Greed Considered Harmful
Nonlinear (in)stabilities in network resource allocation

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Outline

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• Model & Motivation
• Main results
  – Fixed delays
    • Single-user, single-link case
    • Feedback based on instantaneous rate
    • Feedback based on averaged rate
  – State-dependent time-varying delay – single-link case
  – Distributed delays-closed form solution for Gamma Kernel
  – Discrete delays with feedback based on average

• Conclusions
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Motivation- Kelly’s rate-based primal algorithm

- **Rate-based flow control**: each resource charges a price per unit flow (derivative of cost)
  \[ \mu_j(t) := p_j \left( \sum_{i : j \in r_i} x_i(t) \right) \]

- users update their rates according to
  \[ \frac{d}{dt} x_i(t) = \kappa \left( w_i(t) - x_i(t) \cdot \sum_{j \in r_i} \mu_j(t) \right) \]

where

- \( w_i(t) \) - user \( i \)’s willingness to pay at time \( t \)
  (set to \( x_i(t) \cdot U'_i(x_i(t)) \))

- \( x_i(t) \) - user \( i \)’s rate at time \( t \)
The sender fragments a file into groups of bits called packets.

The sender attaches a sequence number to packets and waits for ACKs from the receiver. The receiver sends back an ACK when it receives a packet.

The sender places another packet on the network when it receives an ACK.
Background of rate control in the Internet

Question: How do we share resources fairly and efficiently?
Network users have a great deal of freedom as to how they can share the available bandwidth in the network.

The increasing complexity and size of the Internet renders a centralized rate allocation impractical. A \textbf{distributed algorithm} is desired.

Two classes of flow/congestion control mechanisms:

- \textbf{rate-based}: directly controls the transmission rate based on feedback (example: PFCC)
- \textbf{window-based}: controls the congestion window size to adjust the transmission rate and backlog (example: Transmission Control Protocol (TCP))
Model – Optimization Framework (Kelly 1997)

- Network with a set of users \( I \) and a set of links \( J \)
- User receives a utility \( U_i(x) \) when it gets a rate \( x \)
  - increasing and strictly concave
- \( C_j \) - capacity of link \( j \in J \)
- \( r_i \) - route of user \( i \in I \) (\( \subset J \))
- \( A \) - 0-1 matrix where

\[
A_{ij} = \begin{cases} 
1 & \text{if } j \in r_i \\
0 & \text{otherwise}
\end{cases}
\]
Model - Example

- $r_1 = \{1, 2, 4, 5\}$
- $r_2 = \{3, 2, 6, 7\}$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
Kelly’s Model

- **Optimization framework**

\[
\text{SYSTEM}(U, A, C)
\]

\[
\begin{align*}
\max & \quad \sum_{i \in I} U_i(x_i) \\
\text{s.t.} & \quad A^T x \leq C \\
\text{over} & \quad x \geq 0
\end{align*}
\]

- **impractical** for a centralized system to compute and allocate the user rates
Two parts of original model

- $USER(U_i; \lambda_i)$

$$\begin{align*}
\max & \quad U_i \left( \frac{w_i}{\lambda_i} \right) - w_i \\
\text{s.t.} & \quad w_i \geq 0
\end{align*}$$

- $NETWORK(A, C; w)$

$$\begin{align*}
\max & \quad \sum_{i \in I} w_i \cdot \log(x_i) \\
\text{s.t.} & \quad A^T x \leq C \\
\text{over} & \quad x \geq 0
\end{align*}$$
One can always find vectors $\lambda^* = (\lambda^*_i, i \in I)$, $w^* = (w^*_i, i \in I)$ and $x^* = (x^*_i, i \in I)$ that

1) $w^*_i$ solves $USER(U_i; \lambda^*_i)$ for all $i \in I$

2) $x^*_i$ solves $NETWORK(A, C; w^*)$

3) $w^*_i = \lambda^*_i \cdot x^*_i$

4) $x^*$ is the unique solution to $SYSTEM(U, A, C)$

At the equilibrium between NETWORK and USER problems the system optimum is achieved.
Motivation - Kelly’s rate-based primal algorithm

