Department of physics
Indian Institute of Technology, Kanpur

MKH/SMT/2016-17-II/Phy103 Assignment-7 Week of 20-2-17

Note: The problems assigned are from the text book *Introduction to Electrodynamics* (4th edition) by David J. Griffiths

1. Problem 6.10
2. Problem 6.12
3. Problem 6.13
4. Problem 6.16
5. Problem 6.18
6. Problem 6.23
7. Problem 6.28

Hints:

6.10 Bulk current for the torus is negligible. Follow the hint in the book.

6.12 Bulk and surface bound currents are nonzero. Net bound current should come out to be zero. Follow the hint in the book. Also in the absence of free currents, $\vec{H}$ can be calculated as an electric field due to charge $-\vec{\nabla} \cdot \vec{M}$.

6.13 Think in terms of bound currents to calculate $\vec{B}$. Also $\vec{H}$ can be calculated as an electric field due to charge $-\vec{\nabla} \cdot \vec{M}$.

6.16 $\vec{H}$ can be calculated from free current densities alone. Use cylindrical symmetry to facilitate calculations.

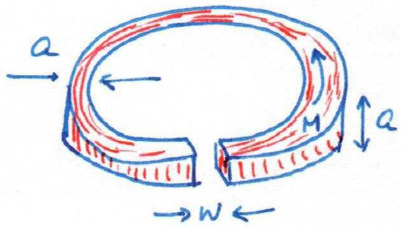
6.18 The applied magnetic field is expected to give a constant magnetization inside the sphere. The magnetic field is the sum of applied field and field arising from the induced magnetization. Finally $\vec{M} = \chi_m \vec{H}$.

6.23 Force on a dipole is $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$.

6.28 Recall the problem of an electric dipole in a dielectric sphere. There, bound charges appeared due to the dipole induced polarization. Here bound currents will appear due to dipole induced magnetization. A good guide will be example 5.11.

SOLUTION

6.10



Total length of the rod is L. $\vec{M}$ in the rod is as follows

$$\vec{M} = M\hat{\varphi}$$

This gives both a bulk current density and surface current density

$$\vec{M} = M\hat{\varphi} \Rightarrow M_s = 0, \quad M_z = 0 \quad M_\varphi = M$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = \frac{\hat{z}}{s}\frac{\partial}{\partial s}(sM_\varphi)$$

$$= \frac{M}{s}\hat{z}$$

$$\vec{K}_b = \vec{M}\times\hat{n} = M\,\hat{\varphi}\times(-\hat{s}) \text{ for inner surface}$$

$$= M\,\hat{\varphi}\times(\hat{z}) \text{ for upper surface}$$

$$= M\,\hat{\varphi}\times(\hat{s}) \text{ for outer surface}$$

$$= M\,\hat{\varphi}\times(-\hat{z}) \text{ for lower surface}$$

So $\vec{K}_b = M\hat{z}$ for inner surface

$= M\hat{s}$ for upper surface

$= -M\hat{z}$ for outer surface

$= -M\hat{s}$ for lower surface

Since there is a bulk current density and surface current density and the radii of the inner and outer surfaces are different, there will be a difference in the currents on the inner and outer surface. However because $a \ll L$, we will ignore this difference and take the currents to be the same. This can be seen easily if we calculate the total current due to $\vec{J}_b$, which is in the $\hat{z}$ direction

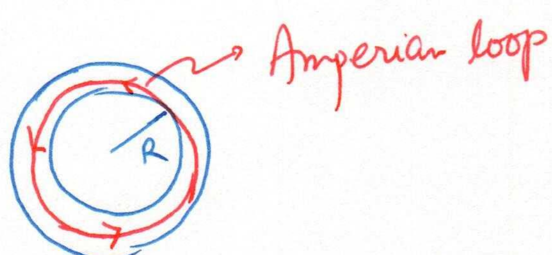
$$I_b = \int 2\pi s\, ds \cdot J_b$$

$$= 2\pi \int ds \,.\, s \,.\, \frac{M}{s}$$

$$= 2\pi M a$$

a being very small, we will neglect it. Now we will calculate field by bound currents as well as by $\vec{H}'$

By Ampere's law, if the gap was not there, we will have



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

$$B \,.\, \cancel{2\pi R} = \mu_0 \cancel{2\pi R}\, M$$

$$B = \mu_0 M \quad (\text{magnitude})$$

$$\vec{B} = \mu_0 M \hat{\varphi}$$

<u>By $\vec{H}'$ calculation</u>:

