Recapitulation of previous concepts

M. Sadhukhan

May 8, 2019

Contents

In most of the course, we will need some notations and definitions. Therefore, it is important to brush up them.

1 Set, countability and intervals

A set is a collection of well-defined objects. For example, a set of all students in this classroom is a set. Each one of us is called a member or element of this set. If we represent the members of a set S as i, the we say i *belongs to* S and write

$$i \in S \tag{1}$$

A set can be a part of a bigger set. In that case the former set is called a subset of the later set, which is called a superset. If $\mathcal{A}$ is a subset of $\mathcal{B}$ then we write

$$\mathcal{A} \subseteq \mathcal{B} \tag{2}$$

If there exist at least one element in $\mathcal{B}$ which is not a part of $\mathcal{A}$, then we say $\mathcal{A}$ is a proper subset of $\mathcal{B}$ and write,

$$\mathcal{A} \subset \mathcal{B} \tag{3}$$

To define these members more precisely, we need the concept of intervals. For example set of all integer/whole numbers from 5 to 10 are 5, 6, 7, 8, 9, 10. We say that these integers lies in the interval between 5 and 10. The lowest value possible in the interval is called lower bound. Here the lower bound is 5. The highest value of the interval is called upper bound. A closed interval contains the end point as a member of the set and represented by a square bracket ("[" or "]"). On the contrary, if the set excludes the lower or upper bounds we say that it is an open interval and represented by parentheses ("(" or ")"). In our example, the interval is $[5, 10]$. It is possible and rather common to have an interval which has one open and one closed side. For example all positive, one and two-digit whole numbers excluding zero lies on the interval $(0, 99]$.

If you can count the elements of the set by integers, then we call that set as countable. It may or may not have finite number of elements. For example, set of all positive whole numbers is countable but infinite in number. If you cannot count them, they are called uncounable. As we will shall see real numbers are uncountable.

2 Numberline

We all know natural numbers (1, 2, 3 , ...) are used for counting. They are positive whole numbers. If we add to that negative whole numbers and zero we have a set of integers. We represent this set by $\mathbb{Z}$. Any integer i is therefore belongs to this set and we write,

$$i \in \mathbb{Z} \tag{4}$$

We notice that we have a notion of *largeness* associated with any two of these integers. This means if we choose any two integers i and j such that $i \neq j$ then either $i < j$ or $j < i$. This allows us to represent all integers to be a part of a linear chain of numbers called number line.

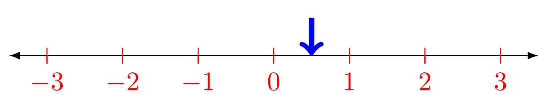

Figure 1: Number line. Integers in red and real numbers in blue.

However, between any two integers lie the fractions. They, along with integers, are called real numbers. The set of all real number is represented by $\mathbb{R}$. Any real number r belongs to $\mathbb{R}$, *i. e.*

$$r \in \mathbb{R} \tag{5}$$

Almost all our discussions will assume real numbers as the variable type unless mentioned otherwise.

3 Function

Suppose we have a set of number $\mathcal{D}$. The elements of that set are x. Then a function f is a machine which takes every x as an input and outputs another number y. In mathematical notation, $y = f(x)$ is a function of $x \in \mathcal{D}$, *i. e.*

$$y = f(x); \forall x \in \mathcal{D} \tag{6}$$

The set $\mathcal{D}$ is called domain of definition of f.

Suppose, all possible y constitute a set $\mathcal{R}$. Then $\mathcal{R}$ is called the range of f.

Therefore a fncion f is a mapping procedure from $\mathcal{D}$ to $\mathcal{R}$. Mathematically,

$$f : x \in \mathcal{D} \rightarrow y \in \mathcal{R} \tag{7}$$

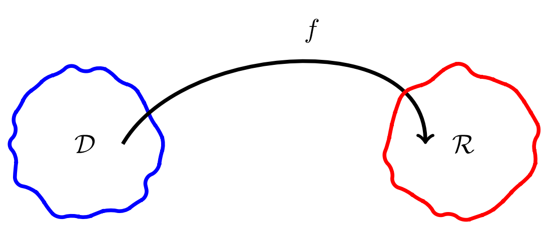

Figure 2: Diagrammatic representation of mapping.

