Department of physics
Indian Institute of Technology, Kanpur

MKH/SMT/2016-17-II/Phy103 Assignment-4 Week of 30-1-17

Note: The problems assigned are from the text book *Introduction to Electrodynamics* (4^{th} edition) by David J. Griffiths

1. Problem 3.32
2. Problem 3.35
3. Problem 3.43
4. Problem 3.58
5. Problem 4.6
6. Problem 4.9
7. Problem 4.11
8. Problem 4.13
9. Problem 4.15
10. Problem 4.16
11. Problem 4.18
12. Problem 4.30
13. Problem 4.31
14. Problem 4.33
15. Problem 4.35

Hints:

3.35: Net charge is zero so the first non-zero contribution to the field comes from the dipole. Dipole moment will be in the z-direction.

3.43: The net field inside the conductor is zero. Surface charge density $\sigma(\theta) = k\cos\theta$ over a spherical shell gives constant field inside the sphere and a dipole field outside.

3.58: Use image problem for a grounded sphere and $\sigma = -\hat{n}\cdot\vec{\nabla}V$ on a conducting surface.

4.6: A perfect dipole $\vec{p} = q\vec{a}$ with $q \to \infty$ and $a \to 0$ or also a spherical shell with surface charge density $\sigma(\theta) = k\cos\theta$. Method of images can be used here to find the net field.

4.11: Find bound charges in each case and calculate the field given by them.

4.13: Two-dimensional analogue of uniformly polarized sphere.

4.15 and 4.16: Look at the bound charges and also at the boundary conditions.

4.18: Displacement remains the same throughout the dielectric.

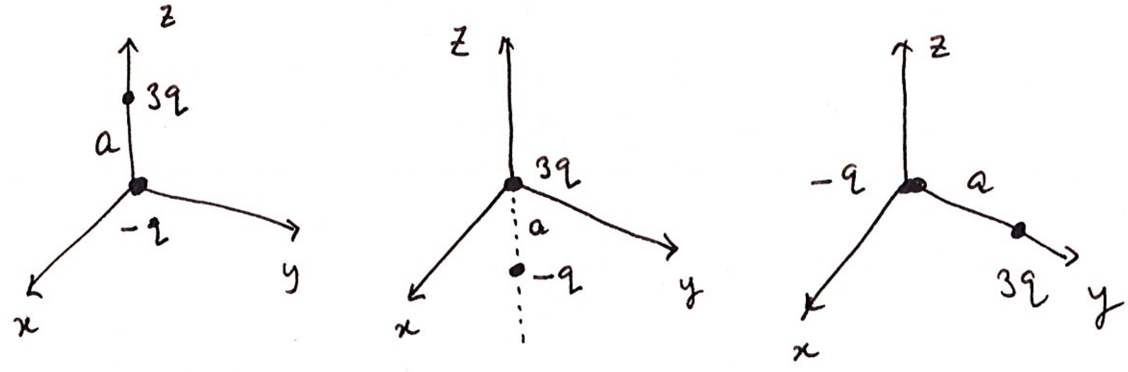
4.30: Draw the field lines and look at the formula $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$

4.31: It will be helpful to use planar polar coordinates here. The dipole experiences both a force and a torque.

4.35: Use Poisson's equation and boundary conditions at the spherical surface.

SOLUTION, ASSGN. 4

3.32



(a) (b) (c)