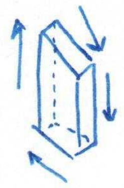
$$\oint \vec{H}' \cdot d\vec{l}' = \mu_0 I_{free} = 0$$

$$H \,.\, 2\pi R = 0 \implies H = 0$$

$$\vec{B}' = \mu_0 (\vec{H}' + \vec{M}') = \mu_0 M \hat{\varphi}$$

However now the field due to the gap is missing

Since $w \ll a$, we can consider the square piece



missing as a current loop carrying current wM

$$I = wM$$

Field at the centre of a square loop carrying current I is (Prob 5.8a)

$$\frac{2\sqrt{2}\,\mu_0 I}{\pi a}$$

where 'a' is the length of the side of the loop

Here $I = wM$ so

$$\vec{B}' = \frac{2\sqrt{2}\,\mu_0 I}{\pi a}\hat{\varphi} = \frac{2\sqrt{2}\,\mu_0 M w}{\pi a}$$

This field will be subtracted from $\vec{B}'$ due to full rod to get the field in the gap so

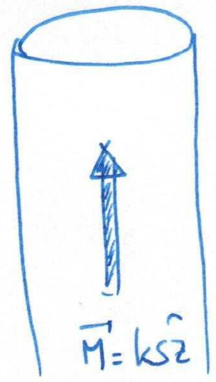
$$\vec{B} = \mu_0 M \hat{\varphi} - \frac{2\sqrt{2}\,\mu_0 M w}{\pi a}\hat{\varphi}$$

$$= \mu_0 M \hat{\varphi} \left(1 - \frac{2\sqrt{2}\,w}{\pi a}\right)$$

For $w \ll a$, we can say that

$$\vec{B}' = \mu_0 M \hat{\varphi}$$

6.12



There is no free current

$$\vec{J}_{free} = 0$$

And

$$\vec{M} = ks\,\hat{z}$$

So

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$= -\frac{\partial}{\partial s}(sk)\,\hat{\varphi}$$

$$= -k\,\hat{\varphi}$$

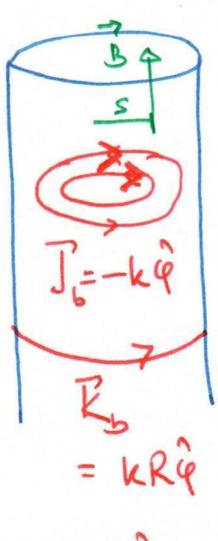
And

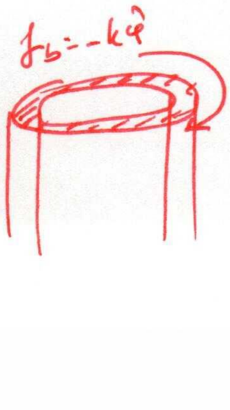
$$\vec{K}_b = \vec{M} \times \hat{n}$$

$$= kR\,\hat{z} \times \hat{s}$$

$$= kR\,\hat{\varphi}$$

(a) To obtain the magnetic field using bound currents





B field $\vec{B}(s)$ at a distance s from the axis of the cylinder is that due to solenoids. Carrying currents. The current per unit length for a solenoid at distance s having thickness ds will be

$$dI = -k\,ds$$

This gives field $= -\mu_0 k\,ds$ for points inside and field $= 0$ for points outside.

In addition the surface current $K_b = kR\,\hat{\varphi}$ also forms a solenoid and gives field

$\mu_0 k R\,\hat{z}$ inside the cylinder.

