Sequence, series and convergence

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1 Sequence and partial sum of sequence

Suppose you are purifying a solution by distillation. Each time your solution becomes reduced to n^{th} fraction of total volume. If the total initial volume of the solution was v, first, we will have $\frac{v}{n}$ as remaining . After second distillation you will retain $\frac{v}{n^2}$ amount. After each successive distillation the volume becomes

Iteration	Volume
0	v
1	$\frac{v}{n}$
2	$\frac{v}{n^2}$
3	$\frac{v}{n^3}$
$\vdots$	$\vdots$
k	$\frac{v}{n^k}$

The successive amounts of volumes *i.e.* $\mathcal{G} = \{v, \frac{v}{n}, \frac{v}{n^2}, \ldots, \frac{v}{n^k}\}$ are, therefore (1) ordered and (2) countable. Such an ordered set of numbers are called a sequence.The sequence generated here is called a geometric progression where each term is a certain multiple of its previous term.

In many applications (will be discussed shortly), we need to encounter such sequences and find out the cumulative effects of certain numbers of terms of a sequence. To do that, let us formally develop the subject. To do that, we will first define a few important concepts and use $\mathcal{G}$ to illuminate them.

Sequence

A sequence is an ordered set of numbers. For example, $\mathcal{G}$ is a sequence.

Series

A series is an expression of the form

$$a_1 + a_2 + \ldots + a_n \tag{1}$$

created from a sequence $\{a_1, a_1, a_1, \ldots, a_n\}$ where $n \in \mathbb{Z}$. For example,

$$S_k = v + \frac{v}{n} + \frac{v}{n^2} + \ldots + \frac{v}{n^k} \tag{2}$$

is a series. We write the Eq.(2) as

$$S_k = \sum_{i=0}^{i=k} \frac{v}{n^i} \tag{3}$$

Partial sum

The sum of first $m, (m < k)$ terms are called the series's m^{th} partial sum. For example,

$$s_m = v + \frac{v}{n} + \frac{v}{n^2} + \ldots + \frac{v}{n^m} \tag{4}$$

where $m < k$ is the m^{th} partial sum of series given in Eq.(3)

Infinite series

A series made from countably infinite number of elements is called an infinite series. In our example,

$$\lim_{k\to\infty} S_k$$

is an infinite series.

Sum of an infinite series

A sum of an infinite series is the sum of all terms of an infinite series.

2 Convergence of infinite series

An infinite series is called a convergent infinite series if its sum is a finite number.

Pictorially, if we plot the partial sums of a convergent series it becomes more and more close to a finite value.

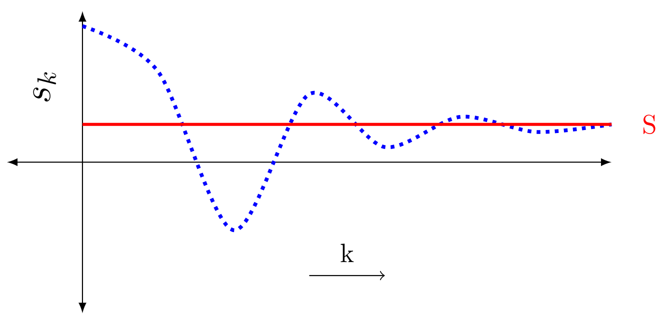

Figure 1: Cauchy convergence of partial sums s_k of a series converging to a value S.

An example of a convergent series is infinite geometric series Eq.(3) where $k \to \infty$. For this series, the sum of the series is $S = \frac{vn}{n-1}$ which is a finite number.

If the partial sums oscillate between more than one finite values as we keep increasing the number of terms, we call that an oscillatory series. Example of which is

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots \infty \tag{5}$$

If we take a partial sum over even number of terms we get zero. If we take odd number of terms we get 1. Clearly, the partial sums do not converge.

If partial sums of a series changes more and more with increasing number of terms we call that series as divergent series. Example :

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \tag{6}$$

and

$$1 + 2 + 3 + 4 + 5 + 6 \dots \quad (7)$$

Eq.(6) is called harmonic progression.

It is now important to learn some techniques to check whether an infinite series is a convergent one or otherwise. These tests will be applied to series with all positive terms. If there are some negative terms in the series, we generate another series taking absolute values of all the terms. If this new series is convergent, the original series is convergent as well. In that case, the original series is absolute convergent.

2.1 Preliminary test

If the terms of a series do not approach to zero as the number of terms tends to infinity the series is divergent. If they approach to zero, further tests should be done to figure out if the series is convergent.

For example, Eq.(7) fails this test but Eq.(6) passes. Therefore, further tests will be required to show that Eq.(6) is also an divergent series.