4 Limit

Sometimes, we need to know how the $f(x)$ behaves as x slowly approach towards a predefined value a. Such necessities may arise as $f(x)$ cannot be found out at $x = a$. Alternatively, we may be interested in knowing the rate of change of $f(x)$ with change in x. For this later case, consider $\mathcal{D}, \mathcal{R} \in \mathbb{R}$ then we can also represent the situation pictorially as :

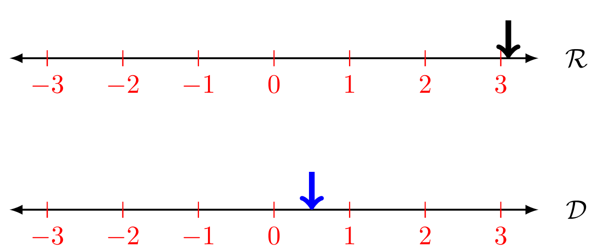

Figure 3: How change in $x \in \mathcal{D}$ changes the value at $y = f(x) \in \mathcal{R}$

It can happen that for a little change in x, $f(x)$ changes significantly r otherwise. This concept of pushing a variable's value closer and closer to a predefined value without really touching it is called limit. When we say x approaches towards a we write

$$\lim_{x \to a} \tag{8}$$

Depending on whether x approaches towards a from right (positive direction of number line) or from left (negative direction of number line), we use notations

$$\lim_{x \to a^+} \text{ or, } \lim_{x \to a^-}, \tag{9}$$

respectively.

5 Differentiation

As discussed in the previous section we may be interested to know how the value of a function $f(x)$ changes with the change of x. It may depend on the value of x as well as how much change in x we are considering. Therefore, we need to now how the value of $f(x)$ changes when we change the value of x by an amount δx. Of course, the new value for $f(x)$ will be $f(x + \delta x)$. Therefore, *the change in $f(x)$ per infinitesimally (very very small) amount of change in x is*

$$\frac{df(x)}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \tag{10}$$

Therefore differentiation concerns how a function changes locally in the vicinity of a certain value. Also, differentiation is connected to the concept of the connection of two points, and thereby is an analytical approach. It can be interpreted as the instantaneous/local slope of a curve.

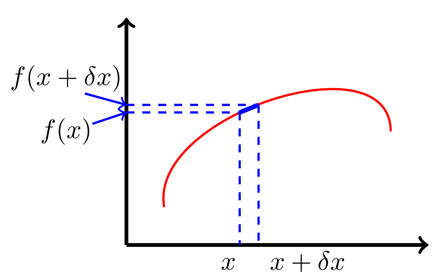

Figure 4: The differentiation as a slope.

6 Integration

Integration can be seen as the overall effect of a variation provided by a function. For example if water flows into a bucket by an amount of δv per δt interval, the total amount of water after N such interval will be

$$V = \sum_{i=1}^{N} \delta v = N\delta v \tag{11}$$

If you are told that the total collection time was t, then

$$N = \frac{t}{\delta t} \tag{12}$$

or,

$$V = \sum_{i=1}^{N} \delta v = \frac{\delta v}{\delta t} t = \frac{\delta v}{\delta t} \sum_{i=1}^{N} \delta t \tag{13}$$

Here, $\frac{\delta v}{\delta t}$ is the rate of change of volume inside the bucket. Now if it changes with time, then the rate depends on the i^{th} time interval $[t_i, t_i + \delta t_i]$:

$$\frac{\delta v}{\delta t} \to \left(\frac{\delta v}{\delta t}\right)_i \tag{14}$$

and the total volume in time t will be :

$$V = \sum_{i=1}^{N} \left(\frac{\delta v}{\delta t}\right)_i \delta t_i \tag{15}$$

In the limit $\delta t_i \to 0$, we express the sum by integration:

$$V = \lim_{\delta t \to 0} \sum_{i=1}^{N} \left(\frac{\delta v}{\delta t}\right)_i \delta t_i = \int_0^t \left(\frac{dv}{dt}\right) dt \tag{16}$$

An integration is also can be seen as the area under the curve. To understand that, let us consider the following example. Suppose, we open the tap for 30 minutes. For each 5 minutes the flow rate changes. The rates of flow for each 5 minutes are $r_1, r_2, r_3, r_4, r_5, r_6$, respectively. Here $r_i = \left(\frac{\delta v}{\delta t}\right)_i$

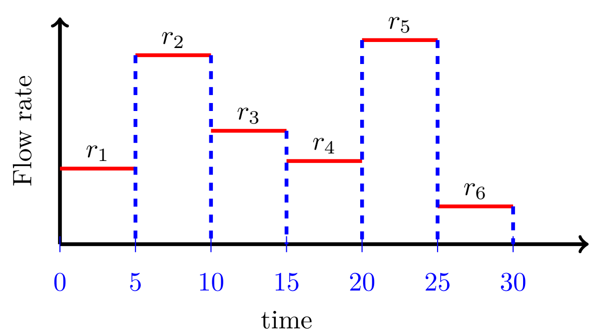

Figure 5: Integration as area under the curve.