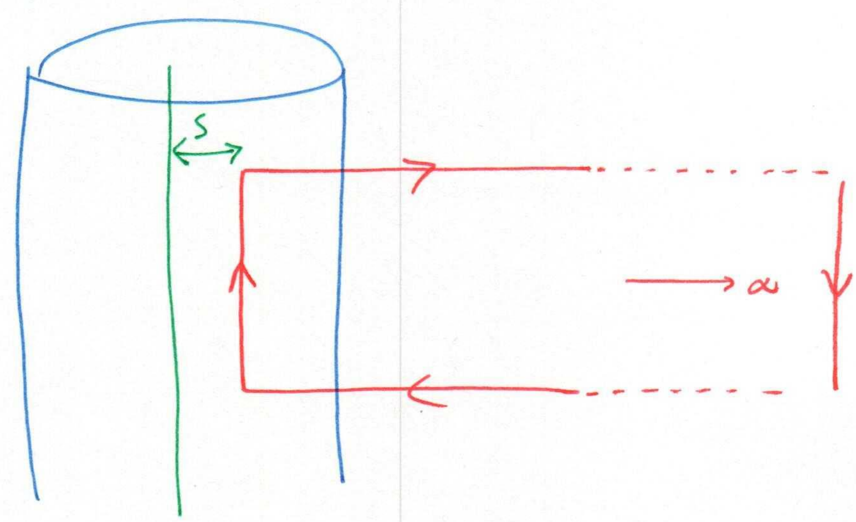
Thus at a distance 's' from the axis, the net magnetic field is

$$\vec{B}' = \vec{B}_{\text{due to current} < s} + \vec{B}_{\text{due to current} > s} + \vec{B}_{\text{due to surface current}}$$

$$= \left[0 + \int_s^R (-\mu_0 k\, ds) + \mu_0 k R \right] \hat{z}$$

$$= \mu_0 k s\,\hat{z}$$

This can easily be checked by taking $\vec{B}(s \to \infty) = 0$ (Problem 5.62) and taking the following Amperian loop



(b) By calculating $\vec{H}'$

By symmetry, $\vec{H}'$ is going to be in the $\hat{z}$ direction.

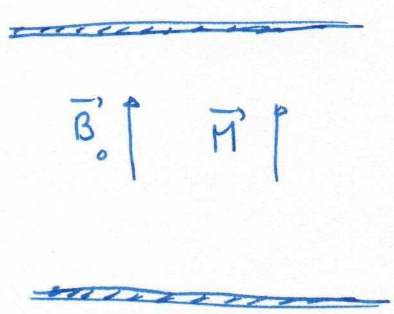
$$\vec{J}'_{free} = 0 \Rightarrow \vec{\nabla} \times \vec{H}' = 0$$

Although $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}' \neq 0$ on the upper and lower surfaces of the cylinder, its contribution to $\vec{H}'$ is going to be negligible as the cylinder is infinitely long. So

$$\vec{H}' = 0$$

$$\Rightarrow \vec{B}' = \mu_0 (\vec{H}' + \vec{M}') = \mu_0 k s \hat{z}$$

<u>6.13</u>



$$\vec{H}_0 = \frac{\vec{B}_0}{\mu_0} - \vec{M}$$

These are constant quantities.

For all the cavities made in the ~~cavity~~ bulk, we wish to find $\vec{B}$, $\vec{H}$ in them.

(a) <u>A small spherical cavity</u>: Let the field at the centre of the cavity be $\vec{B}_1$. Then

$$\vec{B}_1 + \text{field due to sphere} = \vec{B}_0$$

Field due to a sphere of magnetization $\vec{M}$ is (7)

$$\frac{2}{3}\mu_0\vec{M} \; * \quad (\text{Example } 5.11)$$

$$\Rightarrow \quad \vec{B}_1 + \frac{2}{3}\mu_0\vec{M} = \vec{B}_0$$

$$\text{or} \quad \vec{B}_1 = \vec{B}_0 - \frac{2}{3}\mu_0\vec{M}$$

$$\text{And} \quad \vec{H}_1 = \frac{\vec{B}_0}{\mu} - \frac{2}{3}\vec{M}$$

* This can be seen in the following two ways

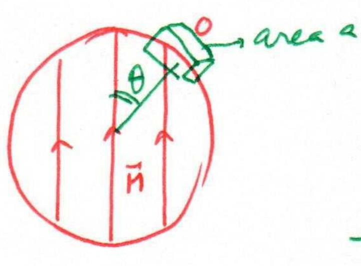
(i) $\quad \vec{\nabla}\times\vec{M} = \vec{J}_b = 0$ here

$$\vec{K} = \vec{M}\times\hat{n} = M\sin\theta\,\hat{\varphi}$$

This gives a field $\frac{2}{3}\mu_0\vec{M}$

(ii) $$\vec{\nabla}\times\vec{H} = 0 \quad (J_f = 0)$$

$$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$



$$= M\cos\theta \quad \text{charge on surface}$$

The pill box in surface gives

$$-\int \vec{\nabla}\cdot\vec{M}\,dV = \sigma_M \cdot a$$

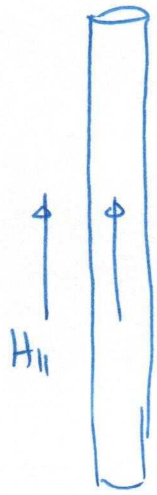
$$-\int \vec{\nabla}\cdot\vec{M}\,dV = -\int \vec{M}\cdot\hat{n}\,ds = M\cos\theta$$