- Rate-based flow control:
  - each resource charges a price per unit flow (derivative of cost)
    \[ \mu_j(t) := p_j \left( \sum_{i : j \in r_i} x_i(t) \right) \]
  - users update their rates according to
    \[ \frac{dx_i(t)}{dt} = \kappa \left( w_i(t) - x_i(t) \cdot \sum_{j \in r_i} \mu_j(t) \right) \]

where

- \( w_i(t) \) - user \( i \)'s willingness to pay at time \( t \)
  (set to \( x_i(t) \cdot U_i'(x_i(t)) \))

- \( x_i(t) \) - user \( i \)'s rate at time \( t \)
Motivation- Kelly’s rate-based algorithm

- **Theorem (Kelly et al. 98):** User rates converge to a point \( x^* \) that maximizes the following expression

\[
U(x) = \sum_{i \in I} U_i(x_i) - \sum_{j \in J} \int_0^{\sum_{i : j \in r_i} x_i} p(y) dy
\]

- Proposed rate-based (primal) algorithm solves a relaxation of the \text{SYSTEM(U, A, C)} problem

**But, wait! Where is the delay?!?!?!**

\[
\frac{d}{dt} x_i(t) = \kappa \left( w_i(t) - x_i(t) \cdot \sum_{j \in r_i} \mu_j(t) \right)
\]
Delays between network elements ignored in Kelly’s original formulation!

- Especially important in wireless environment, (e.g., multi-hop wireless networks and satellite networks)
Motivation

- Delays in a network vary widely and are hard to predict
  - Uncertainty in delays (e.g., multi-hop wireless networks)
  - Large variation (e.g., wireline vs. satellite connections)

- **Robustness** is critical in a large, evolving system
  - Stability of the system independent of the delays or gain parameters ($\kappa$) of the algorithm

  => Improves the portability of the protocol from one technology to another (e.g., cellular to satellite to military networks)
Model with Fixed DELAYS

- Delays between network elements \((\text{FIXED} \text{ for now})\)

  - \(T_{i,j}\) - feedback delay from resource \(j\) to user \(i\)
  
  - \(Z_{i,j}\) - delay of packets from the user \(i\) to resource \(j\)
  
  - \(T_i = T_{i,j} + Z_{i,j}\)

End user algorithm with delays

\[
\frac{d}{dt}x_i(t) = \kappa_i \left( x_i(t)U'_i(x_i(t)) - x_i(t - T_i)\left(\sum_{j \in r_i} \mu_j(t - T_{i,j})\right) \right)
\]

where

\[
\mu_j(t - T_{i,j}) = p_j \left( \sum_{l:j \in r_i} x_l(t - (T_{i,j} + Z_{l,j})) \right)
\]
Single-User, Single-Link Case

- **End user algorithm (no forward delay)**

  \[ \frac{d}{dt}x(t) = \kappa \left( x(t)U'(x(t)) - x(t - T)p(x(t - T)) \right) \]

- Normalize time by \( T \)

  \[ \nu \frac{d}{ds}x(s) = x(s)U'(x(s)) - x(s - 1)p(x(s - 1)) \]

  where \( \nu = \frac{1}{T\kappa} \)
• End user algorithm (no forward delay)

\[
\frac{d}{dt} x(t) = \kappa \left( x(t) U'(x(t)) - x(t - T)p(x(t - T)) \right)
\]

- Normalize time by \( T \)

\[
\nu \frac{d}{ds} x(s) = x(s) U'(x(s)) - x(s - 1)p(x(s - 1))
\]

where \( \nu = \frac{1}{T\kappa} \)
Single-User, Single-Link Case

Sender → Forward Delay D1 → Buffer → Server

Reverse Delay D2

Generic Network Feedback Model
Single-User, Single-Link Case

• Define

\[ y(t) = x(t)U'(x(t)) := g(x(t)) \]

and

\[ f(x(t)) := x(t)p(x(t)) \]