(i) Monopole is $3q - q = 2q$ in all the cases

(ii) Dipole in

(a) $3qa\,\hat{z}$

(b) $3qa\,\hat{z}$

(c) $3qa\,\hat{y}$

$\vec{p} = \Sigma q_i \vec{a}_i$

(iii) Approximate potential in

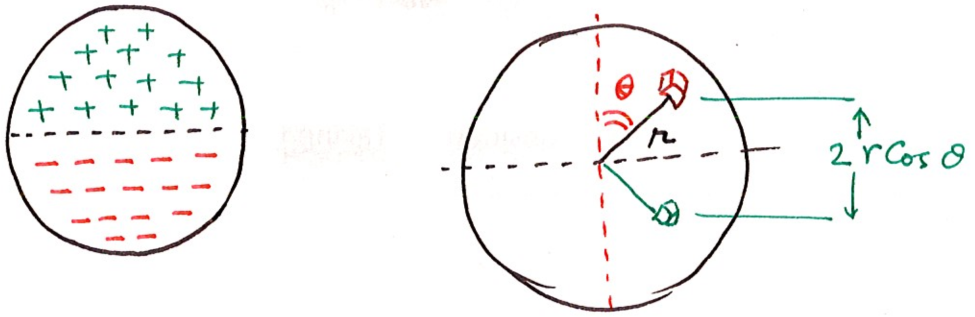
(a) $$\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]$$

(b) $$\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]$$

(c) $$\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2}\right]$$ up to dipole

3.35



The red and gree pair in the right sphere

gives $\quad d\vec{p} = \left(\rho \cdot r^2 dr \cdot d(\cos\theta)\, d\varphi\right) 2r\cos\theta\, \hat{z}$

(This has taken into account the charge in the lower hemisphere too)

So $\quad \vec{p} = 2\rho \int_0^{2\pi} d\varphi \int_0^{1} \cos\theta\, d(\cos\theta) \int_0^{R} r^3 dr$

This integral is only over the upper hemisphere because of the comment above

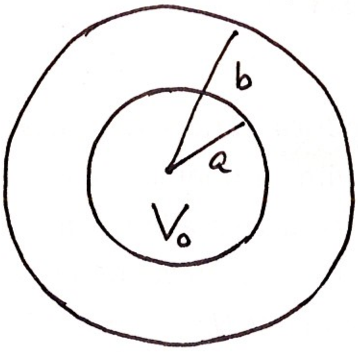
$$= 4\pi\rho \times \frac{R^4}{4} \times \frac{1}{2} = \left(\frac{\pi\rho R^4}{2}\right)$$

NOTE: The same answer can also be obtained by (distance of the two centre of charges) $\times$ (charge in each hemisphere) because this calculation is like the centre of mass calculation.

CM for hemisphere $= \left(\frac{3R}{8}\right)$ above the centre of sphere

$$\Rightarrow \quad \vec{p} = \frac{2\pi R^3}{3}\rho \times 2 \times \frac{3R}{8} = \frac{\pi R^4 \rho}{2}$$

3.43 $\sigma(\theta) = k\cos\theta$



Potential everywhere will have three components:

(i) Potential due to the boundary contion that

$$V(a) = V_0$$

(ii) Potential due to the surface charge density on the outer shell

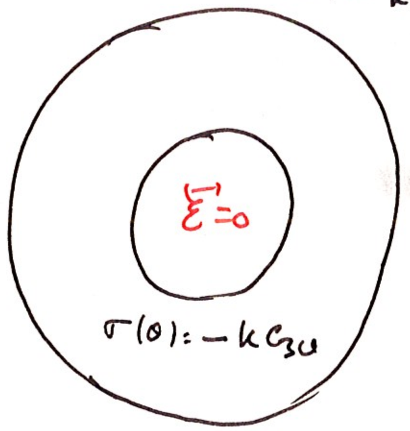
(iii) Potential due to the induced charge

The problem can be solved in two ways:

(a) directly by realizing that field given by $\sigma(\theta) = k\cos\theta$ inside the shell is $-\frac{k}{3\epsilon_0}\hat{z}$ and field inside the ~~shell~~ conducting sphere is zero, and

(b) by the image method.

DIRECT METHOD: To make field vanish inside the conducting sphere, charge $-k\cos\theta$ will get induced on the conducting sphere. Thus ~~after~~ after the charges are induced, the picture is as follows



Also the dipole moment of $\sigma(\theta) = k\cos\theta$ is

$$\vec{p} = \frac{4\pi}{3} k R^3 \hat{z}$$

if the surface charge density is spread over a sphere of radius R

Thus for

$\underline{r > b}$: the potential is the sum of of two potentials — that due to $\vec{p}$ of $\sigma = k\cos\theta$ over sphere of radius b and the other due to $\vec{p}$ of $\sigma = -k\cos\theta$ over sphere of radius a.

