Contraction Theory : A Tool for Design and Analysis of Nonlinear Systems

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Contraction Analysis/Theory or simply Contraction

- Given a system or multiple systems
- Questions: Does the solution or trajectory converges

\[ \dot{x} = f_i(x, t), i = 1, 2, \ldots, r. \]

- Introduction
  - Concepts, Definition
  - Applications to example problems
- Design analysis of frequency estimator
- Synchronisation of a network of nonlinear systems
Contraction analysis/Theory or simply contraction

- Analyze the convergence of nonlinear systems trajectories with respect to each other
  - Property regarding the convergence between two arbitrary system trajectories.
  - If initial conditions or disturbances are forgotten exponentially fast.

- Nonlinear system
  \[ \dot{x} = f(x, t) \]

  where \( x \in \mathbb{R}^m \); \( f : (m \times 1) \) vector function.

- First variation in \( x \)
  \[ \delta \dot{x} = \frac{\partial f(x, t)}{\partial x} \delta x \]

Virtual dynamics of two neighboring trajectories
• Time derivative of squared distance

\[
\Rightarrow \frac{d}{dt}(\delta \dot{x}^T \delta \dot{x}) = 2 \delta \dot{x}^T \delta \ddot{x} = \frac{\partial f}{\partial x} \delta \dot{x} \leq 2 \lambda_m(x, t) \delta \dot{x}^T \delta \dot{x}
\]

If \( \lambda_m(x, t) \) is negative definite in nature, then all solution trajectories of system converge together exponentially

• Contraction Region

For system \( x = f(x, t) \), a region of state space is called a contracting region if the Jacobian \( \frac{\partial f}{\partial x} \) is uniformly negative definite (UND) in that region.

• UND of a Jacobian \( \frac{\partial f(x, t)}{\partial x} \) means \( \exists \alpha > 0, \forall x, \forall t \geq 0 \text{ s.t. } \frac{\partial f}{\partial x} \leq -\alpha I < 0 \)

\[
\Rightarrow \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T \right) \leq -\alpha I < 0
\]

Important features

➤ Utilizes differential framework. A type of incremental stability method
➤ No issues like selection of energy function
➤ Establishes exponential stability of systems
➤ Handles synchronization problems effectively
Generalisation of the convergence analysis

• Coordinate transformation \( \delta z = \theta \delta x \)

\( \theta(x, t) : \) uniformly invertible matrix. Squared distance between trajectories is

\[ \delta z^T \delta z = \delta x^T M \delta x \]

where \( M(x,t) = \theta^T(x,t)\theta(x,t) \) is a uniformly positive definite matrix.

• Time derivative of weighted square length

\[
\Rightarrow \frac{d}{d} (\delta z^T \delta z) = 2 \delta z^T \dot{\delta z} = 2 \delta z^T (\dot{\theta} + \theta \frac{\partial f}{\partial x}) \theta^{-1} \delta z
\]

• Generalized Jacobian matrix \( F \) is defined as

\[
F = (\dot{\theta} + \theta \frac{\partial f}{\partial x}) \theta^{-1}
\]

If matrix \( F \) is U.N.D., exponential convergence of \( \delta Z \) to zero is guaranteed.
**Results:**

For the system in (1), if there exists a uniformly positive definite matrix $M(x, t)$ such that the associated generalized Jacobian matrix

$$ F = (\dot{\theta} + \theta \frac{\partial f}{\partial x})\theta^{-1} $$

is UND, then all system trajectories converge exponentially to a single trajectory with convergence rate $|\tilde{\lambda}_m(x, t)|$ where $|\tilde{\lambda}_m(x, t)|$ is the largest eigenvalue of the symmetric part of $F$. Then the system is said to be contracting.

• Intuitively, if the temporal evolution of a virtual displacement tends to 0 as time goes to infinity, this being true for all state $x$ and at all time, the whole flow will shrink to a point, hence the term contraction.
Results

Given system $\dot{x} = f(x,t)$ any trajectory which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to the given trajectory. Further, global exponential convergence to this given trajectory is guaranteed if the whole state space region is contracting.