• Change of coordinates

\[ \nu \dot{y}(t) = \kappa(y(t)) \left( f(g^{-1}(y(t-1))) - y(t) \right) \quad (1) \]

where

\[ \kappa(y(t)) := -g'(g^{-1}(y(t))) \]
Assumption: (i) The function $g(x)$ is strictly decreasing with $-g'(x) > 0$ for all $x > 0$,

(ii) the function $f(x)$ is strictly increasing for all $x > 0$, and

(iii) both $g(x)$ and $f(x)$ are Lipschitz continuous on $[\epsilon, I \cdot B_{max}]$, where $\epsilon$ is an arbitrarily small positive constant.
Close correspondence between invariance and global stability properties of the discrete time map

\[ y_{n+1} = f(g^{-1}(y_n)) := F(y_n) \]

and those of (1)

- Underlying market structure
- \( g^{-1}(y_n) \) - rates at the equilibrium at given \( y_n \)
- \( f(x_n) \) - prices given \( x_n \)
Invariance and Stability...

**Assumption:** Suppose that $I \subset \{x : x \geq \epsilon\}$ is a closed invariant interval under $F$, i.e., $F(I) \subset I$. In particular, let $I = [a, b]$ be compact.

**Theorem (Invariance)**
If $\phi \in X_I := \{\phi \in C([-1, 0], \mathbb{R}_+) : \phi(s) \in I \ \forall \ s \in [-1, 0]\}$, the corresponding solution $y(t) = y(t; \phi)$ satisfies $y(t) \in I$ for all $t \geq 0$. This means that the set $I$ is invariant under (1).

**Assumption:** The discrete time map $F$ has an attracting fixed point $y^* = g(x^*)$ with immediate basin of attraction $J_0 : F^n y_0 \to y^*$ for any $y_0 \in J_0$.

**Theorem (Stability)** For any $\nu > 0$ and $\phi \in X_{J_0} := C([-1, 0], J_0)$, we have $\lim_{t \to \infty} y_\phi(t) = y^* = g(x^*)$. 
Example

- The utility function given by

\[ U(x) = \frac{-1}{a \cdot x^a}, \quad a > 0 \]

  \[ \frac{p}{x(p)} \frac{dx^*(p)}{dp} = \frac{p}{p^{-\frac{1}{1+a}} \left(1 + a\right)^{-1}} \cdot \frac{-1}{1 + a} \left(1 + a\right)^{-1} \]

  \[ = \frac{-1}{1 + a} \]

- Users with smaller \( a \) are GREEDIER
Example

- Price function given by
  \[ p(x) = c \cdot \left( \frac{x}{C} \right)^b, \quad b > 0 \]
- Marking function imitating an M/M/1 queue

**Theorem** The rate \( x(t) \) converges to \( x^* \) for all \( \nu \) if \( a > b + 1 \).

\[
\left. \frac{d}{dy} F(y) \right|_{y^*} = \left. \frac{df}{dg^{-1}(y)} \right|_{y^*} \left. \frac{dg^{-1}(y)}{dy} \right|_{y^*}
\]

\[
= (b + 1) \left( \frac{g^{-1}(y)}{C^b} \right)^b \cdot \left( -\frac{1}{a} \right) y^{-\frac{1+a}{a}} \quad y^* = C^{\frac{a}{a+b+1}}
\]

\[
= -\frac{b + 1}{a}
\]

Fundamental trade-off between the elasticity of user demand and responsiveness of resources
Discrete-time Map

Numerical Simulation: 2 homogeneous users $a=3$, $b=5$, and $C=5$, Rate control algorithm is unstable since $6/3=2>1$.

$$x_{n+1} = F(x_n)$$

First Return Map

Second Return Map

$45^\circ$ line
Instability: effect of greed

Plot of $x(t)$ ($b = 5$, $T = 200$, $k = 0.2$)

- $a = 2$
- $a = 6.1$
- $a = 10$
Instability: effect of delay

Numerical Simulation: 2 homogeneous users $a=3$, $b=5$, and $C=5$. Unstable since $6/3 = 2 > 1$. $T=1$
Instability: effect of delay

Numerical Simulation: 2 homogeneous users $a=3$, $b=5$, and $C=5$. Unstable since $6/3=2 > 1$. $T=10$
Instability: effect of delay

Numerical Simulation: 2 homogeneous users $a=3$, $b=5$, and $C=5$, Unstable since $6/3=2>1$. $T=50$
Instability: Period Doubling, What happens if \( a < b+1 \)?