2.2 Comparison test

This test has two parts (a) and (b).

Suppose we need to know if

$$a_1 + a_2 + a_3 + a_4 + \dots \quad (8)$$

is convergent or not.

(a) We also know that another infinite series of positive terms

$$m_1 + m_2 + m_3 + m_4 + \dots \quad (9)$$

is convergent. If

$$|a_k| \leq m_k; \forall k \quad (10)$$

then the series in expression (8) is also convergent.

(b) Suppose another infinite series of positive terms,

$$d_1 + d_2 + d_3 + d_4 + \dots \quad (11)$$

is divergent.

If

$$|a_k| \geq d_k; \forall k \quad (12)$$

then the series in expression (8) is divergent.

However, this test cannot check the convergence for $|a_k| \leq d_k; \forall k$ and $|a_k| \geq m_k; \forall k$.

2.3 Integral test

Suppose the sequence of numbers of which an infinite series is comprised, represent points on a plot (Fig. 2). If we take infinite of them we can approximate their sum as the integral of a continuous function. Therefore, if the integral becomes finite for infinite number of terms, we can say the series is convergent as well.

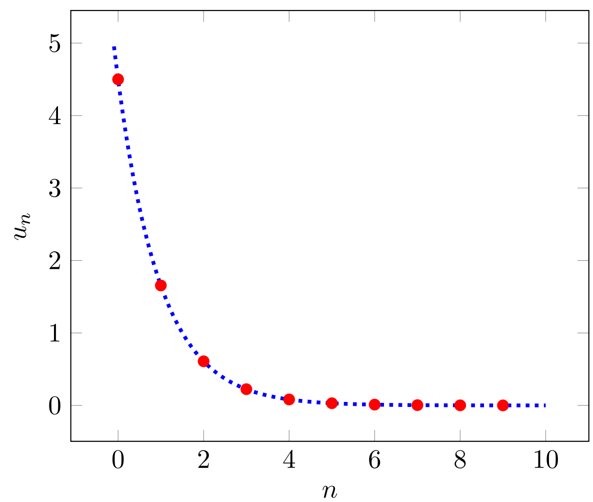

Figure 2: Red points represent the terms of the sequence u_n for $n = 1, 2, \ldots$. Blue dotted curve is the interpolating function through the points.

Therefore, the integral test for convergence of a series can be constructed as follows:

An infinite series

$$S = \sum_{n=0}^{\infty} u_n \tag{13}$$

for which $0 < u_{n+1} \leq u_n; \forall n > N$, $\sum_{n=0}^{\infty} u_n$ converges if $\int^{\infty} u(n)dn$ is a finite number. Remember that only the upper limit is important here.

2.4 Ratio test

For an infinite series Eq.(2), we define the limiting value ρ of the ratio of two successive terms n_{n+1} and n_n as

$$\rho = \lim_{n\to\infty} \rho_n, \tag{14}$$

$$\rho_n = \left|\frac{u_{n+1}}{u_n}\right|. \tag{15}$$

Then

$$\rho < 1; \implies \text{S is } \textit{convergent} \tag{16}$$
$$\rho > 1; \implies \text{S is } \textit{divergent} \tag{17}$$
$$\rho = 1; \implies \text{S is } \textit{inconclusive} \tag{18}$$

3 Alternating series

The most used and important series in natural sciences are alternating series for which the each term's sign alternates between positive and negative.

Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

An alternating series is convergent if

$$|u_{n+1}| \leq |u_n|$$

For an alternating series:

- individual sums of positive and negative terms can be divergent but together they may converge
- the orders of the terms are sacrosanct- they cannot be changes. Otherwise, a same series will give multiple numbers as the sum.

4 Power series and generating functios

An infinite series of the form

$$P(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + \dots \infty = \sum_{n=0}^{\infty} a_n (x-b)^n \tag{19}$$

is called a power series where $a_n \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

Whether a power series $P(x)$ is convergent or divergent depends upon the value of x. The interval of $x \in [a, b]$ for which $P(x)$ converges is called interval of convergence. Here, $[a, b]$ is the interval of convergence for $P(x)$.

Eq. (26) means that the sum in left hand side depends on the value of x. Therefore, the power series takes up a value of x and outputs another number. That is the definition of functions! Therefore, we can actually write a function in terms of a power-series. However, the converse is not rue *i.e.* not every function can be expanded in a power series expansion.[1] Question is can we write more than one power series for one function expanded around the same point?