As we can see in Fig.5, the total amount of water flown in first five minute is $r_1 \times \Delta t$, where $\Delta t = 5$ minutes. Therefore, total volume of water flown out of the tap is

$$V = r_1 \times \Delta t + r_2 \times \Delta t + r_3 \times \Delta t + r_4 \times \Delta t + r_5 \times \Delta t + r_6 \times \Delta t = \sum_{i=1}^{6} r_i \times \Delta t \quad (17)$$

Now clearly, we can see that $r_i \times \Delta t$ is the area of the box under r_i segment. Altogether, V is to total area under all segments. Now, if the flow rate changes more and more smoothly, Δt becomes smaller and smaller. Eventually, for $\Delta t \to 0$, $r_i \to r(t)$ and the sum becomes an integral *i. e.*

$$V = \lim_{\Delta t \to 0} \sum_{i=1}^{N=\frac{t}{\Delta t}} r_i \times \Delta t = \int_0^t r(t)dt \quad (18)$$

Another way of defining an integral is called line integral which we will learn later.

7 Monomial, polynomial, homogeneous functions and algebraic equation

Suppose a variable $x \in \mathbb{R}$. A monomial of x is a function

$$m(x, n) = x^n \quad (19)$$

where $n \in \mathbb{Z}$ and n is a constant. n is called exponent.

A polynomial is a summation of more than one monomials with same variable x *i. e.*

$$P_{n_k}(x) = a_1 x^{n_1} + a_2 x^{n_2} + \ldots + a_k x^{n_k} = \sum_{i=1}^{k} x^{n_i} \quad (20)$$

where the set of exponents $\{n_1, n_2, n_3, \ldots, n_k\} = \mathcal{K}$ is a proper subset of set of all integers *i. e.*

$$\mathcal{K} \subset \mathbb{Z} \quad (21)$$

For most of our course we will assume $a_i \in \mathbb{R}$. The largest exponent of a polynomial is called is its degree.

It is important to note that the degree of a polynomial is finite.

A homogeneous function is a function $f(x)$ which follows :

$$f(\alpha x) = \alpha^k f(x) \quad (22)$$

where k is called the degree of homogeneity of $f(x)$. For a monomial of degree n the degree of homogeneity is n.

An algebraic equation is an equation where we seek *"non-trivial"* values of an algebraic quantity x for which a polynomial $P_k(x)$ vanishes *i. e.*

$$P_k(x) = 0 \tag{23}$$

Values of x for which $P_k(x) = 0$ are called roots of $P_k(x)$. If a root $x = \alpha$ appears m times as a solution of an algebraic equation, we say the multiplicity of root $x = \alpha$ is m.

The number of roots of a polynomial equation of degree n is exactly n[1].

The geometric interpretation of roots of a polynomial comes from plotting $P_k(x)$ as a function of x. Fig. 6 shows a few roots of polynomial $P_k(x)$ marked by red circles. Note that $P_k(x)$ touches x-axis at roots.

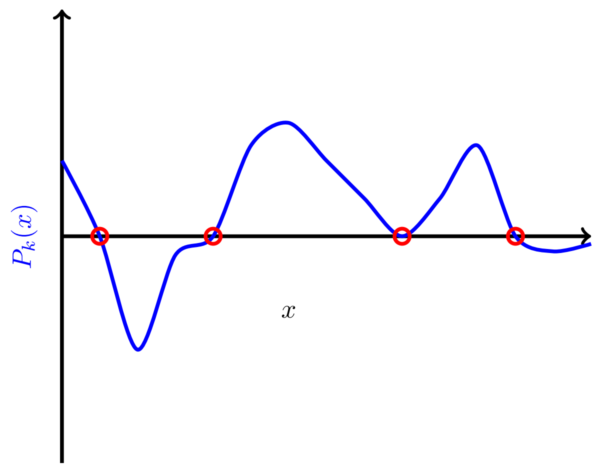

Figure 6: Polynomial and its roots

[1]It is a direct consequence of "Fundamental theorem of algebra"