**Linear Instability Analysis**

Assume that the map \( F \) is locally smooth.

\[
\begin{align*}
\mathbf{z}'(t) &= \kappa(y(t)) F'(y(t-T)) \bigg|_{y=y^*} \mathbf{z}(t-T) \\
&\quad + \left( \kappa'(y(t)) [F(y(t-T)) - y] - \kappa(y(t)) \right) \bigg|_{y=y^*} \mathbf{z}(t) \\
&= \kappa(y(t)) F'(y(t-T)) \bigg|_{y=y^*} \mathbf{z}(t-T) - \kappa(y(t)) \bigg|_{y=y^*} \mathbf{z}(t) \\
&\quad \text{(because } F(y^*) = y^* \text{)} \\
&:= B \mathbf{z}(t-T) + A \mathbf{z}(t)
\end{align*}
\]

where \( B = \kappa(y^*) F'(y^*) \) and \( A = -\kappa(y^*) \).

Local stability of a DDE [Verriest et. al., Hale]

1. Stable for all \( T \geq 0 \) only if: \( A \leq 0 \) and \(-A \geq |B|\)

2. Delay-dependent stability \( -B > |A| \) and \( T \leq T^* : = \frac{\cos^{-1}(\frac{-A}{B})}{B^2 - A^2} \)
Instability: Period Doubling

- What happens if \( a < b+1 \)?

For our case: \(-\kappa(y^*)\) is always negative, which holds due to fact that \(\kappa(\cdot)\) is always positive.

The second condition \(\kappa \geq \kappa|F'|\) is crucial to stability.

Clearly, for the case when \(F' < -1\) (period doubling condition for the map \( F' \)) the linear stability condition is violated and for a large enough \(T\) the constant solution \(y(t) = y^*\) will not be stable.

This has consequence for something called slowly oscillating periodic (SOP) orbit for DDE.
Instability: Period Doubling

Numerical Simulation: 2 homogeneous users $a=3$, $b=5$, and $C=5$, Rate control algorithm is unstable since $6/3=2>1$.

$$X_{n+1} = F(x_n)$$

First Return Map

Second Return Map

45° line
Instability: Period Doubling

• What happens if $a < b + 1$?

**Period Doubling Theorem:** Let $I := [a, b]$ be a closed interval such that $F(I), [a_1, b_1] \subset I$. Let the initial condition $\phi(t) \in Y_I$, where $Y_I = \{\phi^* \in Y \mid \phi^*(s) \in I \forall s \in [-1, 0]\}$ and $Y := C([-1, 0], \mathbb{R}_+)$, and $y(t)$ be the solution of DDE.

Now, if the points $a_1$ and $b_1$ are period two fixed points of $F$, then for all sufficiently small $\epsilon \geq 0$ there exists a finite $T = T(\phi, \epsilon, \kappa)$ such that $y(t) \in [a_1 - \epsilon, b_1 + \epsilon]$ for all $t \geq T$. 
Global Bifurcation: Slowly Oscillating Periodic Periodic Orbit

**Slowly Oscillating Periodic (SOP) orbits.**
Roughly, an SOP is a periodic orbit with its consecutive zeros (zero corresponds to the fixed point $y^*$ in our case) separated by more than one normalized time unit.
We have following conjecture regarding the existence of an SOP:

**Conjecture (SOP):** For all $0 < \nu < 1/T^*$, where $T^*$ is given by in linear stability context, rate control DDE has at least one slowly oscillating periodic solution with period $T(\nu) > 2$. Moreover, $T(\nu) \to 2$ as $\nu \to 0$.

