Department of physics
Indian Institute of Technology, Kanpur

MKH/SMT/2016-17-II/Phy103 Assignment-1 Week of 9-1-17

1. Suppose the electrostatic potential of a point charge were $\varphi(r) = 1/r^{1+\epsilon}$.
   (a) Find the potential $V(r)$ at a point at a distance *r* from the centre of a spherical shell of radius *R* (*R*>*r*) that carries a surface charge density σ per unit area.
   (b) To the first order in ϵ, show that
   $$\frac{V(R)-V(r)}{V(R)} = \frac{\epsilon}{2}\left[\frac{R}{r} ln\left(\frac{R+r}{R-r}\right) - ln\left(\frac{4R^2}{R^2-r^2}\right)\right]$$
   (c) By differentiating the potential with respect to *r*, find the corresponding electric field and plot it.

2. Imagine a two dimensional world. In this world, it is observed that if a metallic disc is given a charge, the entire charge moves to the periphery of the disc. Using geometric arguments similar to those given in the lecture, find the dependence of the electric field on the distance from a unit charge.

3. A cube is filled uniformly with charge. If the electrostatic potential at its centre is V_0 and at one of its corners V_1 then find the ratio V_0/V_1.

Hints:

1. (a) Either do the direct integration over the spherical shell (recommended as it will give you some practice in using spherical coordinates) or use the general expression derived in the class.
   (b) For $\epsilon \to 0$, $x^{1+\epsilon} = x(1 + \epsilon lnx)$. Can you derive it?

2. Electric field must vanish inside a metal.

3. Divide the cube into eight equal cubes and use the dependence (dimensionality) of the potential on charge and distance.

If time permits, do problem 2.18 from the book.
Solution for that problem is also attached.

SOLUTION ASSGN 1

1 (a) From expression derived in Lecture on 6 Jan

$$V(z) = \frac{2\pi R\sigma}{z} \int_{|z-R|}^{z+R} \phi(t)\, t\, dt$$

for potential of a spherical shell charged uniformly by surface charge σ

In the present case

$$\phi(t) = \frac{1}{t^{1+\epsilon}}$$

So $$V(z) = \frac{2\pi R\sigma}{z} \int_{|z-R|}^{z+R} \frac{dt}{t^{\epsilon}}$$

$$= \frac{2\pi R\sigma}{z(1-\epsilon)} \cdot t^{1-\epsilon}\Big|_{|z-R|}^{z+R}$$

One also could have gotten it directly by integrating over a spherical shell the expression (as done in the lecture)

$$V(z) = \int \frac{2\pi R^2\sigma\, d(\cos\theta)}{(R^2+z^2-2zR\cos\theta)^{\frac{1+\epsilon}{2}}}$$

$$V(z) = \frac{2\pi R\sigma}{z(1-\epsilon)} \left[(z+R)^{1-\epsilon} - (R-z)^{1-\epsilon} \right] \qquad z < R$$

Replace z by r to get

$$V(r) = \frac{2\pi R\sigma}{r(1-\epsilon)}\left[(r+R)^{1-\epsilon} - (R-r)^{1-\epsilon}\right] \quad r<R$$

(b) $$V(R) = \frac{2\pi R\sigma}{R(1-\epsilon)}\cdot(2R)^{1-\epsilon}$$

$$= \frac{2\pi\sigma}{(1-\epsilon)}\cdot(2R)^{1-\epsilon}$$

$$\frac{V(R)-V(r)}{V(R)} = \frac{\frac{2\pi\sigma}{(1-\epsilon)}\left[(2R)^{1-\epsilon} - \frac{R}{r}(R+r)^{1-\epsilon} + \frac{R}{r}(R-r)^{1-\epsilon}\right]}{\frac{2\pi\sigma}{(1-\epsilon)}(2R)^{1-\epsilon}}$$

$$= 1 - \frac{R}{r}\left(\frac{1}{2}+\frac{1}{2}\frac{r}{R}\right)^{1-\epsilon} + \frac{R}{r}\left(\frac{1}{2}-\frac{1}{2}\frac{r}{R}\right)^{1-\epsilon}$$

Now use $y = x^{1-\epsilon}$

$$\ln y = (1-\epsilon)\ln x = \ln x - \epsilon\ln x$$

$$y = e^{\ln x}e^{-\epsilon\ln x}$$

$$\simeq x(1-\epsilon\ln x) \qquad \epsilon\to 0$$

Substitute this to get

$$\frac{V(R)-V(r)}{V(R)} = \frac{\epsilon}{2}\ln\left(\frac{R^2-r^2}{4R^2}\right) + \frac{\epsilon}{2}\frac{R}{r}\ln\left(\frac{R+r}{R-r}\right)$$

