Department of physics
Indian Institute of Technology, Kanpur

MKH/SMT/2016-17-II/Phy103 Assignment-3 Week of 23-1-17

Note: The problems assigned are from the text book *Introduction to Electrodynamics* (4^{th} edition) by David J. Griffiths

1. Problem 2.41(do it by explicitly calculating force and also by energy method)
2. Problem 2.42 (do it by explicitly calculating force and also by energy method)
3. Problem 2.50
4. Problem 2.53
5. Problem 3.1
6. Problem 3.4
7. Problem 3.37
8. Problem 3.41
9. Problem 3.51
10. Although in calculating the electrostatic energy of a charge distribution we assume that the charge can be varied continuously, in reality it is not so since the charge comes in units of electronic charge e. Quantized nature of charge is seen if we deal with small number of electrons, for example in a nano-device. (i) Assume a capacitor of capacitance *C* with *N* electrons. What will be its energy if we do not count the self energy of electrons assuming each electron to be uniformly spread over the capacitor (an assumption justified by quantum mechanics which treats an electron as a wave spread over the capacitor). (ii) Plot the number of electrons *N* versus the voltage *V* applied for a nano-capacitor.

Hints:

2.41: Apply the direct formula. For the energy method, move one of the plates by an infinitesimal distance Δx and calculate the corresponding energy change using the energy density formula.

2.42: Apply the direct formula taking appropriate component of the force. For the energy method, move the hemispheres away from each other an infinitesimal distance Δx and calcul ate the corresponding energy change using the energy density formula.

2.50: Keep in mind what the Laplacian of 1/r is.

2.53: You are solving a one dimensional second-order differential equation. Energy conservation gives $\frac{1}{2}mv^2(x) - eV(x) = 0$. Current density is $\rho(x)v(x)$.

3.1 and 3.4: Do the explicit integral over the sphere as has been done many times in the lectures. For 3.4 you will have to write the integrand as a gradient with respect to $\vec{r}'$.

3.37: Take the derivative with respect to R after cancelling R^2 with $1/R^2$ as suggested in the book. Then express dV/dR in terms of gradient of *V* and elemental area $d\vec{S} = dS\hat{r}$.

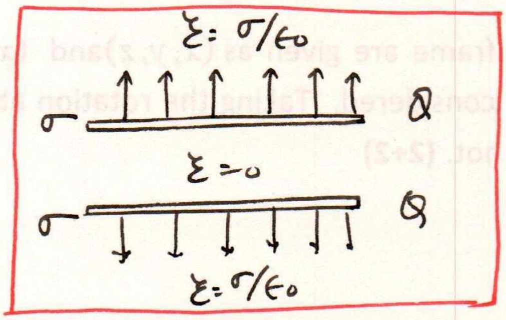
3.41: Image problem with charged sphere.

3.51: (a) one situation is that given in the problem; the other situations is one plate grounded and other plate at potential V.
(b) one situation is the same as the problem; the other situation is one shell at *V* and one grounded.

10. The potential with charge *ne* (n=integer) is given as *ne*/*C*. No charge will go to the capacitor unless it gains sufficient energy from the applied voltage.

SOLUTION ASSGN. 3

2.41



Charge Q, surface charge density σ and electric field for prob 2.41

If area is A then $\sigma = \dfrac{Q}{A}$

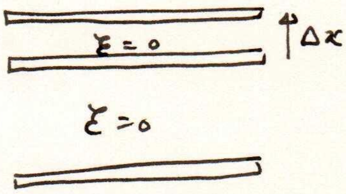
Then electric field $\mathcal{E} = 0$ between the plates and $\mathcal{E} = \sigma/\epsilon_0$ outside (see box above)

By direct formula $\quad F/\text{area} = \dfrac{\sigma^2}{2\epsilon_0}$

So force on each plate is $= \left(\dfrac{\sigma^2}{2\epsilon_0}\right) A$

<u>Energy Method</u>:

Move one of the plates away by a small distance Δx. Then the volume where energy density is non-zero decreases so that



$$\Delta E = -\frac{1}{2}\epsilon_0 \mathcal{E}^2 A \Delta x$$

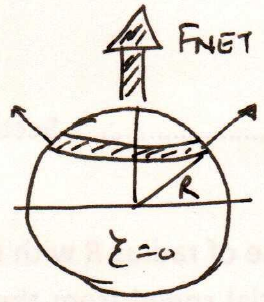
$$F = -\frac{\Delta E}{\Delta x} = \frac{1}{2}\epsilon_0 \mathcal{E}^2 A$$

$$= \frac{1}{2}\epsilon_0 \frac{\sigma^2}{\epsilon_0^2} A$$

$$= \frac{\sigma^2}{2\epsilon_0} A$$

And the force direction is the same as that of Δx.