In addition there is background potential of $\left(\frac{V_0 a}{r}\right)$ because $V = V_0$ at $r = a$

Thus $$V(r > b) = \frac{1}{4\pi\epsilon_0}\left[\frac{\frac{4\pi}{3} k b^3\, \hat{z}\cdot\hat{r}}{r^2} - \frac{\frac{4\pi}{3} k a^3\, \hat{z}\cdot\hat{r}}{r^2}\right] + \frac{V_0 a}{r}$$

$$= V_0 \frac{a}{r} + \frac{(b^3 - a^3)}{r^2} \frac{k\cos\theta}{3\epsilon_0}$$

For $a \le r \le b$, the potential is

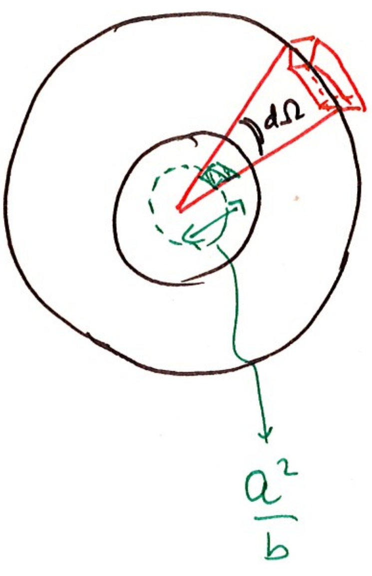
$\frac{V_0 a}{r}$ + potential for electric field $-\frac{k}{3\epsilon_0}\hat{z}$

+ potential of dipole $\vec{p} = -\frac{4\pi}{3} k a^3 \hat{z}$

$$= \frac{V_0 a}{r} + \frac{kz}{3\epsilon_0} - \frac{1}{4\pi\epsilon_0} \cdot \frac{4\pi}{3} \frac{k a^3\, \hat{z}\cdot\hat{r}}{r^2}$$

$$= \frac{V_0 a}{r} + \frac{k r \cos\theta}{3\epsilon_0} - \frac{k a^3 \cos\theta}{3\epsilon_0 r^2}$$

IMAGE METHOD: For the charge contained in solid angle $d\Omega$ (See figure) there is an image inside the conductor



Charge $dq = (d\Omega\, b^2)\,\sigma(\theta)$

Image $dq' = -(d\Omega\, b^2)\,\sigma(\theta) \cdot \frac{a}{b}$

$$\sigma'(\theta) = \frac{dq'}{d\Omega (a^2/b)^2} = -\sigma(\theta)\frac{b^3}{a^3}$$

Radius of the image sphere

$$= -\frac{k b^3}{a^3}\cos\theta$$

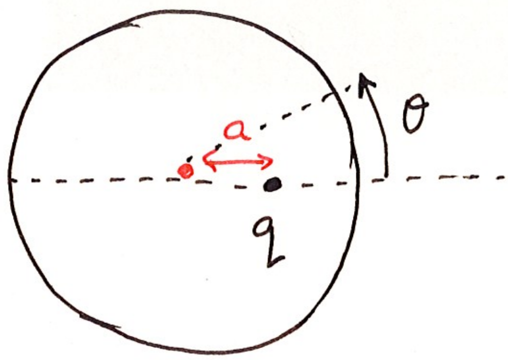
Field for $r \geq a$ by the induced charge will include field due to this image

But this image charge also is equivalent to a dipole $= \frac{4\pi}{3}\left(\frac{a^2}{b}\right)^3 \times \left(-\frac{k b^3}{a^3}\right)\hat{z}$