Contraction of two neighboring trajectories.
Contraction theory and Krasovaskii’s Stability Analysis

- For n-dimensional continuous system having dynamics as
  \[ \dot{x}(t) = f(x(t), t) \]
  \[ \Rightarrow \frac{d}{dt} \delta x(t) = \frac{\partial f}{\partial x}(x(t), t)\delta x(t) = J(x(t), t)\delta x(t) \]

- Transformation is \( \delta z = \Theta(x(t), t) \), where \( \Theta(x(t), t) : (n \times n) \) invertible matrix.
  \[ \frac{d}{dt} \delta z(t) = F(x(t), t)\delta z(t) \]

- Contraction of differential system means exponential stability of actual system.
- Select of Lyapunov function \( V(\delta x(t), t) = \delta x^T(t)M(x(t), t)\delta x(t) \)
- If \( \dot{V}(\delta x(t), t) \leq 2\alpha V(\delta x(t), t) \) it yields contraction condition \( F \leq \alpha I \) with \( \alpha < 0 \).
- If function \( f \) is autonomous & \( M \) is constant, then Krasovaskii’s sufficient condition for asymptotic stability is equivalent to contraction of such systems.
Contraction applied to stability analysis

Actual system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-1 & x_1 \\
-x_1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[\text{Lyapunov function: } V = \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[\dot{V} = -\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < 0 \Rightarrow \text{Easy to conclude}\]

Virtual system

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
-1 & x_1 \\
-x_1 & -1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
\]

\[\text{Jacobian } J = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} < 0\]

• The y-system is contracting and has two particular solutions, namely

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} \text{ and } \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[\Rightarrow x_1 \text{ and } x_2 \text{ will both tend to 0 exponentially}\]
Actual & Virtual System Concept

**Actual System**  
\[ \dot{x} = -\rho(x, p, t)x \]  
where \( x \in \mathbb{R}^{m \times 1} \): state vector, \( p \): parameter vector, \( \rho(x, p, t) \geq \alpha I \), \( I > 0 \).

**Virtual System**

To show exponential stability, define new system as
\[ \dot{\delta y} = -\rho(x, p, t)\delta y \]  
Let \( \delta y \) is virtual differential increments in \( y \), then

As \( \rho(x, p, t) \geq \alpha I > 0 \), virtual system is contracting. Hence, actual system states also converge together, exponentially

- Define virtual system
- Establish contracting nature of virtual system.
- It leads to exponential convergence of virtual system states to actual system states.
Simple Example

Lorenz chaotic system is

\[
\begin{align*}
\dot{x}_s &= -px_s + py_s \\
\dot{y}_s &= -x_s z_s - y_s \\
\dot{z}_s &= -x_s y_s - z_s - (R_0 + u)
\end{align*}
\]

Where \( R = R_0 + u, \ p > 0 \). With coordinate change \( \bar{z}_s = z_s - p \)

\[
\begin{bmatrix}
\dot{x}_s \\
\dot{y}_s \\
\dot{z}_s
\end{bmatrix} =
\begin{bmatrix}
-p & p & 0 \\
-p & -1 & -x_s \\
0 & x_s & -1
\end{bmatrix}
\begin{bmatrix}
x_s \\
y_s \\
\bar{z}_s
\end{bmatrix} -
\begin{bmatrix}
0 \\
0 \\
p + R_0 + u
\end{bmatrix}
\]

Taking simple controller \( u = -(p + R_0) \) system becomes

\[
\begin{bmatrix}
\dot{x}_s \\
\dot{y}_s \\
\dot{z}_s
\end{bmatrix} =
\begin{bmatrix}
-p & p & 0 \\
-p & -1 & -x_s \\
0 & x_s & -1
\end{bmatrix}
\begin{bmatrix}
x_s \\
y_s \\
\bar{z}_s
\end{bmatrix}
\]
Selecting virtual system as

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
-p & p & 0 \\
-p & -1 & -x_s \\
0 & x_s & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

Actual system

\[
\begin{bmatrix}
\dot{x}_s \\
\dot{y}_s \\
\dot{z}_s
\end{bmatrix} =
\begin{bmatrix}
-p & p & 0 \\
-p & -1 & -x_s \\
0 & x_s & -1
\end{bmatrix}
\begin{bmatrix}
x_s \\
y_s \\
z_s
\end{bmatrix}
\]

Dynamics in differential framework will be

\[
\begin{bmatrix}
\delta \dot{x} \\
\delta \dot{y} \\
\delta \dot{z}
\end{bmatrix} =
\begin{bmatrix}
-p & p & 0 \\
-p & -1 & -x_s \\
0 & x_s & -1
\end{bmatrix}
\begin{bmatrix}
\delta x \\
\delta y \\
\delta z
\end{bmatrix}
\]

The symmetric part of Jacobian matrix comes out to be

\[
J_s =
\begin{bmatrix}
-p & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

It is UND, so system is exponentially stabilized to \((0 \ 0 \ p)^T\)
Contraction Theory for Observers

System-I

System-II
Contraction Theory for Observers

Contraction theory based analysis is used for:

• Design of observers/estimators.