Hard to prove existence and asymptotic stability of such periodic orbit.

Advanced functional-analysis tools like construction of ejective fixed point etc. are required.
TCP-RED Dynamics: Firiou’s Model 2001

\[
\bar{q}_{e,k+1} = \begin{cases} 
(1 - w)\bar{q}_{e,k} & \text{if } \bar{q}_{e,k} > b_1 \\
(1 - w)\bar{q}_{e,k} + wB & \text{if } \bar{q}_{e,k} < b_2 \\
(1 - w)\bar{q}_{e,k} + w\left(\frac{NK}{p_{\text{max}}(\bar{q}_{e,k} - q_{\text{min}})} - \frac{R_0C}{M}\right) & \text{otherwise}
\end{cases}
\]

\[N = \text{Number of active TCP connections} \quad w = \text{Exp. averaging weight}
\]

\[C = \text{Bottleneck bandwidth}
\]

\[b_2 = \frac{\left(\frac{NK}{B + \frac{R_0C}{M}}\right)^2}{p_{\text{max}}} (q_{\text{max}} - q_{\text{min}}) + q_{\text{min}}
\]

\[b_1 = \begin{cases} 
p_0\left(\frac{q_{\text{max}} - q_{\text{min}}}{p_{\text{max}}}\right) + q_{\text{min}} & , \text{if } p_{\text{max}} \geq p_0 \\
q_{\text{max}} & , \text{otherwise}
\end{cases}
\]
TCP-RED Dynamics: First Order Nonlinear Map

- Essentially nonlinear first order discrete time dynamical system
- Inverse square root nonlinearity comes from TCP transfer function which has an inverse square root-type dependence on drop prob.
- Model as a self clocking system.
- Piecewise smooth (with respect to state) map description comes from buffer being bounded between 0 and B. Different increasing and decreasing segments hint at chaos like behaviour.
- Map is smooth with respect to exponentially averaging parameter w and other system/RED parameters
TCP-RED Dynamics: Discrete Time Map

First and second return maps and their intersection with 45° degree line show the existence of a fixed point and period two orbit.
TCP-RED-Gentle_ Dynamics: First Order Nonlinear Map

First and second return maps for TCP and RED GENTLE_
TCP-RED Model: PDB and Border Collision

\[ q_{\text{max}} = 100, \quad q_{\text{min}} = 50, \quad c = 1500\text{kbps}, \quad K = \sqrt{8/3} \]

\[ B = 300 \text{ packets}, \quad R_0 = 0.1\text{sec}, \quad M = 0.5\text{kb} \]

\[ n = 20, \quad p_{\text{max}} = 0.1 \quad w = \text{bifurcation parameter} \]
General Case – Not Quite Yet!

- General network with homogeneous delay

\[
\frac{d}{dt} x_i(t) = \kappa_i \left( x_i(t) U_i'(x_i(t)) - x_i(t - T) \left( \sum_{j \in r_i} \mu_j(t - T) \right) \right)
\]

- \( y_i = x_i \cdot U_i'(x_i) := g_i(x_i) \)

- Normalize time by \( T \)

\[
\nu y_i(t) = \kappa_i g_i \left( g_i^{-1}(y_i(t)) \right) \left( y_i(t) - f_i \left( \overline{g}^{-1}(\overline{y}(t - 1)) \right) \right)
\]

where \( \nu = \frac{1}{T}, \overline{y}(t-1) = (y_1(t-1), \ldots, y_N(t-1)) \), and
General Case – Homogeneous Delay

\[ f_i(\bar{g}^{-1}(\bar{y}(t - 1))) = g_i^{-1}(y_i(t - 1)) \left( \sum_{l \in r_i} p_l \left( \sum_{j \in I_l} g_j^{-1}(y_j(t - 1)) \right) \right) \]

