Department of physics
Indian Institute of Technology, Kanpur

MKH/SMT/2016-17-II/Phy103 Assignment-2 Week of 16-1-17

Note: The problems assigned are from the text book *Introduction to Electrodynamics* (4^{th} edition) by David J. Griffiths

1. Problem 1.20
2. Problem 1.57
3. Problem 2.20
4. Problem 2.21
5. Problem 2.26
6. Problem 2.34 and 2.35
7. Problem 2.39
8. Problem 2.40
9. Problem 2.47

Hints:

1.57 Divide the area into two adjacent areas – one in the xy plane and the other in the yz plane. The line integral along the common boundary of the areas along the y-axis cancels.

2.20 Curl of electrostatic field is zero

2.21 Calculate the potential difference as the work done in moving a charge in an electric field

OR

Calculate the potential at r as a sum of the potential due to sphere of charge of radius r and potential of shells of radius r' and of thickness $\Delta r'$ with r' varying from r to R.

2.26 Divide the cone into rings of infinitesimal thickness.

2.39 All points in a conductor are at the same potential. Consider what is the electric field due to a spherical equipotential surface.

2.47 If force is calculated between two infinitesimal charges in the sphere and integration is carried out over one hemisphere, the mutual sum of forces between elements in the hemisphere will cancel and the only force left will be that due to the other hemisphere. Hence one can calculate the force due to a sphere of charge and integrate this force over one hemisphere.

Solution / Assgn. 2

1.20. Constructing a function that has zero divergence and zero curl.

One example is $(y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

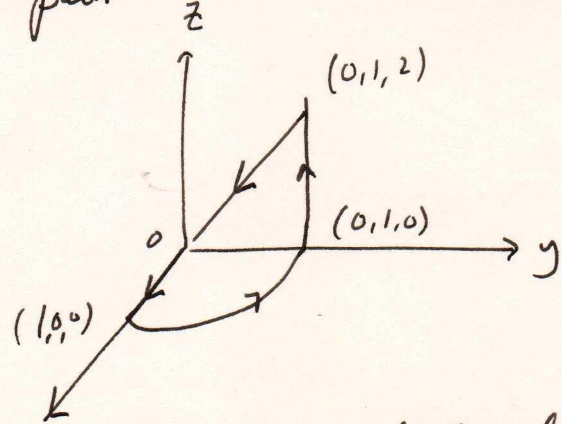
You can construct some more.

For $r \neq 0$ $\quad \dfrac{\hat{r}}{r^2}$ has this property

1.57 Line integral of

$$\vec{v} = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$$

over the path



We will do it using cylindrical coordinates $\left(\begin{array}{l} r\cos\theta = z \\ r\sin\theta = s \end{array}\right)$

From $(0,0,0)$ to $(1,0,0)$ $\quad \vec{v} = 0\hat{s} - 0\hat{\theta} + 3s\hat{\phi}$

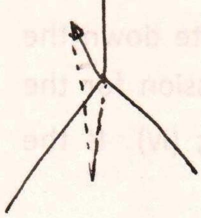
$(r = s$ in $\theta = \frac{\pi}{2}$ plane$)$

$\vec{v}\cdot d\hat{x}$ along x axis $= 0$

Along the arc of radius 1 $\quad \vec{v} = 3\hat{\phi}$

$$\int \vec{v}\cdot d\vec{l} = \frac{3\pi}{2}$$

In the $y\,z$ plane going from $(0,1,0)$ to $(0,1,2)$ $s = r\sin\theta$ is fixed



Also $\hat{r} = \cos\theta\,\hat{z} + \sin\theta\,\hat{s}$

$\hat{\theta} = -\sin\theta\,\hat{z} + \cos\theta\,\hat{s}$

$$\vec{v} = r\cos^2\theta\left(\cos\theta\,\hat{z} + \sin\theta\,\hat{s}\right)$$