[1] Functions with singularities are common example. For example, $f(x) = \frac{1}{x}$ will be equal to ∞ as x tends to 0

For simplicity here we assume that we expand a function $f(x)$ around $x = 0$ and we can have two distinct power series expansion of $f(x)$ *i.e.*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{20}$$

$$f(x) = \sum_{n=0}^{\infty} b_n x^n \tag{21}$$

where at least one of the coefficients are unequal *i.e.* $a_k \neq b_k, \forall k \in \mathcal{K} \subset \mathbb{Z}^+$. Therefore,

$$f(x) - f(x) = 0 = \sum_{n=0}^{\infty} (a_n - b_n)x^n = \sum_{k} (a_k - b_k)x^k \tag{22}$$

However, in that case, $D(x) = \sum_k (a_k - b_k)x^k = 0 \forall x$. Here the $D(x)$ has the lowest degree $k = k_1$. Therefore,

$$x^{k_1} \sum_{k=k_1} (a_k - b_k)x^{(k-k_1)} = 0 \tag{23}$$

For any value of $x \neq 0$,

$$\sum_{k=k_2} (a_k - b_k)x^k = -(a_{k_1} - b_{k_1}) \tag{24}$$

Setting $x = px$, where p is any arbitrary number, we again get

$$\sum_{k=k_2} (a_k - b_k)p^k x^k = -(a_{k_1} - b_{k_1}) \tag{25}$$

Eq. (24) and Eq. (25) cannot be true for any arbitrary p. Therefore, the only way Eq. (24) can be true if $a_{k_1} = b_{k_1}$, which is in direct disagreement with our assumption that Eqs. (20) and (21) are two distinct expansions. Therefore, power series expansion of any function around a constant value is unique!

Suppose a function $g(x)$ has a power series expansion with coefficients $\{a_k, k \in \mathbb{Z}\}$. Then $g(x)$ is said to be the generating function of the sequence $\{a_k\}$. Therefore, any distinct sequence of numbers will have a distinct function as a generating function.

5 Taylor and McClaurine series

Question is how to get these coefficients? Suppose, we have the power series expansion of a function $f(x)$ around $x = b$ as

$$f(x) = \sum_{k=0}^{\infty} a_k (x - b)^k \tag{26}$$

Putting $x = b$ in Eq. (26) gives us

$$f(b) = a_0 \tag{27}$$

Now, let us differentiate Eq. (26) once and put $x = b$. This will give

$$\left.\frac{df(x)}{dx}\right|_{x=b} = a_1 \tag{28}$$

Differentiate Eq. (26) twice and then putting $x = b$ will give

$$\left.\frac{1}{2!}\frac{d^2 f(x)}{dx^2}\right|_{x=b} = a_2 \tag{29}$$

In fact, this way we will be able to see that, in general,

$$a_n = \left.\frac{1}{n!}\frac{d^n f(x)}{dx^n}\right|_{x=b} \tag{30}$$

Combining Eqs.(30) and (26), we obtain

$$f(x) = \sum_{n=0}^{\infty}\left(\left.\frac{1}{n!}\frac{d^n f(x)}{dx^n}\right|_{x=b}\right)(x-b)^n \tag{31}$$

Eq. (31) is known as Taylor series expansion of function $f(x)$. Setting $b = 0$ gives

$$f(x) = \sum_{n=0}^{\infty}\left(\left.\frac{1}{n!}\frac{d^n f(x)}{dx^n}\right|_{x=0}\right)x^n \tag{32}$$

is known as McClaurine series expansion.

It is possible to add and subtract two power series, expanded around same point. Two power series

$$P(x) = \sum_{k=0}^{N} p_k x^k$$

and

$$Q(x) = \sum_{k=0}^{N} q_k x^k$$

can be combined to give

$$R(x) = aP(x) + bQ(x) = \sum_{k=0}^{N} r_k x^k; \iff r_k = ap_k + bq_k \forall k, a, b \in \mathbb{R} \tag{33}$$

It is also possible to express a function as a power series of another power series. For example

$$e^{sin(x)} = \sum_{k=0}^{\infty}\frac{(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!})^k}{k!} \tag{34}$$

6 Accuracy of an approximated power series

Not all functions can be expanded around every point in a power series. For that we need to learn a bit of advanced complex analysis which may or may not be covered here.

The difference between the limiting value of an infinite series

$$S = \sum_{k=0}^{\infty} u_k \tag{35}$$

and its n^{th} partial sum

$$s_n = \sum_{k=0}^{n} u_k \tag{36}$$

is called the remainder

$$R_n = S - s_n. \tag{37}$$

For a convergent infinite series,

$$\lim_{n\to\infty} |R_n| = 0 \tag{38}$$

For a power series expansion of a function, the n^{th} remainder is at least proportional to x^{n+1}. We use a notation for that : the "big O notation". In this notation the lowest power of remainder is represented as

$$\mathcal{O}(x^{n+1})$$