2.42



$\frac{F}{\text{area}} = \frac{\sigma^2}{2\epsilon_0}$

horizontal component of force cancel. Vertical component over the shaded strip add to

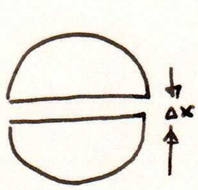
$$2\pi R^2\, d(\cos\theta) \cdot \frac{\sigma^2}{2\epsilon_0} \cos\theta$$

Net force therefore

$$F_{NET} = 2\pi R^2 \frac{\sigma^2}{2\epsilon_0} \int_0^1 \cos\theta \, d(\cos\theta)$$

$$= \left(\frac{\sigma^2}{2\epsilon_0}\right) \pi R^2$$

ENERGY METHOD: Displace the northern and southern hemispheres by small Δx. Since field is zero inside the hemispheres, $\vec{E}=0$ (to the zeroth order in Δx) in a larger volume after hemispheres are seperated



The volume change is $\Delta V = \pi R^2 \Delta x$

So energy decreases by $=$ (Energy density near the sphere surface) $\times \pi R^2 \Delta x$

$$\Delta E = -\frac{1}{2}\epsilon_0 \left(\frac{\sigma}{\epsilon_0}\right)^2 \cdot \pi R^2 \Delta x$$

$$F = -\frac{\Delta E}{\Delta x} = \left(\frac{\sigma^2}{2\epsilon_0}\right) \pi R^2$$

2.50

$$V(r) = \frac{A e^{-\lambda r}}{r}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(r)$$

$$= -\hat{r} \frac{\partial}{\partial r} \left[A \frac{e^{-\lambda r}}{r} \right]$$

$$= -\hat{r} \left[-\lambda A \frac{e^{-\lambda r}}{r} - \frac{A e^{-\lambda r}}{r^2} \right]$$

$$= \frac{A e^{-\lambda r}}{r^2} (1 + \lambda r) \hat{r}$$

Charge density $\rho(\vec{r}) = \epsilon_0 \vec{\nabla} \cdot \vec{E}$

$$= \epsilon_0 \vec{\nabla} \cdot \left\{ \frac{A \hat{r}}{r^2} e^{-\lambda r} (1 + \lambda r) \right\}$$

$$= \epsilon_0 A \left[\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) e^{-\lambda r}(1+\lambda r) + \frac{A}{r^2} \hat{r} \cdot \vec{\nabla} \left\{ e^{-\lambda r} (1 + \lambda r) \right\} \right]$$

$$= \epsilon_0 A \Big[4\pi \delta(\vec{r}) e^{-\lambda r} (1 + \lambda r)$$

$$+ \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ e^{-\lambda r} (1 + \lambda r) \right\} \Big]$$

$$= \epsilon_0 A \Big[4\pi \delta(\vec{r}) e^{-\lambda r} (1 + \lambda r)$$

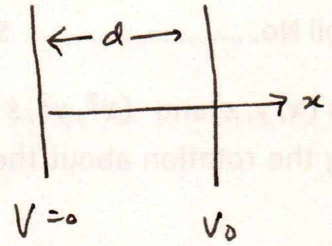
$$+ \frac{1}{r^2} \left\{ -\lambda e^{-\lambda r} (1 + \lambda r) + \lambda e^{-\lambda r} \right\} \Big]$$

For the first term, contribution comes only for $\vec{r} = 0$. So

$$\rho(\vec{r}) = \epsilon_0 A \left[4\pi \delta(\vec{r}) - \frac{\lambda^2 e^{-\lambda r}}{r} \right]$$

Charge can be found easily by integrating.