(from the lower surface)

$$\vec{\nabla}\cdot\vec{H} = M\cos\theta \quad \text{gives} \quad \vec{H} = -\frac{\vec{M}}{3}$$

$$\Rightarrow \vec{B} = \mu_0\left(-\frac{\vec{M}}{3} + \vec{M}\right) = \frac{2}{3}\mu_0\vec{M}$$

(b) <u>Needle shaped cavity</u>: It is like a solenoid



with $K = M$ *

$$\vec{B}_1 + \mu_0 \vec{M} = \vec{B}_0$$

$$\Rightarrow \vec{B}_1 = \vec{B}_0 - \mu_0 \vec{M}$$

$$\vec{H}_1 = \frac{B_0}{\mu_0} - \frac{\mu_0 \vec{M}}{\mu_0}$$

$$= \frac{\vec{B}_0}{\mu_0} - \vec{M}$$

$$= \vec{H}_0$$

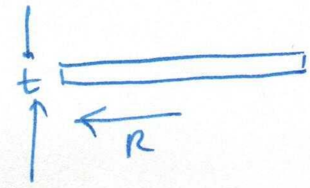
This also follows from the boundary condition across the cavity surface

* Again the field can be calculated by $\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$

$\vec{H}$ is like the field of charges at the upper and lower surfaces so for a long needle-like cavity $\vec{H} = 0$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0 \vec{M}$$

(c)



For a thin wafer, the current on the side will be (tM) with $t \ll R$ (t = thickness and R = radius of wafer)

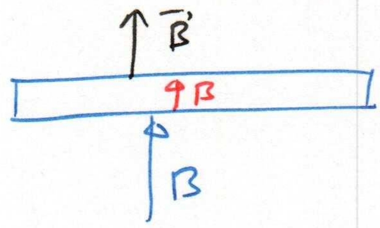
$\Rightarrow$ $\vec{B}$(centre) $\simeq 0$ due to the

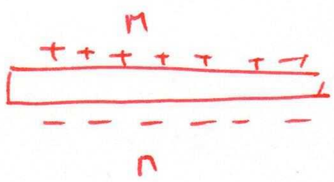
Current in the wafer.

$$\vec{B}_1 + 0 = \vec{B}_0 \quad \text{Ⓐ}$$

$$\Rightarrow \quad \vec{B}_1 = \vec{B}_0$$

This also follows from continuity of $\vec{B}_\perp$ across the large surface of the wafer



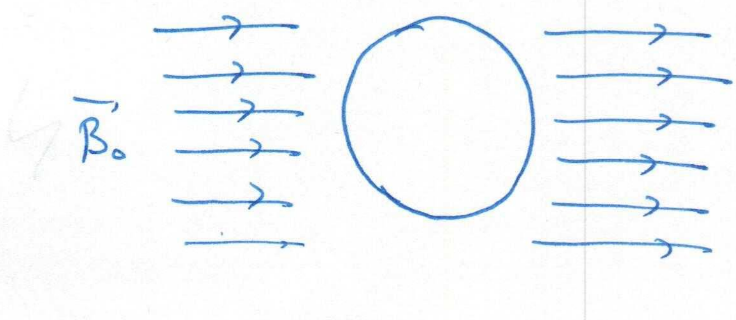


Also $-\vec{\nabla}\cdot\vec{M}$ gives $+M$ charge on top and $-M$ charge at the bottom $\Rightarrow$ $\vec{H} = -\vec{M}$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = 0$$

$$\vec{B}_1 + 0 = \vec{B}_0 \quad \text{or} \quad \vec{B}_1 = \vec{B}_0$$

<u>6.18</u> (6.16 has been solved in the lecture on 17 Feb).



We can do this problem iteratively or directly.

