Department of physics
Indian Institute of Technology, Kanpur

MKH/SMT/2016-17-II/Phy103 Assignment-5 Week of 6-2-17

Note: The problems assigned are from the text book *Introduction to Electrodynamics* (4th edition) by David J. Griffiths

1. Problem 4.22
2. A capacitor carries charge Q when fully charged by a battery of EMF V. The capacitor is disconnected from the battery and a dielectric is introduced in it. (i) Show that the change in the energy E stored in the capacitor is $-\frac{1}{2}Q\Delta V$ where ΔV is the decrease in the potential of the capacitor when the dielectric is introduced. Now reconnect the battery to the capacitor so that some charge ΔQ flows to it and its potential again becomes V. (ii) Up to the first order in ΔQ and ΔV, calculate the energy supplied by the battery in this process. (iii) Show that $V\Delta Q = Q\Delta V$ (iv) Using the results above, show that the change in the energy stored in the capacitor when the dielectric is introduced with the battery remaining connected to it is $\frac{1}{2}Q\Delta V = \frac{1}{2}V\Delta Q$. (This problem is related directly to section 4.4.4). Hence the force F on the dielectric is $F = -\left(\frac{dE}{dx}\right)_Q = \left(\frac{dE}{dx}\right)_V$.
3. Problem 4.28
4. Complete the derivation of force on a dielectric slab (done in lecture on Feb 1) in a parallel plate capacitor.
5. Problem 4.37
6. Problem 4.39
7. Problem 4.40
8. Problem 5.5
9. Problem 5.6
10. Problem 5.7
11. Problem 5.12
12. Problem 5.15
13. Problem 5.44

Hints:

4.22: Either do the iterative method done in the lecture to generate a geometric series for $\chi_e < 1$ ro assume a polarization in the direction of $\vec{E}_0$ and use the fact that field inside will be the sum of $\vec{E}_0$ and the electric field due to the polarization.

4.28: Use result of problem 2

4.37: The field due to the dipole in the middle gives a polarization that has the same dependence on $\vec{r}$ as the field of the dipole. This leads to a bound volume charge. Furthermore, this polarization also gives a surface charge that gives an additional field and polarization.

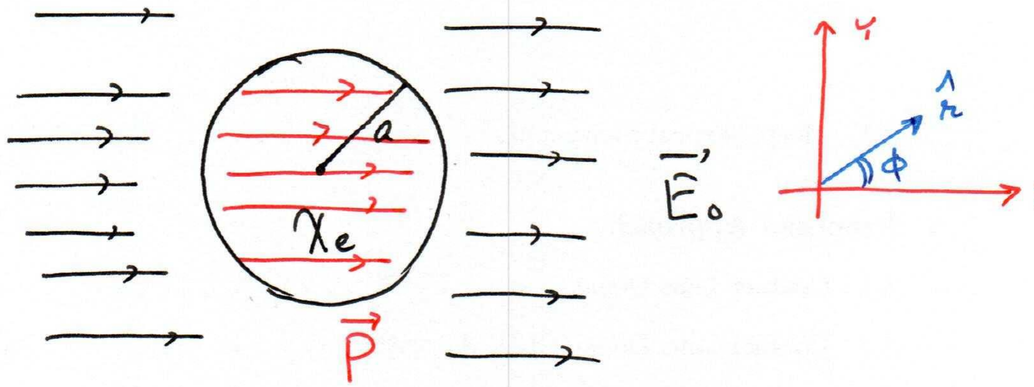
4.39: The potential satisfies the Laplace's equation throughout the space outside the sphere. The boundary condition is also very well specified. That is sufficient to determine the potential.

5.15: Use Ampere's law

5.44: Recall how this force was calculated for a spherical shell charged uniformly by electrical charge.