2.53



This problem is one-dimensional

(a) Poisson's equation $\frac{d^2V}{dx^2} = -\frac{\rho(x)}{\epsilon_0}$

(b) By energy conservation

$$\frac{1}{2} m_e v(x)^2 - eV(x) = 0$$

$$\text{or} \quad v(x) = \sqrt{\frac{2eV(x)}{m_e}}$$

(c)

$$J = \rho(x)\, v(x) = \text{constant} = C$$

(d)

$$\rho(x)\; v(x) = C$$

$$\rho(x) \cdot \sqrt{\frac{2eV}{m_e}} = C$$

$$\rho(x) = C\sqrt{\frac{m_e}{2eV}}$$

$$-\epsilon_0 \frac{d^2V(x)}{dx^2} = C\sqrt{\frac{m_e}{2eV(x)}}$$

Differential Equation for $V(x)$

$$\boxed{\frac{d^2V(x)}{dx^2} = -\frac{C}{\epsilon_0}\sqrt{\frac{m_e}{2eV(x)}}}$$

(e) Assume $V = kx^n$

$$\frac{d^2V}{dx^2} = k\, n(n-1)\, x^{n-2}$$

Equating power of x on both sides gives

$$n-2 = -\frac{n}{2} \quad \text{or} \quad n = \frac{4}{3}$$

So $\quad V(x) = k\, x^{4/3}$

$$V(x=d) = V_0 \Rightarrow k = \frac{V_0}{d^{4/3}}$$

So $$V(x) = V_0 \left(\frac{x}{d}\right)^{4/3}$$

So $$\rho(x) = -\epsilon_0 \frac{d^2 V}{dx^2}$$

$$= -\frac{\epsilon_0 V_0}{d^{4/3}} \cdot \left(\frac{4}{3}\right)\left(\frac{1}{3}\right) x^{-2/3}$$

$$= -\frac{4\epsilon_0 V_0}{9 d^{4/3}} x^{-2/3}$$

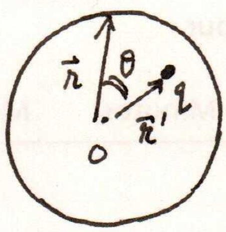
And $$v(x) = \sqrt{\frac{2eV(x)}{m_e}} = \sqrt{\frac{2e}{m_e} \cdot V_0 \left(\frac{x}{d}\right)^{4/3}}$$

$$= x^{2/3} \sqrt{\frac{2e V_0}{m_e d^{4/3}}}$$

So $$j(x) = \rho(x)\, v(x) = -\frac{4\epsilon_0 V_0}{9 d^{4/3}} \sqrt{\frac{2e V_0}{m_e d^{4/3}}} = -K_1 V_0^{3/2}$$

Current $$\boxed{I = A\, j(x)\, v(x) = A K_1 V_0^{3/2}}$$

3.1



Calculate the average of potential due to a charge 'q' at $\vec{r}'$

$(|\vec{r}'| < R)$

$$V_{av} = \frac{k}{4\pi R^2} \int \frac{q}{|\vec{R}-\vec{r}'|} \, ds \qquad k = \frac{1}{4\pi\epsilon_0}$$

$$= \frac{qk}{4\pi R^2} \int_{-1}^{1} \frac{2\pi R^2 \, d(\cos\theta)}{\sqrt{R^2 + r'^2 - 2Rr'\cos\theta}}$$

$$= \frac{qk}{2} \left[-\frac{1}{\cancel{2}Rr'} \times \cancel{2} \sqrt{R^2+r'^2-2Rr'\cos\theta} \right]_{-1}^{1}$$

$$= \frac{qk}{2} \left[\frac{1}{Rr'} \left\{ (R+r') - (R-r') \right\} \right.$$

$$= \frac{qk}{R} = \frac{q}{4\pi\epsilon_0 R}$$

Since each charge gives the same average irrespective of its position inside the sphere

$$V_{av} = \frac{Q_{enclosed}}{4\pi\epsilon_0 R}$$

$$V_{av} \text{ (due to charges outside)} = V_{centre}$$

So $$V_{av} = V_{centre} + \frac{Q_{enclosed}}{4\pi\epsilon_0 R}$$

Physical Interpretation:

The integral $k\int \frac{ds}{|\vec{R}'-\vec{r}'|}$

$=$ The electrostatic potential of a uniform surface charge of $\sigma = 1$ at $\vec{r}'$ inside the shell

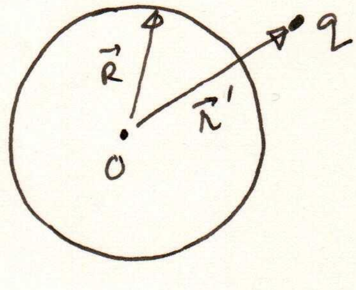
$$= \frac{k\, 4\pi R^2 \sigma}{R} = 4\pi k R$$