- **Matrix form**

\[ \nu \dot{\bar{y}}(t) = \kappa(\bar{y}(t)) \left( F(\bar{y}(t - 1)) - \bar{y}(t) \right) \]

where \( \kappa(\cdot) \) is a state dependent diagonal gain matrix with \( \kappa_{ii} = -\kappa_i g_i'(g_i^{-1}(y_i(t))) \) and \( F_i(\bar{y}) = f_i(\bar{g}^{-1}(\bar{y})) \)
General Case – Homogeneous Delay

**Discrete time map**

\[ \bar{y}_{n+1} = F(\bar{y}_n), \]

where \( n \in \mathbb{Z}_+, \bar{y}_n \in \mathbb{R}_+^N \), and

\[ F_i(\bar{y}) = f_i(g_1^{-1}(y_1), \ldots, g_N^{-1}(y_N)) \]

- Establish the correspondence between the stability of the above discrete time map and the delay-differential equations
- Underlying market mechanism
Assumption 1  (i) The function $g_i(x_i)$ is strictly decreasing with $g'_i(x_i) < 0$ for all $x_i > 0$, (ii) the function $f_i(x)$ is strictly increasing in $x_j$ for all $j \in I^i = \{l \in I \mid r_i \cap r_l \neq \emptyset\}$ and $\overline{x} > 0$ and does not depend on $x_j$ for all $j \in I \setminus I^i$, and (iii) both $g_i(x_i)$ and $f_i(x)$ are Lipschitz continuous on $[\epsilon, B_{max}]$ and $[\epsilon, N \cdot B_{max}]^N$, respectively, where $\epsilon$ is an arbitrarily small positive constant.
Theorem (Invariance) Suppose that $D \subset \mathbb{R}_+^N$ is a closed product space invariant under $F(\cdot)$, i.e., $D = \prod_{i=1}^N \text{proj}_i(D)$ where $\text{proj}_i(\cdot)$ denotes the $i$-th component projection operator. Then, for any initial function $\phi \in C([-1, 0], D) := X_D$ the resulting $\bar{y}(t)$ belongs to the domain $D$ for all $t \geq 0$ and $\nu > 0$. 
**Assumption:** Multidimensional map $F : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$ has a fixed point $y^* = \bar{g}(x^*)$, where $x^*$ is the solution to the relaxation problem, and there exists an open convex neighborhood $\text{int}(D_0)$, which is an open product space. Also, assume that there is a sequence of closed product spaces $D_n, n \geq 0$, such that $F(D_n) \subset \text{int}(D_{n+1}) \subset D_{n+1} \subset \text{int}(D_n)$ and $\{y^*\} = \cap_{n \geq 0} D_n$. 
• **Theorem** (Stability) All solutions $\overline{y}(t)$ starting with initial functions $\phi \in \mathcal{Y}_{D_0}$ converge to $y^*$ for all $\nu > 0$ under the previous assumption.
Example

• Utility functions: \( U_i(x_i) = \frac{-1}{a_i \cdot x_i^{a_i}}, \quad a_i > 0 \)

• Price functions: \( p_j(x) = c_j \cdot \left( \frac{x}{C_j} \right)^{b_j}, \quad j \in J \)

• Suppose that \( \bar{\alpha} > 1 \) and \( \bar{\beta} \) satisfies \( \frac{\alpha^{1/\sigma}}{\bar{\beta}} < \bar{\beta} < \bar{\alpha}^\sigma \), where \( \sigma = -\max_{i \in I} \frac{b_{i_{\text{max}}}+1}{a_i} - \varepsilon \), where

\[ b_{i_{\text{max}}} = \max_{j \in r_i} b_j \] and
\[ 0 < \varepsilon < 1 - \max_{i \in I} \frac{b_{i_{\text{max}}}+1}{a_i} \]
Problem: Define

\[ D_n = \begin{cases} \prod_{i=1}^{N} [\alpha^{\sigma_n} x_i^*, \beta^{\sigma_n} x_i^*] , & n \text{ odd} \\ \prod_{i=1}^{N} [\beta^{\sigma_n} x_i^*, \alpha^{\sigma_n} x_i^*] , & n \text{ even} \end{cases} \]

Lemma: Suppose that \( a_i > b_{\text{max}}^i + 1 \) for all \( i \in I \). Then, \( \hat{F}(D_{n-1}) \subset \text{int}(D_n) \subset D_n \subset \text{int}(D_{n-1}) \), where \( \text{int}(D_{n-1}) \) is the interior of \( D_{n-1} \), and \( \cap_{n=0}^{\infty} D_n = \{x^*\} \).