For $\epsilon\to 0$ $\quad V(R) = V(r)$ for $r<R$

(c)
$$\vec{E}(r) = \hat{r}\left(-\frac{\partial V(r)}{\partial r}\right)$$

$$= \frac{2\pi\sigma R}{r^2(1-\epsilon)}\left[(R+r)^{1-\epsilon} - (R-r)^{1-\epsilon}\right]$$

$$- \frac{2\pi\sigma R}{r}\left[\frac{1}{(R+r)^{\epsilon}} + \frac{1}{(R-r)^{\epsilon}}\right]$$

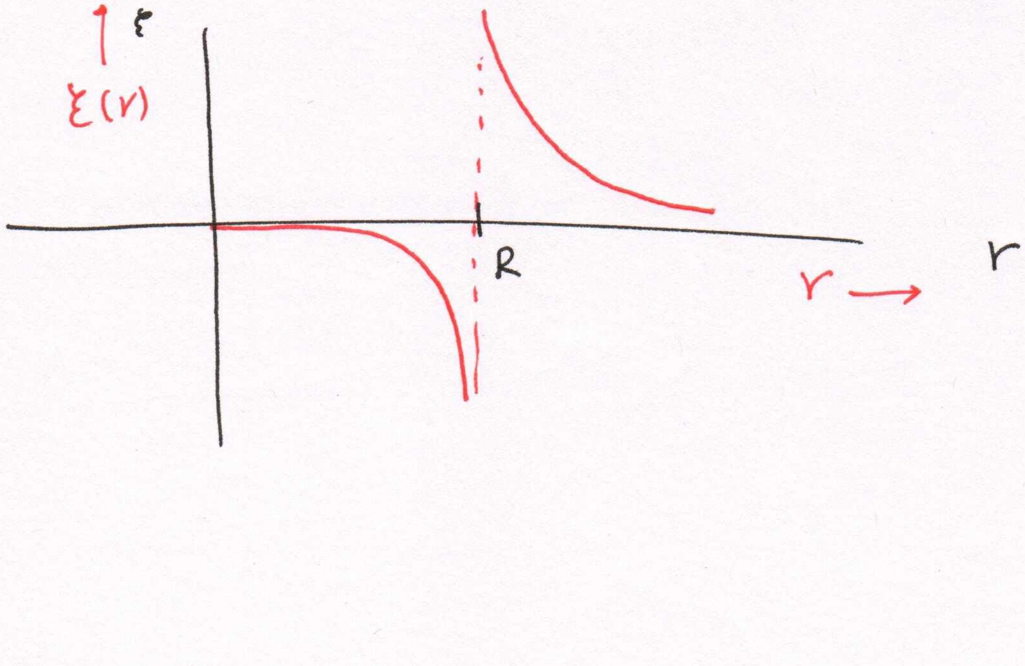
for $r < R$

$$\vec{E}(r) = \hat{r}\left(-\frac{\partial V(r)}{\partial r}\right)$$

$$= \frac{2\pi\sigma R}{r^2(1-\epsilon)}\left[(R+r)^{1-\epsilon} - (r-R)^{1-\epsilon}\right]$$

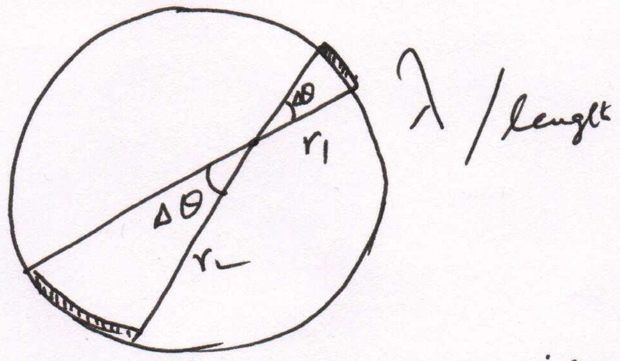
$$- \frac{2\pi\sigma R}{r}\left[\frac{1}{(R+r)^{\epsilon}} - \frac{1}{(r-R)^{\epsilon}}\right] \quad \text{for } r > R$$

The field looks like



(2) Since the field is always zero inside a metal, and the disc has a cylindrical symmetry

The charge that goes out on the periphery is going to be distributed uniformly over the circumference



Field due to two arcs on opposite sides must cancel.