SOLUTION

4.22



Cylinder top view:

Let the cylinder develop a polarization $\vec{P}$ in the same direction as $\vec{E}_0$

$\vec{P}$ produces a bound charge $\rho_b = -\vec{\nabla}\cdot\vec{P}$
$= 0$

And a bound surface charge density

$$\sigma_b = \vec{P}\cdot\hat{r}$$
$$= P\cos\phi$$

This in turn gives an electric field $-\dfrac{P}{2\epsilon_0}$ in the cylinder. Thus

$$E = E_0 - \frac{P}{2\epsilon_0}$$

$$= E_0 - \frac{\chi_e \epsilon_0 \cdot E}{2\epsilon_0}$$

$$\Longrightarrow \quad E\left(1 + \frac{\chi_e}{2}\right) = E_0$$

or
$$E = \frac{2}{2+\chi_e} E_0$$
$$= \left(\frac{2}{K+1}\right) E_0$$
$$\vec{E}' = \left(\frac{2}{K+1}\right) \vec{E_0}$$

~~Also~~ One can also do the problem iteratively

$\vec{E_0}$ gives rise to $\vec{P_0}' = \epsilon_0 \chi_e \vec{E_0}$

$\vec{P_0}'$ gives $\sigma_{b0} = \epsilon_0 \chi_e E_0 \cos\phi$

σ_{b0} gives $\vec{E_1} = -\frac{\chi_e \vec{E_0}}{2}$

$\vec{E_1}$ gives new $\vec{P_1}' = -\frac{\chi_e^2 \epsilon_0}{2} \vec{E_0}$

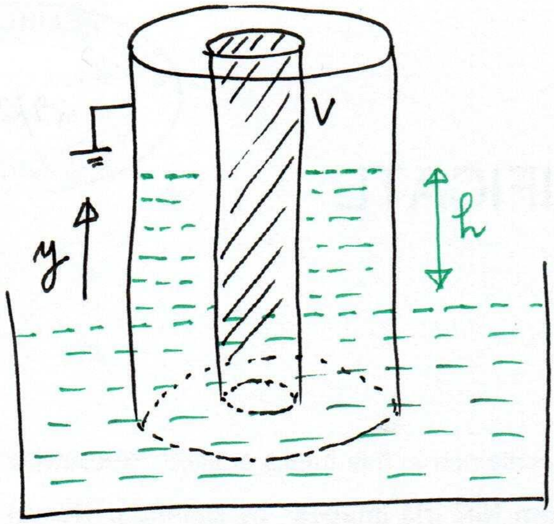
$\vec{P_1}'$ gives $\sigma_{b1} = -\frac{\chi_e^2 \epsilon_0 E_0}{2} \cos\phi$

σ_{b1} gives $\vec{E_2} = \frac{\chi_e^2}{4} \vec{E_0}$

And so on with the result that

$$\vec{E}_{net} = \vec{E_0} + \vec{E_1} + \vec{E_2} + \cdots$$
$$= \vec{E_0}\left(1 - \frac{\chi_e}{2} + \frac{\chi_e^2}{4} - \frac{\chi_e^3}{8} + \cdots\right)$$
$$= \vec{E_0}\left(\frac{1}{1+\chi_e/2}\right)$$
$$= \left(\frac{2}{2+\chi_e}\right)\vec{E_0}$$

4.28



③

For a system consisting of two metallic tubes as in this problem, the capacitance per unit length is

$$\frac{C}{\ell} = \frac{\lambda}{\frac{\lambda}{2\pi\epsilon_0 K}\ln(b/a)} = \frac{2\pi\epsilon_0 K}{\ln(b/a)}$$

When liquid is filled in the space between the tubes, the system can be considered as two capacitors connected in parallel — one with liquid in between and the other that has air. Let the length of the tubes be L. Then if the liquid is filled up to height y, the total capacitance will be

$$C = \frac{2\pi\epsilon_0 K y}{\ln(b/a)} + \frac{2\pi\epsilon_0 (L-y)}{\ln(b/a)}$$