$$= -\frac{4\pi}{3} a^3 k \hat{z}$$

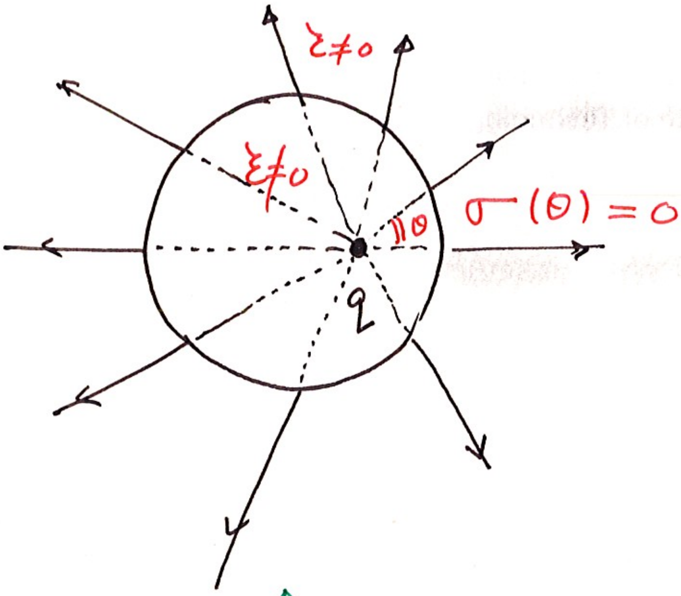
This is the same dipole moment as obtained earlier and leads to the same answers.

3.58



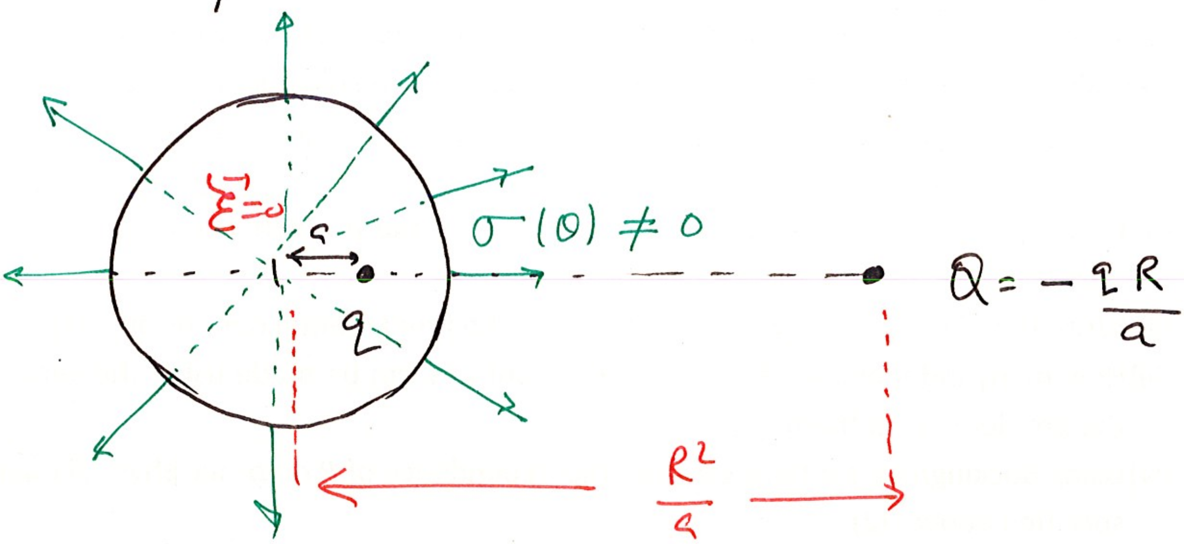
We wish to find a $\sigma(\theta)$ that gives the same field outside that is given by 'q' shown above.

For this consider the following situation.



Field outside is that given by charge q

$\sigma(\theta) = 0$ by Gauss' law



Field outside is that given by the sum of charge 'q' and its image $-\frac{qR}{a}$ at $\frac{R^2}{a}$ distance

OR

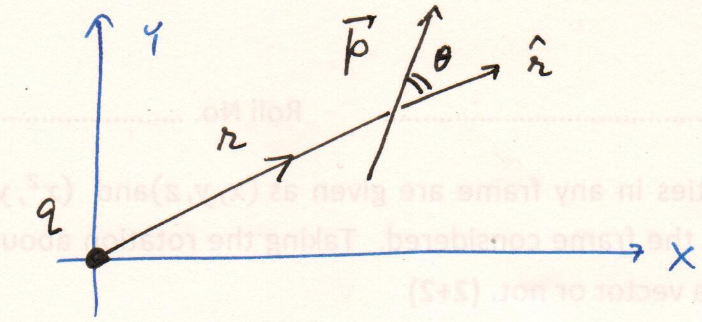
Field outside is that given by the sum of field given by $\sigma(\theta)$ and $-\frac{qR}{a}$ at R^2/a