$$- r\cos\theta\sin\theta\left(-\sin\theta\,\hat{z} + \cos\theta\,\hat{s}\right)$$

$$+ 3r\,\hat{\phi}$$

$\left(r = \sqrt{z^2+s^2}\right)$

$$= \frac{z^3}{(z^2+s^2)}\hat{z} + \frac{z^2 s}{(z^2+s^2)}\hat{s}$$

$$+ \frac{z s^2}{(z^2+s^2)}\hat{z} - \frac{z^2 s}{(z^2+s^2)}\hat{s}$$

$$+ 3\sqrt{z^2+s^2}\,\hat{\phi}$$

$$= z\,\hat{z} + 3\sqrt{z^2+s^2}\,\hat{\phi}$$

Along straight line $(0,1,0)$ to $(0,1,2)$

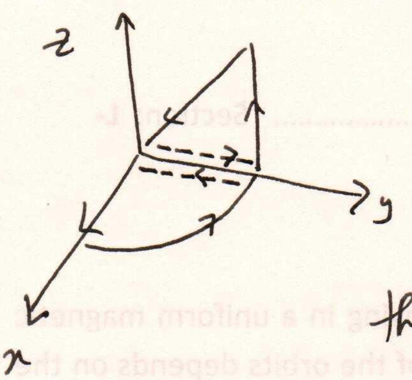
$$\vec{v}\cdot d\vec{l}' = \int_1^2 z\,dz = \frac{3}{2}$$

Along straight line $(0,1,2)$ to $(0,0,0)$ $d\vec{l}' = dz\,\hat{z} + ds\,\hat{s}$

As $\vec{v}\cdot d\vec{l}' = \int_2^1 z\,dz = -\frac{3}{2}$

So $\oint \vec{v}\cdot d\vec{l}' = \frac{3\pi}{2}$

Checking Stokes' theorem



Calculate surface integral of $(\vec{\nabla}\times\vec{v})$ over the two surfaces shown. The line integral along the dashed line cancels so the final result is the line integral around the path

$$\vec{\nabla}\times\vec{v}\Big|_{\text{cylindrical}} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{z}$$

$$= \cancel{+\frac{1}{s}}\left(-\frac{\partial}{\partial z}\, 3\sqrt{z^2+s^2}\right)\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}\left(3s\sqrt{z^2+s^2}\right)\hat{z}$$

$$= -\frac{3z}{\sqrt{z^2+s^2}}\hat{s} + \frac{1}{s}\left\{3\sqrt{z^2+s^2} + \frac{3s^2}{\sqrt{z^2+s^2}}\right\}\hat{z}$$

$$= -\frac{3z}{\sqrt{z^2+s^2}}\hat{s} + \frac{3}{s}\left(\frac{z^2+2s^2}{\sqrt{z^2+s^2}}\right)\hat{z}$$

Over the area in the y-z plane $d\vec{s}$ is $\perp$ to $\vec{v}$

so no contribution to $\int(\vec{\nabla}\times\vec{v})\cdot d\vec{s}$

Over the area in the xy plane

$$\vec{ds} = r\,dr\,d\phi\,\hat{z} = s\,ds\,d\phi\,\hat{z}$$

$z = 0$ so

$$\vec{\nabla} \times \vec{v} = \left(\frac{3}{s} \times \frac{2s^2}{s}\right)\hat{z} = 6s\,\hat{z}$$

$$\int (\vec{\nabla} \times \vec{v}) \cdot \vec{ds} = \int_0^1 6s\,ds \int_0^{\pi/2} d\phi$$