So when connected to a voltage V across the tubes the energy stored in the capacitor is

$$E = \frac{1}{2}CV^2 = \frac{2\pi\epsilon_0}{2\ln(b/a)}\left[Ky + (L-y)\right]$$

$$E = \frac{\pi \epsilon_0}{\ln(b/a)} \left[(K-1)\, y + L \right] V^2$$

$$= \frac{\pi \epsilon_0}{\ln(b/a)} \left[\chi_e\, y + L \right] V^2$$

Force on the dielectric when voltage is kept constant is

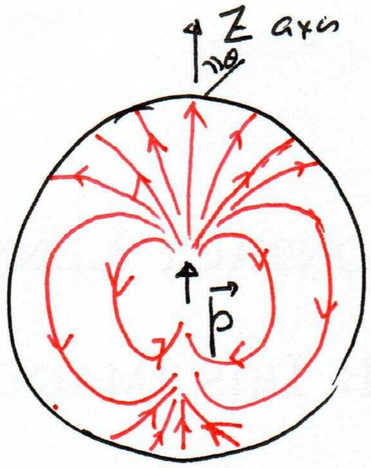
$$F = +\left(\frac{dE}{dy}\right)_V = \frac{\pi \epsilon_0 \chi_e}{\ln(b/a)} V^2$$

And the direction of the force is upward because of the positive sign. This balances the weight of the liquid. So

$$\frac{\cancel{\pi} \epsilon_0 \chi_e V^2}{\ln(b/a)} = \cancel{\pi} (b^2 - a^2) \rho g h$$

$$\boxed{h = \frac{\epsilon_0 \chi_e V^2}{(b^2 - a^2) \ln(b/a) \rho g}}$$

4.37



$(\vec{p} = p\hat{z})$

The dipole $\vec{p}$ create an electric field

$$\vec{E}_0 = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}\right]$$

So it is expected that this will create a polarization of the form

$$(\vec{p}_1 = p_1\hat{z}) \qquad \vec{P}_0 = \frac{3(\vec{p}_1\cdot\hat{r})\hat{r} - \vec{p}_1}{4\pi r^3}$$

(Here 4π is put for later convenience; it does not affect the answer)

However this polarization gives a surface charge

$$\sigma_b = \vec{P}_0\cdot\hat{r} = \frac{2p_1\cos\theta}{4\pi R^3} = \frac{p_1\cos\theta}{2\pi R^3}$$

This in turn will give an electric field which is constant inside the sphere and therefore will give a constant polarization also. Thus the net polarization inside the dielectric sphere can be taken to be

$$\vec{P} = \frac{3(\vec{p}_1\cdot\hat{r})\hat{r} - \vec{p}_1}{4\pi r^3} + \vec{P}_2$$

$$= \frac{3(\vec{p}_1\cdot\hat{r})\hat{r} - \vec{p}_1}{4\pi r^3} + P_2\hat{z}$$

This polarization gives a bound charge

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left[\frac{3(\vec{p}_1\cdot\hat{r})\hat{r}-\vec{p}_1}{4\pi r^3}\right] + 0$$

Interpretation of ρ_b

Dipole $\vec{p}$ at centre gives $\vec{E_0} = \dfrac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{(4\pi\epsilon_0)r^3}$

This gives a charge density

$$\rho(r) = \epsilon_0\vec{\nabla}\cdot\vec{E_0}$$

which is equivalent to a dipole $\vec{p}$

Thus $-\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left[\dfrac{3(\vec{p}_1\cdot\hat{r})\hat{r}-\vec{p}_1}{4\pi r^3}\right]$ will

give charge density ρ_b which is equivalent to a dipole $-\vec{p}_1$

Thus the first correction to $\vec{E_0}$ will be due to $-\vec{p}_1$ So

$$\vec{E}_0 + \vec{E}_1 = \frac{1}{4\pi\epsilon_0}\,\frac{3\{(\vec{p}-\vec{p}_1)\cdot\hat{r}\}\hat{r}-(\vec{p}-\vec{p}_1)}{r^3}$$

$$\vec{E} = \vec{E}_0+\vec{E}_1 + \text{additional terms.}$$

The additional terms arise due to the surface charge arising from the polarization

$$\sigma_b = \vec{P}\cdot\hat{r} = \frac{p_1\cos\theta}{2\pi R^3} + P_2\cos\theta$$