• Actual System
  \[ \dot{x}_s = x_s + u \]

• Observer for the System
  \[ \dot{\hat{x}} = \hat{x} + u + L(x_s - \hat{x}) \]

  \( u = \)control input, \( L = \)gain selected s.t. \( L > 1 \).

• Define virtual system as
  \[ \dot{x} = x + u + L(x_s - x) \]

• Establish contracting nature of virtual system.
• It leads to exponential convergence of observer states to actual system states.
Numerical Example

**Actual system**
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_1 - 3x_2 + u
\end{align*}
\]

**Observer**
\[
\begin{align*}
\dot{y}_1 &= y_2 + L(x_1 - y_1) \\
\dot{y}_2 &= -2y_1 - 3y_2 + u
\end{align*}
\]

**Virtual System**
\[
\begin{align*}
\dot{z}_1 &= z_2 + L(x_1 - z_1) \\
\dot{z}_2 &= -2z_1 - 3z_2 + u
\end{align*}
\]

**Jacobian of virtual system**
\[
J = \begin{bmatrix}
-L & 1 \\
-2 & -3
\end{bmatrix}
\]

**Symmetric part of Jacobian**
\[
J_s = \frac{1}{2}(J + J^T) = \begin{bmatrix}
-L & -0.5 \\
-0.5 & -3
\end{bmatrix}
\]

**Js to be UND, if** \( L > 1/12 \)
Figure: (a)-(b) Convergence of actual and observer system states and (c) Estimation error variation.

\[ x(0) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T, \quad y(0) = \begin{bmatrix} 0.4 & -0.2 \end{bmatrix}^T \]

\[ z(0) = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T \]

\[ u = 0.5 + 0.1 e^{-0.1t} \sin(t) \quad \text{&} \quad L = 1/6 \]
A measurable sinusoidal signal involving multiple frequencies

\[ y(t) = \sum_{i=1}^{n} \left[ A_{1i} \sin(\alpha_i t + \theta_{1i}) + A_{2i} \cos(\alpha_i t + \theta_{2i}) \right] \]

\( A_{1i}, A_{2i} \) : Non-zero amplitudes; \( \theta_{1i}, \theta_{2i} \) : phases and \( \alpha_i \) : Non-zero frequencies s.t.

\[ \alpha_i \neq \alpha_j \ \forall \ i \neq j \]

- Signals are mostly nonlinear functions of frequency.
- Dynamic estimator is proposed using contraction theory.
- Provides simultaneous online estimation of frequencies.
- Designing asymptotic frequency estimators.
– **Design Steps for Estimator:**

– Select a dynamical system to generate given signal.

– Design a auxiliary system by evolving unknown frequencies as new states. Design a suitable estimator for new system considering the measurement of signal.

– Apply contraction theory to select the gains of estimator so as to achieve asymptotic stability.

– Compute the frequency estimates from the steady state value of the additional states.
Frequency Estimator Design

Sinusoid with two Unknown Frequencies

\[ y(t) = \sum_{i=1}^{2} [A_i \sin(\alpha_i t + \theta_{1i}) + A_{2i} \cos(\alpha_i t + \theta_{2i})] \]

Amplitudes \( A_{1i}, A_{2i} \) and phases \( \theta_{1i}, \theta_{2i} \) are known. \( \alpha_1 & \alpha_2 \) : Unknown frequencies.

A dynamical system generating given sinusoid

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= -c_1 \xi_1 - c_2 \xi_2
\end{align*}
\]

Parameters \( c_i \) are the coefficients of the characteristic polynomial \( q(s) = s^4 + c_2 s^2 + c_1 \)

where \( c_1 = \prod_{i=1}^{2} \alpha_i^2; c_2 = \sum_{i=1}^{2} \alpha_i^2 \)

Output of dynamical system

\[ y = k_1 \xi_1 + k_2 \xi_2 + k_3 \xi_3 + k_4 \xi_4 \]

Question: Estimate \( c_1 & c_2 \)
Frequency Estimator Design

Dynamical system

\[
\begin{align*}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \zeta_3 \\
\dot{\zeta}_3 &= \zeta_4 \\
\dot{\zeta}_4 &= -\zeta_5 \zeta_1 - \zeta_6 \zeta_3 \\
\dot{\zeta}_5 &= 0 \\
\dot{\zeta}_6 &= 0
\end{align*}
\]