$\Rightarrow$ Field given by 'q' and $\sigma(\theta)$ are the same

$$\Rightarrow \sigma(\theta) = -\epsilon_0 \hat{n} \cdot \vec{\nabla} V = -\epsilon_0 \frac{\partial}{\partial r} \left[\frac{q}{|\vec{r} - a\hat{z}|} - \frac{qR/a}{|\vec{r} - \frac{R^2}{a}\hat{z}|} \right]_{r=R}$$

$=$ Answer

4.9



Let $\vec{p}$ be in the XY plane

$$\vec{p} = p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$$

$\vec{E}$ at 'q' due to the dipole

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}\right]$$

Although direction of $\vec{r}$ from $\vec{p}$ to q is opposite to $\vec{r}$, $3(\vec{p}\cdot\hat{r})\hat{r}$ term remains unchanged because of two '−' signs occuring there

$$\text{Force on } q = q\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{3p\cos\theta\,\hat{r} - p\cos\theta\,\hat{r} - p\sin\theta\,\hat{\theta}}{r^3}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{2p\cos\theta\,\hat{r} - p\sin\theta\,\hat{\theta}}{r^3}\right]$$

$$\text{Force on dipole} = (\vec{p}\cdot\vec{\nabla})\,\vec{E}_{\text{due to } q}$$

Use $\frac{\partial}{\partial r}\hat{r} = 0$

$\frac{\partial}{\partial \varphi}\hat{r} = \hat{\varphi} = \hat{\theta}$

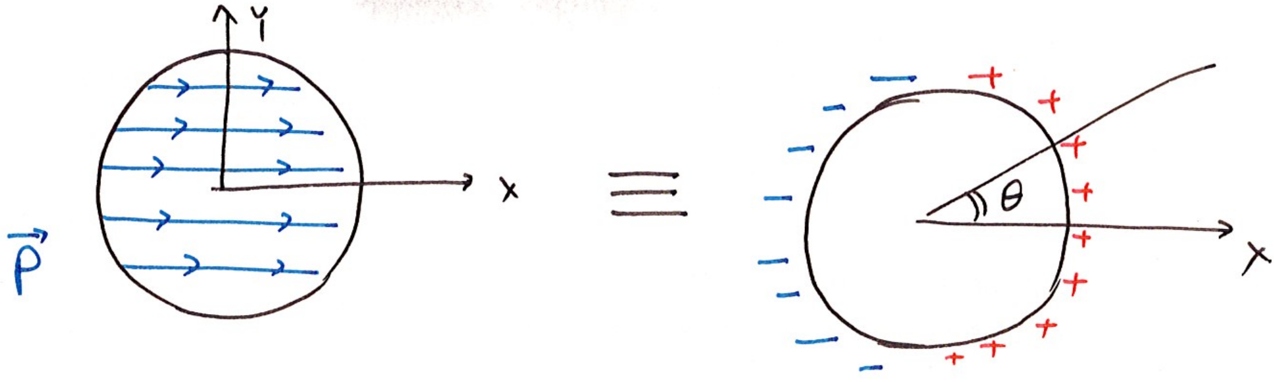
$$= \left(p_r\frac{\partial}{\partial r} + p_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\frac{kq}{r^2}\hat{r}$$

$$= \left(p\cos\theta\frac{\partial}{\partial r} + \frac{p\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\frac{kq}{r^2}\hat{r}$$