This surface charge gives an electric field

$$\vec{E}_2 = -\frac{1}{3\epsilon_0}\left[\frac{p_1}{2\pi R^3} + P_2\right]\hat{z}$$

Thus

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\,\frac{3\{(\vec{p}'-\vec{p}_1)\cdot\hat{r}\}\hat{r} - (\vec{p}'-\vec{p}_1)}{r^3}$$

$$-\frac{1}{3\epsilon_0}\left[\frac{p_1}{2\pi R^3} + P_2\right]\hat{z}$$

Now use the fact that $\vec{P} = \chi_e \epsilon_0 \vec{E}$

to get

$$\frac{3(\vec{p}_1\cdot\hat{r})\hat{r} - \vec{p}_1}{4\pi r^3} + P_2\,\hat{z}$$

$$= \frac{\chi_e}{4\pi r^3}\left[3\{(\vec{p}-\vec{p}_1)\cdot\hat{r}\}\hat{r} - (\vec{p}-\vec{p}_1)\right]$$

$$-\frac{\chi_e}{3}\left[\frac{p_1}{2\pi R^3} + P_2\right]\hat{z}$$

For the equality to be satisfied, we should have

$$\vec{p}_1 = \chi_e(\vec{p}-\vec{p}_1)$$

or $$\vec{p}_1 = \frac{\chi_e}{1+\chi_e}\vec{p} = \frac{\chi_e}{K}\vec{p}$$

and $$P_2 = -\frac{\chi_e}{3}\left(\frac{p_1}{2\pi R^3}\right) - \frac{\chi_e}{3}P_2$$

$$P_2\left(1+\frac{\chi_e}{3}\right) = -\frac{\chi_e}{3}\left(\frac{p_1}{2\pi R^3}\right)$$

$$\Rightarrow \quad P_2 = -\frac{\chi_e}{(3+\chi_e)}\cdot\frac{p_1}{2\pi R^3}$$

$$= \frac{-\chi_e^2}{K(3+\chi_e)}\cdot\frac{p}{2\pi R^3} \quad \left(p_1 = \frac{\chi_e}{K}p\right)$$

So the net electric field is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\,\frac{3\left\{\left(\vec{p}-\frac{\chi_e}{K}\vec{p}\right)\cdot\hat{r}\right\}\cdot\hat{r} - \left(\vec{p}-\frac{\chi_e}{K}\vec{p}\right)}{r^3}$$

$$-\frac{1}{3\epsilon_0}\left[\frac{\chi_e p}{K\cdot 2\pi R^3} - \frac{\chi_e^2}{(3+\chi_e)K}\cdot\frac{p}{2\pi R^3}\right]\hat{z}$$

$$(\epsilon=\epsilon_0 K) \quad = \frac{1}{4\pi\epsilon}\,\frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{r^3} - \frac{\chi_e p}{3\epsilon_0 K(2\pi R^3)}\left(1-\frac{\chi_e}{3+\chi_e}\right)\hat{z}$$

factor of 2 multiplied on upper & lower side

$$= \frac{1}{4\pi\epsilon}\,\frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{r^3} - \frac{2\chi_e p}{4\pi\epsilon.\,R^3}\,\frac{\hat{z}}{(3+\chi_e)}$$

The corresponding potential therefore is

$$V(r) = \frac{1}{4\pi\epsilon}\cdot\frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{2\chi_e p\, r\cos\theta}{4\pi\epsilon R^3 (3+\chi_e)}$$

$$= \frac{p\cos\theta}{4\pi\epsilon r^2}\left(1+2\frac{r^3}{R^3}\,\frac{\chi_e}{3+\chi_e}\right)$$