To do it iteratively

Correction: Inside the material, $H = B/\mu$ And NOT B/μ_0

$$\vec{H}_0 = \cancel{\frac{\vec{B}_0}{\mu_0}}\ \frac{\vec{B}_0}{\mu} = \frac{\vec{B}_0}{\mu_r \mu_0} \quad \text{where } \mu_r = (1+\chi_M) = \mu/\mu_0$$

induces a magnetization $\vec{M}_0 = \left(\frac{\chi_M}{\mu_r}\right)\frac{\vec{B}_0}{\mu_0}$

This in turn produces a magnetic field

$$\vec{B}_1 = \frac{2}{3}\left(\frac{\chi_M}{\mu_r}\right)\vec{B}_0 \quad \left(= \frac{2}{3}\mu_0 \vec{M}_0\right)$$

which gives

$$\vec{M}_1 = \frac{2}{3}\left(\frac{\chi_M^2}{\mu_r^2}\right)\frac{\vec{B}_0}{\mu_0}$$

and

$$\vec{B}_2 = \left(\frac{2}{3}\frac{\chi_M}{\mu_r}\right)^2 \vec{B}_0$$

So

$$\vec{B} = B_0 + \frac{2}{3}\frac{\chi_M}{\mu_r}\vec{B}_0 + \left(\frac{2}{3}\frac{\chi_M}{\mu_r}\right)^2 \vec{B}_0 + \cdots$$

$$= \frac{\vec{B}_0}{1 - \frac{2}{3}\frac{\chi_M}{\mu_r}} = \cancel{\frac{3}{3-2\chi_M}\vec{B}_0}\quad \frac{3(1+\chi_M)}{3+\chi_M}\vec{B}_0$$

One can also solve it directly be assuming that the applied field creates a constant magnetization $\vec{M}$ in the same direction. Then

$$\vec{B} = \vec{B}_0 + \frac{2}{3}\mu_0 \vec{M}$$

$$\vec{H} = \frac{\vec{B}_0}{\mu_r \mu_0} + \frac{2}{3}\frac{\vec{M}}{\mu_r} \quad \left(\vec{H} = \frac{\vec{B}}{\mu};\ \mu = \mu_r \mu_0\right)$$

And $\vec{M} = \chi_M \vec{H}$

$$= \left(\frac{\chi_M}{\mu_r}\right)\frac{\vec{B}_0}{\mu_0} + \frac{2}{3}\left(\frac{\chi_M}{\mu_r}\right)\vec{M}$$

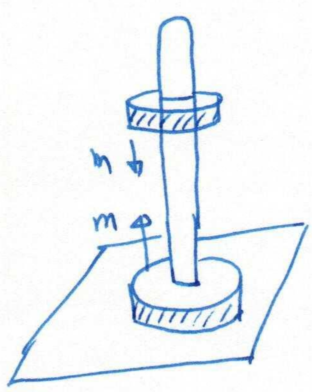
or $$\vec{M}\left(\frac{3+\chi_M}{3}\right) = \chi_M \frac{\vec{B}_0}{\mu_0}$$

$$\vec{M} = \left(\frac{3\chi_M}{3+\chi_M}\right)\frac{\vec{B}_0}{\mu_0}$$

Thus $$\vec{B} = \vec{B}_0 + \frac{2}{3}\mu_0 \vec{M}$$

$$= \left(\frac{3+3\chi_M}{3+\chi_M}\right)\vec{B}_0$$

6.23



Force on magnet

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

$$\vec{m}_{lower} = m\hat{z}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{z})\hat{z} - \vec{m}}{z^3}\right]$$

$$= \frac{\mu_0}{4\pi}\cdot\frac{2m\hat{z}}{z^3}$$

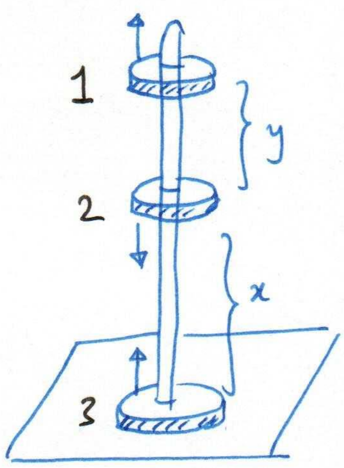
$$\vec{m}_{upper} = -m\hat{z}$$

So $$\vec{F} = \vec{\nabla}\left(-\frac{\mu_0}{2\pi}\frac{m^2}{z^3}\right) = \frac{3\mu_0}{2\pi}\frac{m^2}{z^4}$$