$$= -\text{ the force above}$$

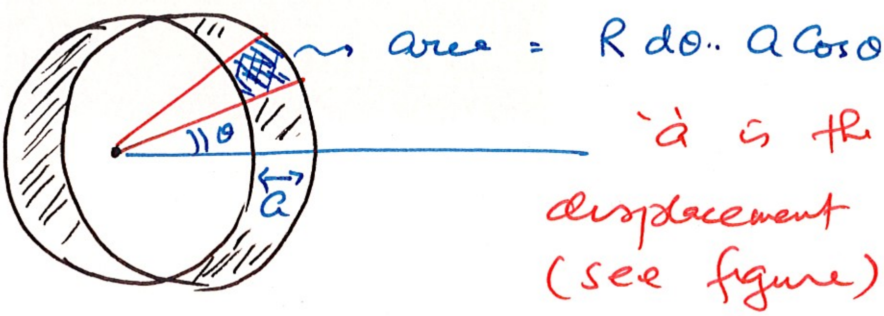
4.11 Done in Lecture

4.13



$$\sigma(\theta) = \vec{P}\cdot\hat{n}$$
$$= P\cos\theta$$

The charge distribution above can be generated by displacing two long cylinders carrying charge densities ρ and $-\rho$ with respect to each other



Charge in the shaded area (blue shaded area)

$$= l \times R\,d\theta\,.\,a\cos\theta\,\rho$$

for length 'l' of the cylinder

$$dq = (Rl\,d\theta)\rho\,a\cos\theta$$

$$\sigma(\theta) = \frac{dq}{(Rl\,d\theta)} = \rho a\cos\theta$$

$R\ell\, d\theta$ is the surface area along the length of the cylinder

Ref $\rho \to \infty$ and $a \to 0$ such that

$$\rho a \to \sigma_0 \ (\text{or } P)$$

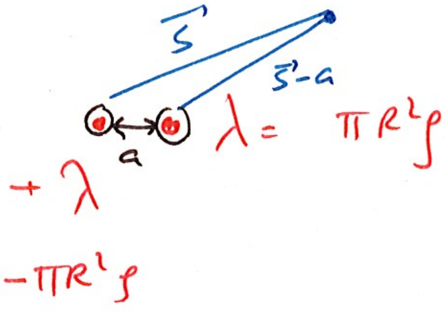
$$\sigma(\theta) = \sigma_0 \cos\theta = P\cos\theta$$

So field inside will be (by Gauss' law)

$$= -\frac{P}{2\epsilon_0}\hat{x}$$

Field outside will be superposition of fields due to two long wires carrying charge per unit length $\lambda = \pm \pi R^2 \rho$ and displaced with respect to each other by 'a'

Field outside the cylinders



$$\vec{E}(s) = -\frac{\lambda \vec{s}}{2\pi\epsilon_0 s^2} + \frac{\lambda(\vec{s}-a\hat{x})}{2\pi\epsilon_0 |\vec{s}-a\hat{x}|^2}$$

$$\lambda = \pi R^2 \rho$$

$$\vec{\mathcal{E}}(\vec{s}) = -\frac{\lambda \vec{s}}{2\pi\epsilon_0 s^2} + \frac{\lambda}{2\pi\epsilon_0}\frac{(\vec{s}-a\hat{x})}{(s^2+a^2-2sa\cos\theta)}$$

$$= -\frac{\lambda \vec{s}}{2\pi\epsilon_0 s^2} + \frac{\lambda}{2\pi\epsilon_0}\frac{(\vec{s}-a\hat{x})}{s^2}\left[1+\frac{a^2}{s^2}-\frac{2a}{s}\cos\theta\right]^{-1}$$

$$= \cancel{-\frac{\lambda \vec{s}}{2\pi\epsilon_0 s^2}} + \cancel{\frac{\lambda \vec{s}}{2\pi\epsilon_0 s^2}} + \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{2a\cos\theta}{s^3}\vec{s}$$

$$- \frac{\lambda}{2\pi\epsilon_0}\frac{a\hat{x}}{s^2}$$

(Keeping terms up to order 'a' in the expansion of $\left[1+\frac{a^2}{s^2}-\frac{2a}{s}\cos\theta\right]^{-1}$)

$$= \frac{1}{2\pi\epsilon_0}\pi R^2\left[2\rho a\cos\theta\frac{\vec{s}}{s^3} - \frac{\rho a\hat{x}}{s^2}\right]$$

using $\lambda = \pi R^2 \rho$

Now $\rho a \to P$ $\qquad P\hat{x} = \vec{P}$

$P\cos\theta = \vec{P}\cdot\hat{s} \qquad \frac{\vec{s}}{s^3} = \frac{\hat{s}}{s^2}$