NOTE $\chi_e = K-1$ and $3+\chi_e = 2+K$

Outside the field (potential) will be that due to a dipole $(\vec{p}' - \vec{p}_1)$ + dipole due to the surface charge

$$\vec{p}'_{net} = \vec{p}' - \vec{p}_1 + \left(\frac{p_1}{2\pi R^3} + P_2\right) \cdot \frac{4\pi}{3} R^3 \, \hat{z}$$

Note: Surface charge is $\left(\frac{p}{2\pi R^3} + P_2\right)\cos\theta$ which is equivalent to that arising from a polarization

$$\vec{P} = \left(\frac{p_1}{2\pi R^3} + P_2\right)\hat{z}$$

$$= \vec{p} - \frac{\chi_e}{K}\vec{p} + \frac{4\pi}{3}R^3\left[\frac{\chi_e p}{K \cdot 2\pi R^3} - \frac{\chi_e^2 \, p}{K(3+\chi_e)2\pi R^3}\right]\hat{z}$$

$$(K = 1 + \chi_e)$$

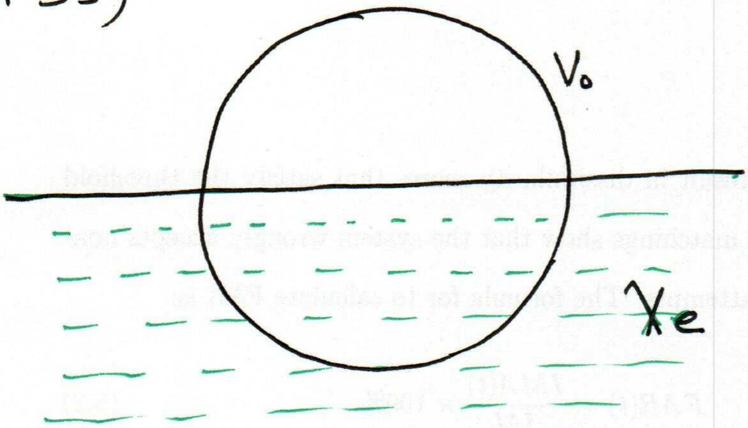
$$= \frac{\vec{p}}{K} + \frac{\overset{2}{\cancel{4\pi R^3}}}{3} \times \frac{\chi_e p}{K \cdot \cancel{2\pi R^3}}\left[1 - \frac{\chi_e}{3+\chi_e}\right]\hat{z}$$

$$= \frac{\vec{p}}{K} + \frac{2\chi_e p}{\cancel{3}K} \times \frac{\cancel{3}}{(3+\chi_e)}\hat{z} \qquad (\vec{p} = p\hat{z})$$

$$= \frac{\vec{p}}{K}\left(1 + \frac{2\chi_e}{3+\chi_e}\right) = \frac{3\vec{p}}{(3+\chi_e)}$$

So $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}\left(\frac{3}{3+\chi_e}\right).$

(4.39)



(a) Equation satisfied by the potential is the Laplace equation

$$\nabla^2 V(r) = 0$$

It is to be solved with the boundary condition

$$V(r=R) = V_0 \quad \text{and} \quad V(z=0^+) = V(z=0^-)$$

So the unique solution is $V(r) = \left(\dfrac{V_0 R}{r}\right)$

Surface charge on the sphere

$$\sigma_f = \hat{n}\cdot\vec{D}\Big|_{\text{surface}} = -\epsilon_0 K\left(\frac{\partial V}{\partial r}\right)\Big|_{\text{surface}}$$

$$\left(\vec{E} = -\vec{\nabla} V, \quad \hat{n} = \hat{r}, \quad \vec{D} = \epsilon_0 K \vec{E}\right)$$

$$\sigma_f \underset{z>0}{(\text{upper half})} = \frac{\epsilon_0 V_0}{R}$$

$$\sigma_f \underset{z<0}{(\text{lower half})} = \frac{\epsilon_0 K V_0}{R}$$

Electric field

$$\vec{E} = -\vec{\nabla} V = \frac{V_0 R}{r^2}\,\hat{r}$$

11

Polarization

$$\vec{P} = 0 \quad \text{for} \quad z > 0$$

$$\vec{P} = \chi_e \epsilon_0 \vec{E} = \frac{\chi_e \epsilon_0 V_0 R}{r^2} \hat{r} \quad \text{for } z < 0$$