(a) To balance
$$\frac{3\mu_0}{2\pi}\frac{m^2}{Z^4} = m_d g$$

$$\Rightarrow \quad Z = \left[\frac{3\mu_0 m^2}{2\pi m_d g}\right]^{1/4}$$

(b)



For magnet 1

$$\frac{3\mu_0}{2\pi}\left[\frac{m^2}{y^4} - \frac{m^2}{(x+y)^4}\right] = m_d g \quad \text{— (I)}$$

For magnet 2

$$\frac{3\mu_0}{4\pi}\left[\frac{m^2}{x^4} - \frac{m^2}{y^4}\right] = m_d g \quad \text{— (II)}$$

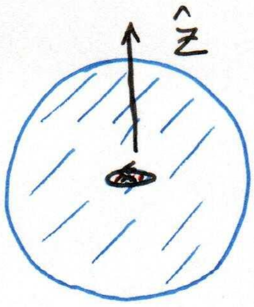
Take $x/y = R$. Then

$$\frac{(I)}{(II)} \Rightarrow \quad 1 - \frac{1}{(R+1)^2} = \frac{1}{R^4} - 1$$

or
$$2 = \frac{1}{R^4} + \frac{1}{(R+1)^4}$$

This is to be solved numerically.

$R = \cdot 85$ gives the right answer for satisfaction of the equation.



The dipole at the centre is like a current loop. This gives a field

$$\vec{B}'(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}$$

This in turn will produce a magnetization like

$$\vec{M}_1(\vec{r}) = \frac{3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1}{4\pi r^3}$$

(4π is put for convenience)

This in turn produces a bulk current $\vec{J}_b$

$$\vec{J}_b = \vec{\nabla}\times\vec{M}_1$$

which is equivalent to a magnetic dipole $\vec{m}_1$ at the centre. It also produces a surface current

$$\vec{K}_b = \vec{M}_1\times\hat{n} = \vec{M}_1\times\hat{r}$$

$$= -\frac{\vec{m}_1\times\hat{r}}{4\pi R^3}$$

$$= -\frac{m_1 \sin\theta\,\hat{\phi}}{4\pi R^3}$$

This current will give a constant magnetization $\vec{M}_2$. Thus we take the net magnetization to be

$$\vec{M}(\vec{r}) = \frac{3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1}{4\pi r^3} + \vec{M}_2 \quad (\vec{M}_2 = M_2\hat{z})$$

The corresponding current are

$$\vec{J}_b \equiv (\text{equivalent to}) \quad \vec{m}_1$$

$$\vec{K}_b = -\frac{\vec{m}_1\times\hat{r}}{4\pi R^3} + \vec{M}_2\times\hat{r}$$

$$= \left(M_2 - \frac{m_1}{4\pi R^3}\right)\sin\theta\,\hat{\varphi}$$

The magnetic field it gives rise to is

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1}{r^3} + \frac{2}{3}\mu_0\left(M_2 - \frac{m_1}{4\pi R^3}\right)\hat{z}$$

NOTE: We do not calculate $\vec{H}_1$ from $\vec{m}_1$ because $\vec{m}_1$ arises from bound currents and $\vec{H}_1$ does not.

Although it is tempting to write

$$\vec{H}_1 = \frac{1}{4\pi}\frac{3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1}{r^3} + \left[\text{that due to } \cancel{M_2}\ \hat{r}\cdot\left\{\frac{3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1}{R^3} + \vec{M}_2\right\}\right]$$

but it will be wrong to do so because

$$\vec{\nabla}\times\vec{H}_1 = \vec{J}_b \quad \text{whereas it should be}$$

zero.