Theorem: If \( a_i > b_{\text{max}}^i + 1 \) for all \( i \in I \), the rate vector starting away from zero converges to the solution of the relaxation.
General Case (Finally!!)

- **Theorem:** If the network is stable under the homogeneous delay assumption, it is stable with arbitrary delays with an appropriate initial functions if there is a common divisor of the delays.

- Devices driven by local oscillators with fixed frequencies
Fairness

\[
\max_{x \geq 0} \sum_i U_i(x_i) = \max_{x \geq 0} \sum_i \frac{-1}{a \cdot x_i^a}
\]

s.t. \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix} x \leq \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
Outline

- Background
- Model & Motivation
- Main results
  - Single-user, single-link case
  - Multiple links case
- Fairness
- Conclusions
Distributed delay case: Physical Motivation
Distributed delay case:

- Imagine a strip with a continuum of sensors (e.g. smart dust, amorphous computing etc.)
- Feedback weighted by distributed delay

\[ \dot{x}(t) = \kappa \left( g(x(t)) - \int_{0}^{\infty} f(x(t - s - T))K(s)ds \right) \]

where \( K(u) \) is delay kernel and often assumed to be Gamma function.

\[ K(u) = \frac{\alpha^{r+1}u^r}{r!}e^{-\alpha u} \quad \text{and mean delay} \]

\[ E[K] = \frac{r + 1}{\alpha} \]
Distributed delay case: Gamma Function Used in Chemostat modeling
\[ \omega_i(t) = \int_{-\infty}^{t} f(x(\theta - T)) G^i(t - \theta) d\theta, \quad i = 0, 1, \ldots, r, \quad \text{(1)} \]

where \( G^i(u) = \frac{\alpha^{i+1} u^i}{i!} e^{-\alpha u}, \quad u \geq 0. \)

Note that for any \( i \geq 1, \)

\[ \frac{d}{du} G^i(u) = -\alpha G^i(u) + \alpha G^{i-1}(u), \quad \text{and} \quad \frac{d}{du} G^0(u) = -\alpha G^0(u). \]

\( x(t) := (x(t), \omega_r(t), \omega_{r-1}(t), \ldots, \omega_0(t)) \) satisfies

\[ \frac{d}{dt} x(t) = \kappa (g(x(t)) - \omega_r(t)), \quad \frac{d}{dt} \omega_i(t) = -\alpha \omega_i(t) + \alpha \omega_{i-1}(t), \quad i = 1, \ldots, r \quad \frac{d}{dt} \omega_0(t) \]

\[ = -\alpha \omega_0(t) + \alpha f(x(t - T)) \]
Similar to Hale and Ivanov’s Result (1993, JMAA)

By substituting \( t = T \cdot s \) and redefining \( \beta = T \cdot \alpha \)

\[
\nu \frac{d}{dt} x(s) = g(x(s)) - \omega_r(s), \quad \frac{d}{dt} \omega_i(s) = -\beta \omega_i(s) + \beta \omega_{i-1}(s), \quad i = 1, \ldots, r
\]

\[
\frac{d}{dt} \omega_0(t) = -\beta \omega_0(t) + \beta f(x(t-1))
\]

where \( \nu = 1/T \cdot \kappa \). Similar to a high dimensional DDE studied by Ivanov and Hale.

\[
\left( \epsilon_m \frac{d}{dt} + 1 \right) \ldots \left( \epsilon_m \frac{d}{dt} + 1 \right) = f(y(t-1)) \quad (1)
\]

This equation is equivalent to the system

\[
\epsilon_0 \dot{x}(t) = -x_0(t) + x_1(t)
\]

\[
\ldots
\]

\[
\epsilon_m \dot{x}(t) = -x_m(t) + f(x_0(t-1))
\]
Hale and Ivanov’s Result (1993, JMAA)