Bound charges

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0 \quad \text{everywhere}$$

$$\sigma_b = (\vec{P} \cdot \hat{n})$$

($\hat{n} = -\hat{r}$ for volume outside sphere)

$$= 0 \quad \text{for } z > 0$$

$$= -\frac{\chi_e \epsilon_0 V_0}{R} \quad \text{for } z < 0$$

(b)

$$(\sigma_f + \sigma_b) = \frac{\epsilon_0 V_0}{R} \quad \text{for } z > 0$$

$$= \frac{\epsilon_0 V_0}{R} (K - \chi_e)$$

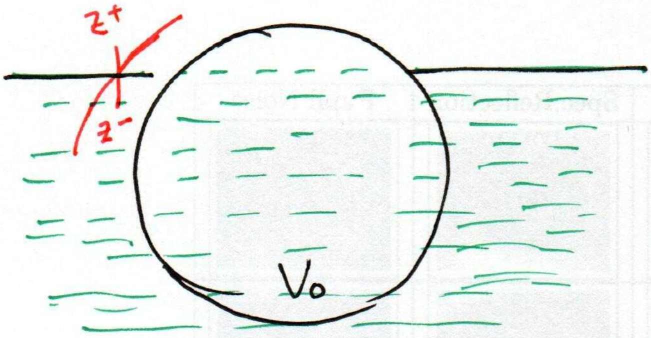
$$= \frac{\epsilon_0 V_0}{R} \quad \text{for } z < 0$$

So net charge giving the field is the same for both the upper and lower halves.

(d) We had solved the problem earlier with the boundary condition that

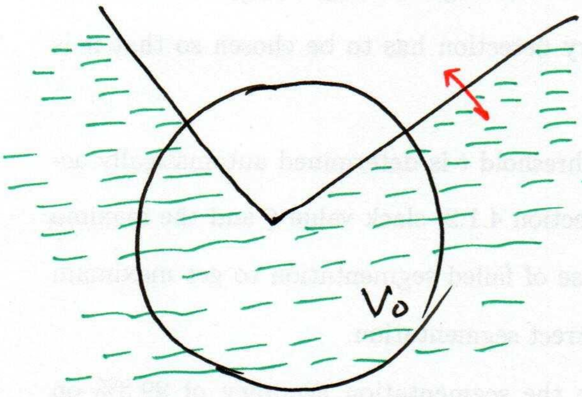
$V = V_0$ at $r = R$ and $V(z = 0^+) = V(z = 0^-)$

In Configuration (a), $V(r) = \frac{V_0 R}{r}$ will not satisfy the boundary condition that V be continuous across all boundaries, Hence $V(r) = \frac{V_0 R}{r}$ cannot be the solution.



Physically $V(r) = \frac{V_0 R}{r}$ mean that radial field will produce σ_b on the surface that will in turn alter the potential

In Configuration (b), however, $V(r) = \frac{V_0 R}{r}$ does satisfy the boundary condition that V be continuous across all boundaries Hence this is the solution. Physically radial field



due to $V(r) = \frac{V_0 R}{r}$ cannot produce σ_b here

So the solution remains the same.