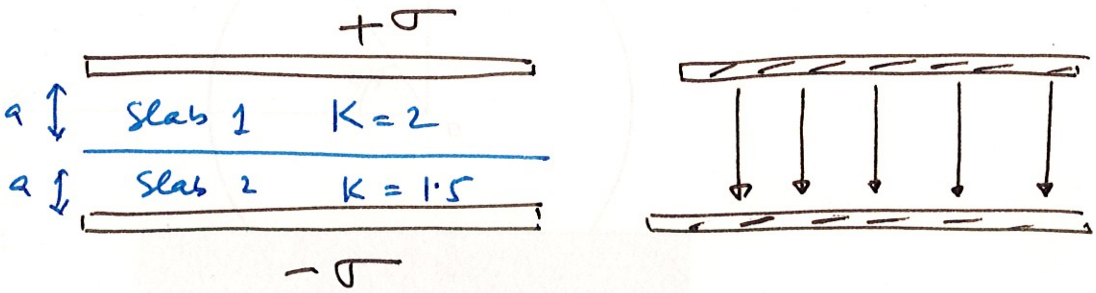
$$\Rightarrow \vec{\mathcal{E}}(\vec{s}) = \frac{R^2}{2\epsilon_0}\left[\frac{2(\vec{P}\cdot\hat{s})\hat{s} - \vec{P}}{s^2}\right]$$

4.15 — Done in Lecture

4.16 Cavity can be thought of as the original bulk material + material of cavities' shape with $-\vec{P}$

Now use principle of superposition.

4.18



$\vec{\nabla}\cdot\vec{D} = \rho_{free}$; By symmetry $\vec{\nabla}\times\vec{D} = 0$

because $\vec{D} \perp$ plates (like $\vec{E}$ without dielectrics)

(a) $\Rightarrow \vec{D} = -\sigma\,\hat{z}$

$D_\perp$ is continuous so D is going to be the same in both slabs. ($E = D/\epsilon_0 K$)

(b) $E_{upper\ slab} = -\frac{\sigma}{2\epsilon_0}\hat{z}$, $E_{lower\ sl} = -\frac{2\sigma}{3\epsilon_0}\hat{z}$

(c) $\vec{P} = \vec{D} - \epsilon_0\vec{E}$

$$\vec{P}_{\text{upper slab}} = \left(-\sigma + \frac{\sigma}{2}\right)\hat{z} = -\frac{\sigma}{2}\hat{z}$$

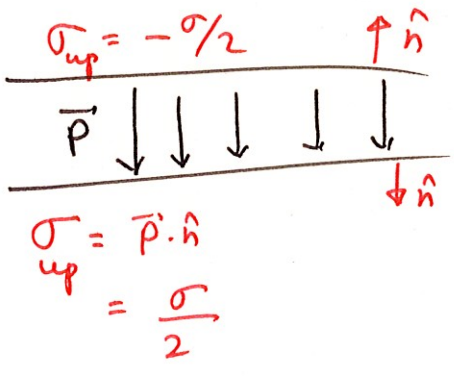
$$\vec{P}_{\text{lower slab}} = \left(-\sigma + \frac{2\sigma}{3}\right)\hat{z} = -\frac{\sigma}{3}\hat{z}$$

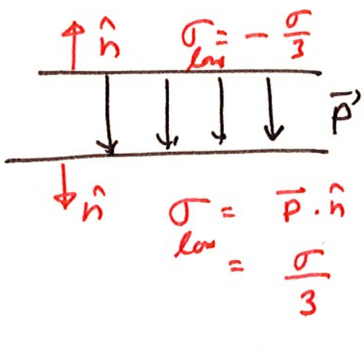
(d) $$\Delta V = E_{\text{upper}} \times a + E_{\text{lower}} \times a$$