So

$$\frac{k}{4\pi R^2}\int \frac{q}{|\vec{R}'-\vec{r}'|}\, ds = \frac{q}{4\pi R^2}\cdot 4\pi k R$$

$$= \frac{kq}{R} = \frac{q}{4\pi\epsilon_0 R}$$

3.4



(a) $\vec{\mathcal{E}}'_{av} = \frac{kq}{4\pi R^2}\int \frac{ds\,(\vec{R}'-\vec{r}')}{|\vec{R}'-\vec{r}'|^3}$

$$= \frac{kq}{4\pi R^2}\left[+\vec{\nabla}'\int \frac{ds}{|\vec{R}'-\vec{r}'|}\right]$$

One can again calculate the integral as done above for $|\vec{r}'| > R$ or take the physical interpretation given above and get the result

$$k\int \frac{ds}{|\vec{R}'-\vec{r}'|} = \frac{4\pi k R^2}{r'}$$

So $\quad \vec{\mathcal{E}}'_{av} = +\dfrac{q}{4\pi R^2}\,\vec{\nabla}'\left(\dfrac{4\pi k R^2}{r'}\right)$

Also see direct physical interpretation of $k\int \dfrac{ds\,(\vec{R}-\vec{r}')}{|\vec{R}-\vec{r}'|^3}$ below.

$= -\dfrac{kq}{r'^2}\hat{r}'$

$=$ Electric field at the centre of sphere due to charge at $\vec{r}'$

So for all the charges outside the sphere

$\vec{\mathcal{E}}_{av} = \sum \mathcal{E}_{centre}$ due to each charge

$= \vec{\mathcal{E}}'_{centre}$

(b) for charge inside $|\vec{r}'| < R$ so we get

$$\int \frac{ds}{|\vec{R}-\vec{r}'|} = 4\pi R$$

So $\quad \vec{\nabla}'\int \dfrac{ds}{|\vec{R}-\vec{r}'|} = \vec{\nabla}'(4\pi R) = 0$

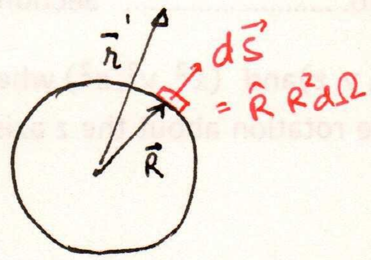
So $\vec{\mathcal{E}}_{av}$ (due to charges inside) $= 0$

By physical interpretation also

$k\int \dfrac{ds\,(\vec{R}-\vec{r}')}{|\vec{R}-\vec{r}'|^3} =$ Field at $\vec{r}'$ due to uniform surface charge $\sigma = 1$ over a sphere of radius R

So for $\vec{R} > |\vec{r}'|$ this integral will vanish (Field for such $\sigma = 0$)

3.37 :



$$V_{av} = \frac{1}{4\pi R^2} \int_{\text{sph.}} \frac{dS}{|\vec{R} - \vec{r}\,'|}$$

$$= \frac{1}{4\pi R^2} \int_{\text{sph}} \frac{R^2\, d\Omega}{|\vec{R} - \vec{r}\,'|}$$

$$= \frac{1}{4\pi} \int \frac{d\Omega}{|\vec{R} - \vec{r}\,'|}$$

$$\frac{dV_{av}}{dR} = \frac{1}{4\pi} \int d\Omega \, \frac{\partial}{\partial R} \frac{1}{|\vec{R} - \vec{r}\,'|}$$

$$= \frac{1}{4\pi} \int d\Omega \; \hat{R} \cdot \vec{\nabla} \frac{1}{|\vec{R} - \vec{r}\,'|}$$

$$= \frac{1}{4\pi R^2} \int \underbrace{R^2\, d\Omega\, \hat{R}}_{d\vec{S}'} \cdot \vec{\nabla} \frac{1}{|\vec{R} - \vec{r}\,'|}$$

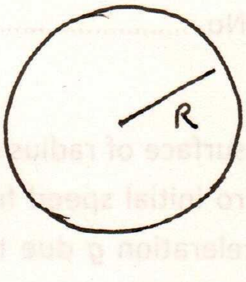
$$= \frac{1}{4\pi R^2} \int d\vec{S}' \cdot \vec{\nabla} \frac{1}{|\vec{R} - \vec{r}\,'|}$$