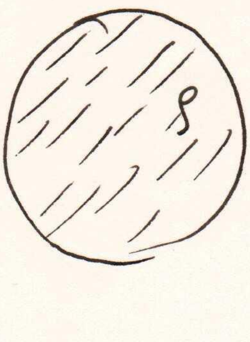
$$= \frac{3\pi}{2}$$

So it checks

2.20 Check for curl of $\bar{E}$ (Calculation is straightforward)

2.21 Potential inside and outside a uniformly charged sphere of radius R.

Let the uniform charge density be ρ



$$\Delta V = -\int \bar{E} \cdot \vec{dl} = \text{work done in moving a charge}$$

$$V(R) - V(\infty) = -\int_\infty^R \frac{kQ}{r^2}\,dr = \frac{kQ}{R}$$

$$V(\infty) \to 0 \Rightarrow V(R) = \frac{kQ}{R}$$

$$V(r) \quad r > R = \frac{kQ}{r}$$

Inside the sphere $(r<R)$

$$V(r) - V(R) = -\int_R^r \frac{\rho r'}{3\epsilon_0}\, dr'$$

$$= \frac{\rho}{3\epsilon_0}\left(-\frac{r^2}{2} + \frac{R^2}{2}\right)$$

$$V(r) = V(R) + \frac{\rho R^2}{6\epsilon_0} - \frac{\rho r^2}{6\epsilon_0}$$

$$\rho = \frac{3Q}{4\pi R^3}$$

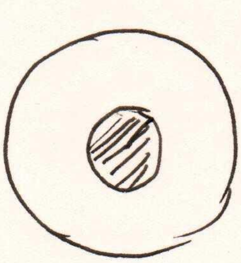
$$V(r \leq R) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R} + \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R} - \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Qr^2}{R^3}$$

$$= \frac{3}{2}\cdot\left(\frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{R}\right) - \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{Qr^2}{R^3}\right)$$

Second Method

For $r \geq R$, as earlier, $V(r) = \frac{kQ}{r}$

For $r < R$



The potential is due to charge inside the sphere of radius r and due to all shells of radius $r' > r$

The potential due to a shell of radius r' and charge dq, the potential at $r < r'$ is

$$k \frac{dq}{r'}$$

$\therefore$ Net potential at $r < R$

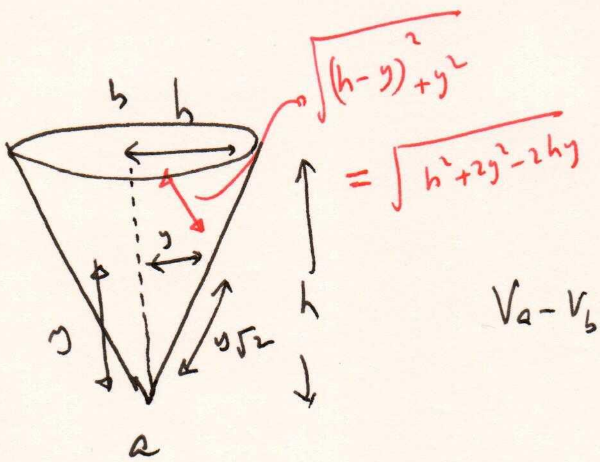
$$= \frac{k \frac{4\pi}{3} r^3 \rho}{r} + k \int_r^R \frac{4\pi r'^2 \, dr' \, \rho}{r'}$$

$$= k \cdot \frac{Q r^2}{R^3} + k \cdot 4\pi\rho \left(\frac{R^2}{2} - \frac{r^2}{2} \right)$$

$$= k \cdot \frac{3}{2} \frac{Q}{R} + \frac{kQr^2}{R^3} - \frac{3}{2} k \frac{Q r^2}{R^3}$$

$=$ Same as earlier.