Proposed estimator structure

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 + g_1 (y - \hat{y}) \\
\dot{x}_4 &= -x_5 x_1 - x_6 x_3 + g_2 (y - \hat{y}) \\
\dot{x}_5 &= -g_3 x_1 (y - \hat{y}) \\
\dot{x}_6 &= -g_4 x_3 (y - \hat{y})
\end{align*}
\]

Output of estimator

\[
\hat{y} = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4
\]

Simulations

Two frequency sinusoid

\[ y = A[\sin(\alpha_1 t) + \sin(\alpha_2 t)] \]

Amplitude \( A = 10 \); unknown frequencies \( \alpha_1 = 1 \) & \( \alpha_2 = 2 \)

- Initial conditions \( x(0) = [1 \ 0.5 \ 0.5 \ 1 \ 0 \ 0]^T \).
- Gains \( g_1 = 2, g_2 = 2 \).
- Select \( k_i \) for \( i = 1,2,3 \) s.t. polynomial is \( p(s) = s^3 + g_1 k_1 s^2 + g_1 k_2 s + g_1 k_3 \) stable.
- Select \( k_4 \) s.t. \( k_4 g_3 = 1, k_4 g_4 = 1, k_4 g_1 = 1 \) and \( k_4 g_2 > 0 \).
Simulations

Figure: Two frequencies estimation case: (a) Phase portrait in three dimensions, (b) Variation of state $x_4$, (c) Variation of states $x_5$ & $x_6$ and (d) Signal estimation error.
Synchronization of Networks

- A network of \( N \) identical nonlinear systems (may be oscillators)
- Coupling can be either linear or nonlinear function of all or some of the system states
- Potential application running from coordination problems in robotics to mechanisms for secure communications. Important in the area of biology, robotics, complex systems in nature and technology etc.
- Synchronization in the context of networks means *states of each and every node should match to each other*.
- The network is said to be synchronised if all the systems converge toward the same synchronous state.
- Find conditions to guarantee the synchronisation of a network of identical systems
Different Configurations

- One way coupling: signal flow is permitted in one direction only.
- Master-Slave synchronisation
- Transmitter-receiver synchronisation (secure communication)
Different Configurations

- One way coupling: signal flow is permitted in one direction only.
- Two way (Bidirectional coupling means signal flow is permitted in both the directions)
- Chain network have systems (one way or two way) connected in cascade.
- Ring network have systems (one way or two way) connected in cascade and forming a closed loop.
- Networks with all-all topology
- Networks with arbitrary topology
Partial Contraction

For a nonlinear system with dynamics

\[ \dot{x} = f(x, x, t) \]

let an auxiliary system (Observer-like system) is defined as

\[ \dot{y} = f(y, x, t) \]

If this auxiliary system is contracting w.r.t. \( y \), and verifies a smooth specific property, then all trajectories of the original \( x \)-system verify this property, exponentially. The original system is said to be partially contracting

- Specific Properties can be matching of states, convergence to equilibrium points, convergence to manifolds etc.
Example

\[ \dot{x}_1 = x_2 - x_1^7 \left( x_1^4 + 2x_2^2 - 10 \right) \]
\[ \dot{x}_2 = -x_1^3 - 3x_2^5 \left( x_1^4 + 2x_2^2 - 10 \right) \]

\[ \Rightarrow \frac{d}{dt} \left( x_1^4 + 2x_2^2 - 10 \right) = -\left( 4x_1^{10} + 12x_2^6 \right) \left( x_1^4 + 2x_2^2 - 10 \right) \]

y-system: \[ \dot{y} = -\left( 4x_1^{10} + 12x_2^6 \right) y \]

- The y-system is contracting as long as not both \( x_1(t) \) and \( x_2(t) \) equal to 0.

- The y-system has two particular solutions \( y = x_1^4 + 2x_2^2 - 10 \) and \( y = 0 \)

\[ \Rightarrow \] The system stays at the origin if \( x_1(0) = x_2(0) = 0 \). Otherwise, it tends to reach \[ x_1^4(t) + 2x_2^2(t) = 10. \]
Construction of auxiliary / virtual system

- Depends on state variables of nonlinear system and some virtual state variables
- The substitution of the i-th system state variables into the virtual state variables returns the dynamics of the i-th system of the network
- Contraction property with respect to these virtual state variables immediately implies synchronisation

**Two coupled nonlinear systems**
\[
\begin{align*}
\dot{z} &= f(z) + h(w) - h(z) \\
\dot{w} &= f(w) + h(z) - h(w)
\end{align*}
\]