If $\quad E \propto f(r)$ then

$$\lambda \cdot \Delta\theta \cdot r_1 \, f(r_1) = \lambda \, \Delta\theta \, r_2 \, f(r_2)$$

for $\quad E_{net} = 0$

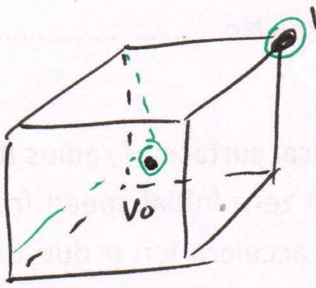
$$\Rightarrow \quad r_1 f(r_1) = r_2 f(r_2) = \text{Constant}$$

$$f(r) \propto \frac{1}{r}$$

So electric field will depend as $\frac{1}{r}$ on the distance from a unit charge

(5)

(3)



Let the field at a corner be V_1

If the cube is divided into eight equivalent cubes, charge on each cube will be $\frac{Q}{8}$ and the distance (edge) of the smaller cube $\left(\frac{1}{2}\right)$

So V_{corner} (smaller cubes) $= \frac{V_1}{8} \times 2$

$= \left(\frac{V_1}{4}\right)$

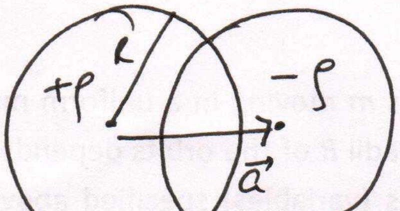
$\Rightarrow \quad V_0 = 8\, V_{corner}$ (smaller cube)

$= 2\, V_1$

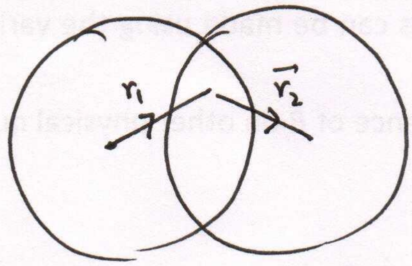
$\frac{V_0}{V_1} = 2$

(4) A possible problem if time permits (2.18) from the text book.

Electric field due to two overlapping spheres of radius R and carrying charge density $\pm\rho$



Let $\vec{a}$ be vector from centre of $+\rho$ sphere to $-\rho$ sphere



$$\vec{E}_+ = \frac{\rho \vec{r}_1}{3\epsilon_0} \text{ (by Gauss' law)}$$

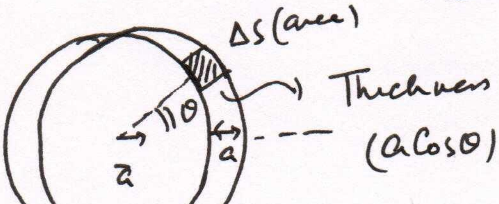
$$\vec{E}_- = \frac{\rho \vec{r}_2}{3\epsilon_0} \text{ (by Gauss law)}$$

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{\rho}{3\epsilon_0}(\vec{r}_1 + \vec{r}_2) = \frac{\rho \vec{a}}{3\epsilon_0}$$

Interesting limit $(\vec{a} \to 0$, $\rho \to \infty)$

Measure charge in the shaded region



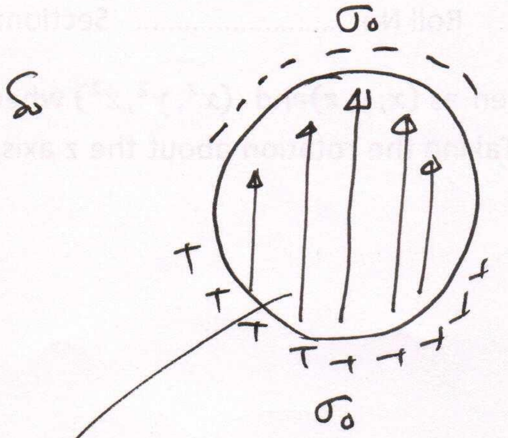
$$\Delta q = -\Delta S . \rho . a\cos\theta$$

$\Rightarrow$ Surface charge density

$$= \frac{\Delta q}{\Delta S} = -\rho a \cos\theta$$

$$= \sigma_0 \cos\theta$$

7



$\sigma' = -\sigma_0 \cos\theta$

give $\vec{E}' = \left(\frac{\rho a}{3\epsilon_0}\right)$

$= \left(\frac{\sigma_0}{3\epsilon_0}\right)$

in the direction of Z axis

Constant electric field.