2.26



$$V_a - V_b = \int_0^h \left(2\pi\sigma y \, dy \sqrt{2} \right) \cdot k \left[\frac{1}{y\sqrt{2}} - \frac{1}{\sqrt{h^2 + 2y^2 - 2hy}} \right]$$

where $2\pi\sigma y \, dy \sqrt{2} =$ charge on the ring at height y

$$= 2\pi\sigma \left[h - \int_0^h \frac{y \, dy \sqrt{2}}{\sqrt{h^2 + 2y^2 - 2hy}} \right]$$

The integral is performed by writing

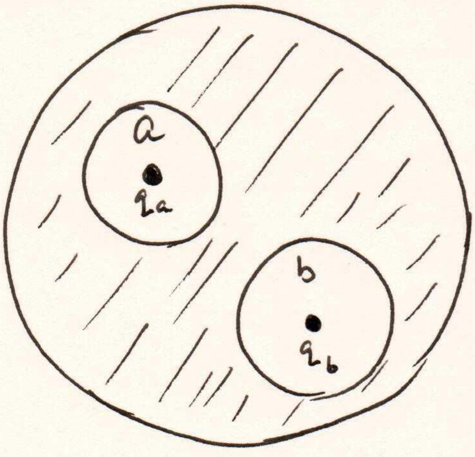
$$\sqrt{h^2 + 2y^2 - 2hy} = \sqrt{\left(\frac{h}{\sqrt{2}} - y\sqrt{2}\right)^2 + \frac{h^2}{2}}$$

The answer then is

$$V_a - V_b = 2\pi\sigma h \left[1 - \frac{1}{2} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \right]$$

2·34 and 2·35 (Not to be done in the tutorial)

2·39



Both cavities 'a' and 'b' are spherical cavities with their surface being equipotential surfaces.

$\Rightarrow$ • There is no field inside the cavities due to charges induced on their surfaces

• Furthermore a spherical surface gives zero field inside if charge is distributed uniformly over the surface (Keep in mind that q_a or q_b already give spherical equi-potential surface)

• By Gauss' law net charge on the surfaces of two cavities is $-q_a$ and $-q_b$, respectively

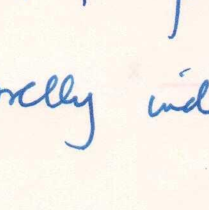
$$\Rightarrow \sigma_a = -\frac{q_a}{4\pi r_a^2} \qquad \sigma_b = -\frac{q_b}{4\pi r_b^2}$$

One could also argue it by saying that the Charges induced on the Cavity Surface will be such that the field immediately outside the Surface is zero. Then first make Cavity 'a', put charge q_a inside. The rest of the Conductor is electrically neutral to cavity 'a' and charge present inside it. So Cavity 'b' when made does not feel the presence of cavity 'a'. The two are prelly independent electrically. This also leads to a uniform

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

Force on q_a and q_b is obviously zero.

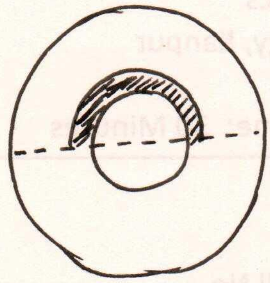
(2.40)



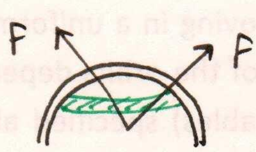
(a) Net force on a charge inside a cavity need not be zero

(b) Leave it for students to read the reference given and discuss

2.47



As explained in the hint, calculate the force on the hemispherical shell due to the charge inside and integrate



$$\vec{E} = \frac{\rho \vec{r}}{3\epsilon_0}$$

Force on hemisphere

$$= \underbrace{\left(2\pi r^2 \, d(\cos\theta)\, \rho \, dr\right)}_{\text{charge on a strip shown on hemisphere}} \; \underbrace{\frac{\rho r \cos\theta}{3\epsilon_0}}_{\text{vertical component of the electric field}}$$

$$= \frac{2\pi \rho^2}{3\epsilon_0} \int_0^R r^3 \, dr \int_0^1 \cos\theta \, d(\cos\theta)$$

$$= \frac{2\pi}{3\epsilon_0} \frac{9Q^2}{16\pi^2 R^6} \times \frac{R^4}{4} \times \frac{1}{2}$$

$$= \left(\frac{3Q^2}{64\pi\epsilon_0 R^2}\right)$$ Answer