$$= \frac{a\sigma}{\epsilon_0}\left(\frac{1}{2} + \frac{2}{3}\right)$$

$$= \frac{7a\sigma}{6\epsilon_0}$$

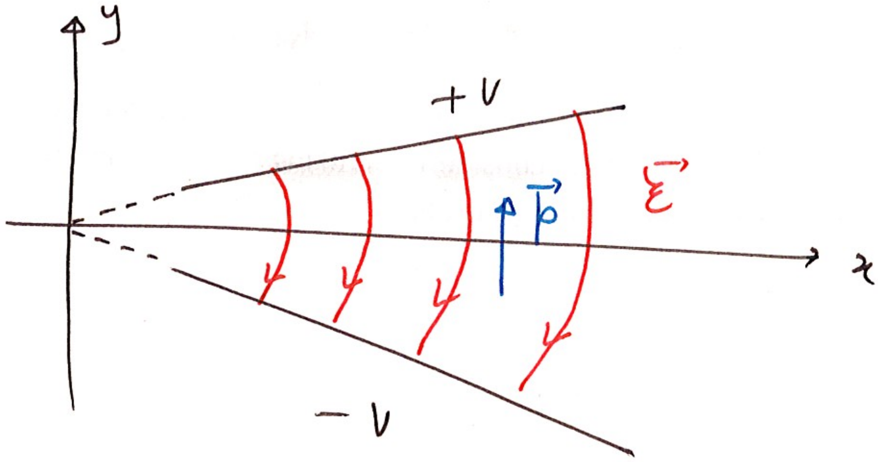
(e) $$\sigma_b = \vec{P}\cdot\hat{n} \qquad (\rho_b = \vec{\nabla}\cdot\vec{P} = 0)$$





(f) Use standard sheet of charge formula

4.30

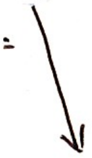


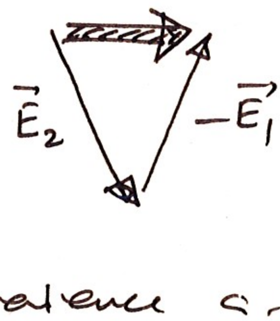
Field lines are going to be as shown ($\perp$ to the equipotential plates)

$$\vec{F}_{\text{on dipole}} = (\vec{p}\cdot\vec{\nabla})\,\vec{\mathcal{E}}$$

$$= p\,\frac{\partial}{\partial y}\vec{E} \qquad \vec{p} = p\,\hat{y}$$

As y changes through the $\vec{p}$'s position (x axis) $\vec{\mathcal{E}}$ changes direction from

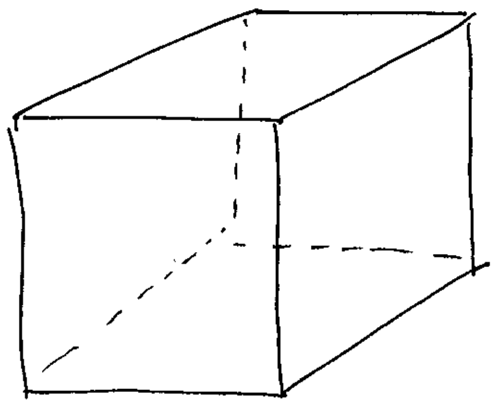
$E_1 =$  to $\bar{E}_2 =$ 

$\vec{E}_2 - \vec{E}_1 =$ 

So $\vec{p}$ will experience a force pointing in the $+\hat{x}$ direction

4.31 Done in Lecture

4.33



$\vec{P} = k\vec{r}$ $\rho_b = -\vec{\nabla}\cdot\vec{P} = -3k$

$$\sigma_b = \vec{P}\cdot\hat{n} = +k\vec{r}\cdot\hat{x}\Big|_{x=\frac{a}{2}}$$

$$= +kx\Big|_{a/2}$$

$$= +\frac{ka}{2} \text{ on the surfaces}$$

$\perp$ to the x-axis

The same answer for face $\perp$ to y- and z-axis

$$\text{Net charge} = 6\cdot\frac{ka}{2}\cdot a^2 - 3ka^3 = 0$$

4.35: Done in Lecture