$= 0$ (By Gauss' theorem since there is no charge inside)

$$\Rightarrow \quad \frac{dV_{av}}{dR} = 0$$

The rest follows as given in the problem itself

3.41



$R = 3.5\,\mathring{A} = 3.5\times10^{-10}\,m$

charge on sphere $= -e$

$= -1.6\times10^{-19}\,C$

When the other electron is at distance r, the net force will be due to the (image charge q' on ~~neutral~~ grounded sphere + charge $(q-q')$ spread uniformly over the sphere)

$$F = -\frac{1}{4\pi\epsilon_0}\frac{q^2 R r}{(r^2-R^2)^2} + \frac{1}{4\pi\epsilon_0}\frac{\left(q+q\frac{R}{r}\right)^2}{r^2}$$

Force due to image charge

$q' = -q\frac{R}{r}$

Force due to charge $\left(q+q\frac{R}{r}\right)$ at the centre of the sphere

$$F = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{Rr}{(r^2-R^2)^2} + \frac{(r+R)^2}{r^4}\right]$$

So $F = 0 \implies$

$$(R+r)^2(r^2-R^2)^2 = r^5 R$$

Taking $r/R = z$, we have

$$\boxed{(1+z)^2(z^2-1)^2 = z^5}$$

Thus gives $z = 1.404 \implies r = 3.86\,\mathring{A}$

(b) To push an electron, one has to do work against the force on electron

$$\text{Work} = -\int_{\infty}^{r_0} F\, dr$$

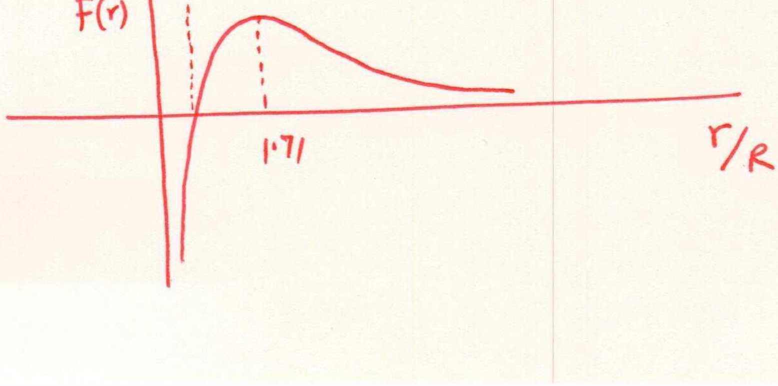
$$= \frac{q^2}{4\pi\epsilon_0}\left[\int_{\infty}^{r_0} \frac{Rr\, dr}{(r^2-R^2)^2} - \int_{\infty}^{r_0} \frac{r^2+R^2+2rR}{r^4}\, dr\right]$$

where $r_0 = 1.404\, R$

$$= \frac{q^2}{4\pi\epsilon_0}\left[\int_{\infty}^{\sqrt{r_0^2-R^2}} \frac{R\, y\, dy}{y^4} - \int_{\infty}^{r_0} \frac{dr}{r^2} - R^2\int_{\infty}^{r_0} \frac{dr}{r^4} - 2R\int_{\infty}^{r_0} \frac{dr}{r^3}\right]$$

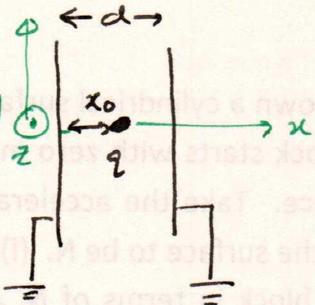
Carrying out the integral gives work $= 3.4\,eV$.

The force vs. distance looks as follows

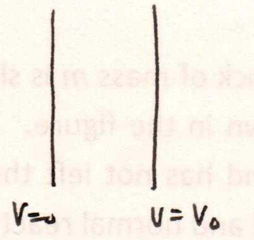


3.51

(a)



Situation 1



Situation 2

By reciprocity theorem

$$\int V_1 \rho_2 \, d\tau + \int V_1 \sigma_2 \, ds = \int V_2 \rho_1 \, d\tau + \int V_2 \sigma_1 \, ds$$

