$m\,(\ddot{x}+l\,\ddot{\theta})=H \tag{6}$$

$$0 = V - mg \tag{7}$$

$$M \ddot{x} = u - H \tag{8}$$

From equations (6) and (8), we get

$$(M + m) \ddot{x} + m l \ddot{\theta} = u \tag{9}$$

From equations (5) and (7), we get

$$(I + m l^2) \ddot{\theta} + m l \ddot{x} = mgl \theta \tag{10}$$

Equations (9) and (10) may be put in the form of state space model (assuming $I = 0$) as follows:

$$\dot{x} = A x + B u$$

or

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m) g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{bmatrix} u \tag{11}$$

u is the force applied to the cart in N. It can be expressed in terms of input voltage V_i as follows:

$$u = \frac{K_m K_g}{Rr} V_i - \frac{K_m^2 K_g^2}{Rr^2} \dot{x} \tag{12}$$

Substituting u in (11) from (12), we get

$$\begin{array}{c} 0 \\ 1 \end{array}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & g \\ \frac{-K_{\text{¿}}^{2K}\,2}{Rr^{2M}} & \frac{-mg}{M} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ g & \frac{K_{\text{¿}}^{2K}\,2}{Rr^{2Ml}} & \frac{(M+m)\,g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_m K_g}{MRr} \\ 0 \\ \frac{-K_m K_g}{MlRr} \end{bmatrix} V_i \qquad (13)$$

Substituting the values of system parameters, we get state matrix A and input matrix B as follows:

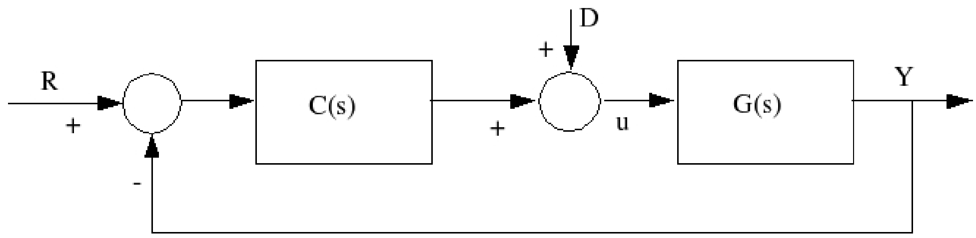
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -17.02 & -4.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 53.08 & 44.6 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 3.8 \\ 0 \\ -11.85 \end{bmatrix}$$

Problems

1. Consider the linear model (13). Compute the transfer functions $G_1(s) = \frac{\Theta(s)}{V_i(s)}$ and $G_2(s) = \frac{X(s)}{V_i(s)}$ where θ and x are angular position of pendulum and linear position of cart respectively.

2. Design a lead compensator to control the angular position θ using root locus technique such that the root locus of compensated system passes through dominant poles $s = -1.5 \pm 1.5i$.

   - Plot the root locus of open loop system.
   - Plot the root locus of compensated system.
   - Plot the response of the compensated system for a step disturbance at the control input as shown in the following diagram.



3. Find the state feedback gain vector K such that the control input $u = -K\,x$ stabilizes the system around its equilibrium point and minimizes the following cost function:

$$J = \int (x'\,Qx + r\,V_i^2)\,dt$$

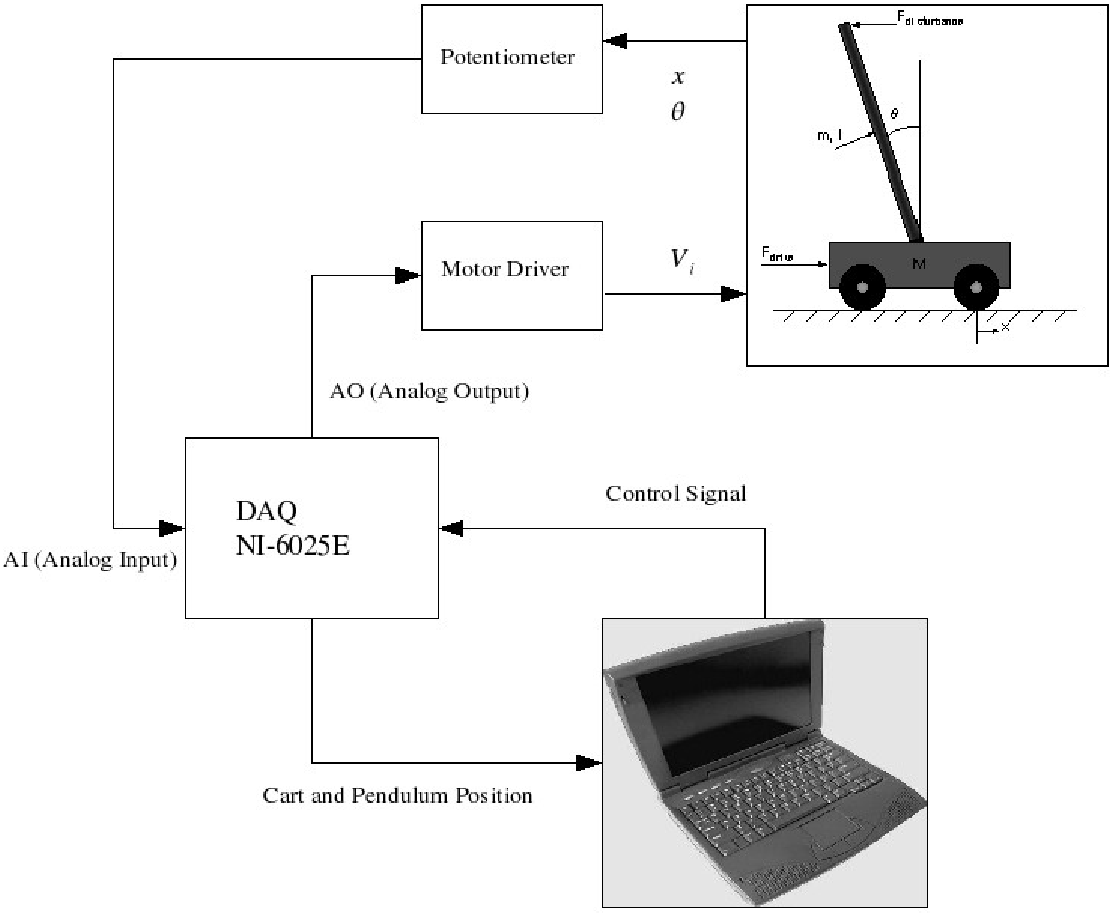
where

$$Q = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad r = 0.0005$$

The resulting closed loop system is termed as linear quadratic regulator (LQR).

- Plot the time response of the closed system for initial condition $x_0 = [0.2 \;\; 0 \;\; 0.2 \;\; 0]$
- Also plot the control input.

4. Interface the cart pole system to a personal computer using a data acquisition card as per the block diagram shown below. Implement the LQR controller designed in Problem 3 in real time. Collect real time data for control input, cart position and pole angular position and plot them.



System Specifications

DC servomotor Rated Voltage : 6V

DC servomotor maximum continuous current: 1 A

Cart Potentiometer sensitivity: 0.0783 m/V

Pole Potentiometer sensitivity: 0.325 rad/V

Potentiometer bias voltage: ± 12 V

Cart Potentiometer measurement range: ± 5 V

Following Matlab Commands would be helpful for performing this experiment. Please go through on-line help for more details.

```
margin, rlocus, feedback, step, lqr, ss2tf, tf,
ode45, plot, function, pole, zero, pzmap.
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