Total $\vec{B}$ therefore is

$$\vec{B} = \vec{B}'_{\text{due to } \vec{m}} + \vec{B}'_1$$

$$= \frac{\mu_0}{4\pi} \frac{3\{(\vec{m}+\vec{m}_1)\cdot\hat{r}\}\hat{r} - (\vec{m}+\vec{m}_1)}{r^3}$$

$$+ \frac{2}{3}\mu_0 \left(M_2 - \frac{m_1}{4\pi R^3}\right)\hat{z}$$

The corresponding auxiliary field $\vec{H}$ is

$$\vec{H}' = \frac{\vec{B}'}{\mu} = \frac{\mu_0}{4\pi\mu} \frac{3\{(\vec{m}+\vec{m}_1)\cdot\hat{r}\}\hat{r} - (\vec{m}+\vec{m}_1)}{r^3}$$

$$+ \frac{2\mu_0}{3\mu}\left(M_2 - \frac{m_1}{4\pi R^3}\right)\hat{z}$$

$$\vec{M}' = \chi_m \vec{H}' = \frac{\mu_0 \chi_m}{4\pi\mu} \cdot \frac{3\{(\vec{m}+\vec{m}_1)\cdot\hat{r}\}\hat{r} - (\vec{m}+\vec{m}_1)}{r^3}$$

$$+ \frac{2\mu_0\chi_m}{3\mu}\left(M_2 - \frac{m_1}{4\pi R^3}\right)\hat{z}$$

$$\frac{3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1}{4\pi r^3} + M_2\hat{z}$$

Comparison of two sides implies that

$$\frac{\mu_0\chi_m}{\mu}(m+m_1) = m_1$$

$$\mu = (1+\chi_m)\mu_0$$

$$\Rightarrow \quad \chi_m (m + m_1) = (1+\chi_m) m_1$$

$$\Rightarrow \quad \boxed{m_1 = \chi_m m}$$

This can also be obtained from

$\vec{\nabla} \times \vec{H} = \vec{J}_{free}$ which will give

$$\frac{\mu_0}{\mu} (\vec{m} + \vec{m}_1) = \vec{m}$$

because $\vec{J}_{free}$ is equivalent to $\vec{m}$

$$\Rightarrow \quad (\vec{m} + \vec{m}_1) = (1+\chi_m)\vec{m}$$

$$\boxed{\vec{m}_1 = \chi_m \vec{m}}$$

And

$$M_2 = \frac{2\chi_m}{3(1+\chi_m)} \left(M_2 - \frac{m_1}{4\pi R^3} \right)$$

$$3M_2 + 3\chi_m M_2 = 2\chi_m M_2 - \frac{2\chi_m m_1}{4\pi R^3}$$

or $$(3+\chi_m) M_2 = -\frac{2\chi_m m_1}{4\pi R^3}$$

$$M_2 = -\frac{2\chi_m}{(3+\chi_m)} \frac{m_1}{4\pi R^3}$$

As from page (15), $\overline{B}$ is

$$\overline{B}' = \frac{\mu_0}{4\pi} \frac{3\left((\vec{m}' + \chi_m \vec{m}) \cdot \hat{r}\right)\hat{r} - (1+\chi_m)\vec{m}}{r^3}$$

$$+ \frac{2}{3}\mu_0 \left(- \frac{2\chi_m}{(3+\chi_m)} \cdot \frac{m_1}{4\pi R^3} - \frac{m_1}{4\pi R^3} \right) \hat{z}$$

$$= \frac{\mu_0 (1+\chi_m)}{4\pi} \frac{3(\vec{m}' \cdot \hat{r})\hat{r} - \vec{m}'}{r^3}$$

$$+ \frac{2}{3}\mu_0 \left(\frac{-2\chi_m - 3 - \chi_m}{3+\chi_m} \right) \frac{m_1}{4\pi R^3} \hat{z}$$

$$= \frac{\mu}{4\pi} \cdot \frac{3(\vec{m}' \cdot \hat{r})\hat{r} - \vec{m}'}{r^3}$$

$$+ \frac{2}{\cancel{3}}\mu_0 \frac{-\cancel{3}(1+\chi_m)}{(3+\chi_m)} \cdot \frac{\chi_m \cdot m}{4\pi R^3} \hat{z}$$

$$= \frac{\mu}{4\pi} \left[\frac{3(\vec{m}' \cdot \hat{r})\hat{r} - \vec{m}'}{r^3} - \frac{2\chi_m \vec{m}'}{(3+\chi_m) R^3} \right]$$

$$\chi_m = \frac{\mu - \mu_0}{\mu_0}$$

$$\Rightarrow \overline{B}' = \frac{\mu}{4\pi} \left[\frac{3(\vec{m}' \cdot \hat{r})\hat{r} - \vec{m}'}{r^3} - \frac{2(\mu - \mu_0)}{(2\mu_0 + \mu)} \frac{\vec{m}'}{R^3} \right]$$