Stability if $x_{n+1} = F(x_n)$
Delay Differential Equation as a Discrete Time Map

\[ x(t) = g^{-1}(y(t)) \Rightarrow \frac{d}{dt} x(t) = \frac{\frac{d}{dt} y(t)}{g'(g^{-1}(y(t)))} \]

\[ \frac{d}{dt} y(t) = \frac{\kappa(y(t))}{\nu} \left( \int_{-\infty}^{t} f(g^{-1}(y(\theta - T))) K(t - \theta) d\theta - y(t) \right) \]

Close correspondence between invariance and global stability properties of map

\[ y_{n+1} = f(g^{-1}(y_n)) := F(y_n) \]
Invariance and Stability Result

**Theorem 1** (Invariance) If initial function $\phi \in Y_I$ and $\omega_r(0), \ldots, \omega_0(0) \in I$, the corresponding solution $(y(t), \omega_r(t), \ldots, \omega_0(t); \phi, \omega_r(0), \ldots, \omega_0(0))$ satisfies $y(t) := (y(t), \omega_r(t), \ldots, \omega_0(t)) \in I^{r+2}$ for all $t \geq 0$. This means that set $I$ is invariant under the action of dynamical system.

Suppose map $F$ has an attracting fixed point $y^*$ with immediate basin of attraction $J_0 : F^n y_0 \rightarrow y^*$ with $y_0 \in J_0$.

**Theorem 2** (Stability) If $\nu > 0$, initial function $\phi \in Y_{J_0}$ and $\omega_r(0), \ldots, \omega_0(0) \in I$ then $\lim_{t \rightarrow \infty} (y(t), \omega_r(t), \ldots, \omega_0(t); \phi, \omega_r(0), \ldots, \omega_0(0)) = (y^*, \ldots, y^*)$. 
Stochastic stability in the presence of perturbations

Consider the network model with stochastic perturbations. One can think of this kind of stochastic behavior associated with UDP traffic.

This kind of models are very relevant to model the behavior of network control systems.

\[ \nu dx(t) = [\kappa(x(t), x(t-1)) [-x(t) + f(x(t-1))]] \, dt + \sigma(x(t-1)) \, dw(t) \]

which is equivalent to

\[ \nu x^\nu(t) = \nu x(0) + \int_0^t \kappa(x(u), x(u-1)) [-x(u) + f(x(u-1))] \, du \]
\[ + \int_0^t \sigma(x(s-1)) \, dw(s) \]

First term from elastic traffic and second from inelastic traffic.
Stochastic stability in the presence of perturbations: Results

Assume that there exists a set $I$ such that $f(I) \in I$ with usual existence and uniqueness conditions.

**Theorem 1** *(Stochastic Invariance Principle):* Let $\phi \in C_I$ and let $\phi$ be a deterministic function. Then $Ex_\phi^\nu(t) \in I$ for all $t \geq 0$ and all fixed $\nu > 0$.

Now assume as before that $x^*$ is an attracting fixed point of the map $F$ with immediate basin $I : \lim_{n \to +\infty} F^n(x_0) = x^*$, for any $x_0 \in I_0$.

**Theorem 2** *(Stochastic Global Stability):* Let $\phi \in C_I$ then $Ex_\phi^\nu(t) = x^*$ for all fixed $\nu > 0$. 
Conclusions

- Discrete time maps is an efficient tool for DDEs
- Robust Stability derived using invariance principle
- Useful for a information rich-world where feedback may be based on data-fused from many sources
- Modeling distributed delay is achieved using Gamma Kernel
- The stability of a system with arbitrary delays can be studied by that of a corresponding discrete time system
  - Much easier to analyze and simulate
- There is a close relationship between the responsiveness of the price functions and the elasticity of user demand
Journal Publications:

Many conference publications on this topic…
People Researching on Global Bifurcations in DDE: Nussbaum, Mallet-Paret, Walther, Ivanov etc.

Thanks for your patience!!