**Virtual systems**
\[
\begin{align*}
\dot{x} &= f(x) - 2h(x) + h(z) + h(w) = \phi(x, z, w)
\end{align*}
\]
- Trajectories of the nodes are particular solutions (x-solutions) of the virtual system.
  \[
  \phi(z, z, w) = f(z) + h(w) - h(z), \quad \phi(w, z, w) = f(w) + h(z) - h(w)
  \]
- If the virtual system is contracting with respect to the x state variable, the two particular solutions, namely z and w, will converge to each other. Synchronisation is then attained

**Jacobian of virtual system**
\[
J = \frac{\partial f(x)}{\partial x} - 2 \frac{\partial h(x)}{\partial x}
\]
Important Features

• New method to study dynamic behavior of coupled nonlinear oscillators/systems.
• Extends contraction analysis to include the convergence to specific properties.
• Specific Properties can be matching of states, convergence to equilibrium points, convergence to manifolds etc.
• Provides general stability analysis framework for large complex systems.
• Powerful tool to study synchronization behavior.
• Synchronization results are global in nature.
Two systems in one way coupled form

- **Master System** \( \dot{x} = f(x,t) \)

- **Slave System** \( \dot{y} = f(y,t) + kC(x - y) \)
  \( C \) = coupling matrix of dimension
  \( k \) = scalar gain to be selected

**Virtual System** \( \dot{z} = f(z,t)z + kC(x - z) \)

- Selection of suitable gain \( k \) ensures UND nature of \( J_s \).
- As virtual system in (44) is particular solution of (45), so UND nature of (45) ensures exponential convergence of system in (45).
- As both master and slave are particular solutions of the virtual system (44), their corresponding states will converge to each other, exponentially.
Two Systems with Bidirectional Coupling

Dynamics of two bidirectional coupled systems

\[
\begin{align*}
\dot{x}_1 &= f(x_1, t) + kC[(x_2 - x_1)] \\
\dot{x}_2 &= f(x_2, t) + kC[(x_1 - x_2)]
\end{align*}
\]

\(x_1, x_2 \in \mathbb{R}^n; C: (n \times n) \text{ coupling matrix}\)

\(k: \text{ coupling gain to be designed}\)

virtual system

\[
\dot{\varphi} = f(\varphi, t) - 2kC\varphi + kC(x_1 + x_2)
\]
Network with all-to-all coupling topology

Dynamics of network with all-to-all coupling

\[
\dot{x}_1 = f(x_1, t) + kC[(x_2 - x_1) + (x_3 - x_1) + \ldots + (x_n - x_1)] \\
\dot{x}_2 = f(x_2, t) + kC[(x_1 - x_2) + (x_3 - x_2) + \ldots + (x_n - x_2)] \\
\vdots \\
\dot{x}_n = f(x_n, t) + kC[(x_1 - x_n) + (x_2 - x_n) + \ldots + (x_{n-1} - x_n)]
\]

\[x_i \in \mathbb{R}^n; k: \text{common coupling gain}; \]
\[C: (n \times n) \text{ coupling matrix.}\]

Virtual System

\[
\dot{\phi} = f(\phi, t) - knC\phi + kC(x_1 + x_2 + \ldots + x_n)
\]
Representative example: Lorenz chaotic system

\[
\begin{align*}
\dot{x}_1 &= -sx_1 + sx_2 \\
\dot{x}_2 &= rx_1 - x_1x_3 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3
\end{align*}
\]

Chaotic behaviour for \( r = 30, s = 10 \) & \( b = 8/3 \).

• Five such systems are interconnected through single variable
• Initial conditions of are different
• At \( t=2 \) sec, coupling gains were switched
Figure: Synchronization of five chaotic Lorenz systems with all to all bidirectional coupling through single variable only ($c_{11} = 1; c_{22} = 0; c_{33} = 0$): (a), (b) & (c) Variation of state trajectories of different system.
Figure: Synchronization of five chaotic Lorenz systems with nearest neighbour bidirectional coupling through single variable only ($c_{11} = 1; c_{22} = 0; c_{33} = 0$): (a), (b) & (c) Variation of state trajectories of different system.
Synchronization of Networks

Key Observations

• Synchronization of networks can be achieved quite easily using virtual system concept.
• Stability results are global in nature as compared to the traditional MSF based stability.
• Provides a common platform to address synchronization of networks in different configurations.

Applications

• Observers/Estimators in under actuated surface vessels, robotic, applications
• Cooperative control and target tracking in multi-agent systems.
• Analysis of networked systems with group leaders
• Group cooperation analysis in animal aggregation applications like bird flocks, fish schools etc.
• Synchronization in complex networks like pacemaker cells in heart, neural networks in brain, synchronized chirping etc.
References:


References:


Thank you