Given

$$V_1 = 0 \text{ on both plates}$$

$$\rho_1 = q\,\delta(x-x_0)\,\delta(y)\,\delta(z)$$

$\int \sigma_1 ds =$ we want to find

$$\left.\begin{array}{l} V_2 = 0 \text{ on left plate} \\ \quad = V_0 \text{ on right plate} \end{array}\right\} \Rightarrow V_2 = \frac{V_0 x}{d}$$

$$\rho_2 = 0$$

$\sigma_2 =$ some finite quantity but irrelevant since $V_1 = 0$

$$\left(\sigma_2 = \pm \frac{V_0}{d\epsilon_0}\right)$$

Substituting we get

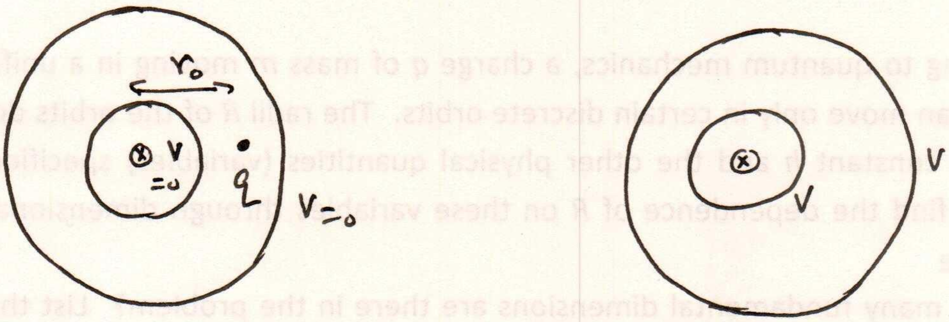
$$0 = \frac{V_0 x_0 q}{d} + V_0 \underbrace{\left(\int \sigma_1 ds\right)_{\text{right plate}}}_{\text{induced charge on right plate}}$$

$\Rightarrow$ Induced charge on right plate $= \left(\dfrac{q x_0}{d}\right)$

(13)

Similarly charge on left plate can be found by taking $V = V_0$ on left plate and $V = 0$ on right plate

(b)



Situation 1 Situation 2

Given $V_1 = 0$ at both plates

$\rho_1 = q\,\delta(\vec{r} - \vec{r}_0)$

$\int \sigma_1 \, ds$ = We want to know

$$\left.\begin{array}{l} V_2 = V \text{ at } r = a \\ \quad = 0 \text{ at } r = b \end{array}\right\} \Rightarrow V(r) = V - \underbrace{\frac{bV}{r}\left(\frac{r-a}{b-a}\right)}$$

This is derived by take $E = \frac{C}{r^2}$ and finding C by $V(a) = V$ & $V(b) = 0$

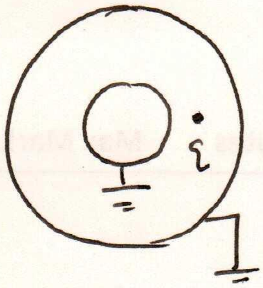
$\rho_2 = 0$

σ_2 : irrelevant although it can be found

By reciprocity theorem

$$0 = Vq - \frac{bVq}{r_0}\left(\frac{r_0 - a}{b-a}\right) + V\,Q_{\text{inner-shell}}$$

$$\Rightarrow Q_{\text{inner-shell}} = -\frac{qa}{r_0}\left(\frac{b - r_0}{b-a}\right)$$



Since $E(r>b)=0$

by Gauss's law

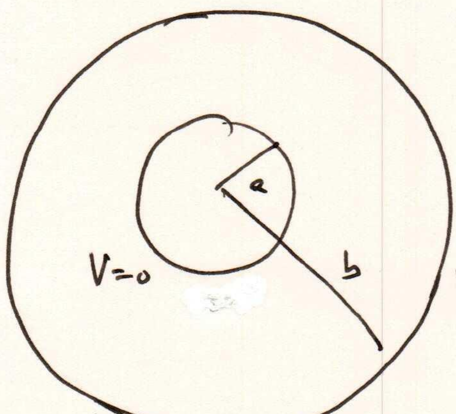
$$Q_{inner\cdot surface} + q + Q_{outer\ sur} = 0$$

$$\Rightarrow \quad Q_{outer\ surface} = -q - Q_{inner\cdot surface}$$

$$= -q + \frac{qa}{r_0}\left(\frac{b-r_0}{b-a}\right)$$

$$= -\frac{qb}{r_0}\left(\frac{r_0-a}{b-a}\right)$$

The result can also be obtained by taking situation 2 as follows



Situation 2

10 $V(i^{th}\text{ charge}) = \dfrac{ie}{C}$

when i numba of elections are on the

work done in bringing $(i+1)^{th}$ election

$$= \frac{ie^2}{C}$$

(i) So total energy

$$= \sum_{i=1}^{N-1} \frac{ie^2}{C}$$

$$= \frac{1}{2} \frac{(N-1)Ne^2}{C}$$

(ii)