5.5

(a) Surface current density

$$K = \frac{\text{current}}{\text{length}} = \frac{I}{2\pi a}$$

(b)

$$J(s) \propto \frac{1}{s} \Rightarrow J(s) = \frac{J_0}{s}$$

Then
$$I = \int_0^a \frac{J_0}{\cancel{s}} \cdot 2\pi \cancel{s}\, ds$$

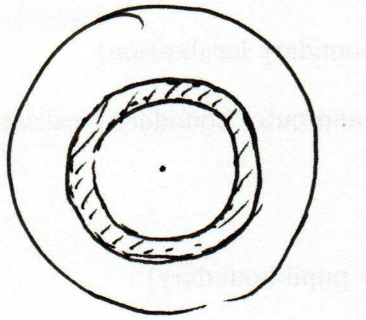
$$= 2\pi a J_0$$

$$J_0 = \frac{I}{2\pi a}$$

$$J(s) = \frac{I}{2\pi a s}$$

5.6

(a)



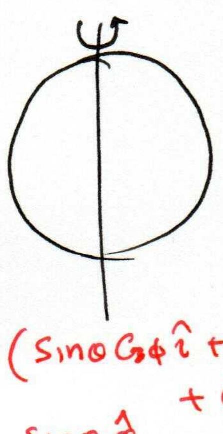
Current in the shaded region

$$= (\cancel{2\pi} r\, dr)\sigma \cdot \frac{\omega}{\cancel{2\pi}}$$

$$dI = \sigma \omega\, dr$$

$$K = \frac{dI}{dr} = \sigma \omega$$

(b)



Current density $\vec{J} = \rho v$

$$\vec{J}(r, \theta, \phi) = \rho(\vec{\omega} \times \vec{r})$$

$$= \rho\omega(\hat{z} \times \hat{r})\, r$$

$$= \rho\omega r \sin\theta\, \hat{\phi}$$

$$\hat{z} \times \hat{r} = \hat{z} \times (\sin\theta\cos\phi\, \hat{i} + \sin\theta\sin\phi\, \hat{j} + \cos\theta\, \hat{k})$$

$$= \sin\theta\, \hat{\phi}$$

5.7

$$\int \vec{J}\, d\tau = \frac{d\vec{p}}{dt}$$

$$\int \vec{\nabla}\cdot(x\vec{J})\, d\tau = \int x\vec{J}\cdot d\vec{s}$$

$$= 0 \quad \text{for } \vec{J} \to 0 \text{ as } r \to \infty$$

$$\int \vec{\nabla}\cdot(x\vec{J})\, d\tau = \int \left(x\,\vec{\nabla}\cdot\vec{J} + \vec{\nabla}x\cdot\vec{J} \right) d\tau$$

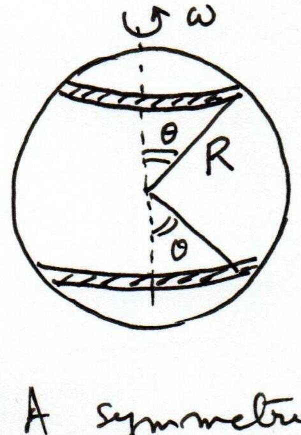
$$0 = \int \left(x\left(-\frac{\partial \rho}{\partial t}\right) + \hat{i}\cdot\vec{J} \right) d\tau$$

$$0 = \frac{-dp_x}{dt} + \int J_x\, d\tau$$

$$\rightarrow \frac{dp_x}{dt} = \int J_x\, d\tau$$

or $\dfrac{d\vec{p}}{dt} = \int \vec{J}\, d\tau$ (taking each component)

5.12



Current due to shaded ring

$$= \frac{\omega}{2\pi} \times 2\pi R^2 d(\cos\theta)\,\sigma\, \hat{\varphi}$$

$$= \omega R^2 \sigma\, d(\cos\theta)\, \hat{\varphi}$$

A symmetric ring is on the other side also. By example 5.6, the magnetic field at the centre due to these two rings is

$$d\vec{B} = 2 \times \frac{\mu_0 I}{2} \frac{R^2 \sin^2\theta}{(R^2\sin^2\theta + R^2\cos^2\theta)^{3/2}} \hat{z}$$

$$= 2 \times \frac{\mu_0 \cdot \omega R \sigma \, d(\cos\theta) R^2 \sin^2\theta}{2R^3} \hat{z}$$

$$= 2 \times \frac{\mu_0 \omega \sigma R}{2} \sin^2\theta \, d(\cos\theta) \, \hat{z}$$

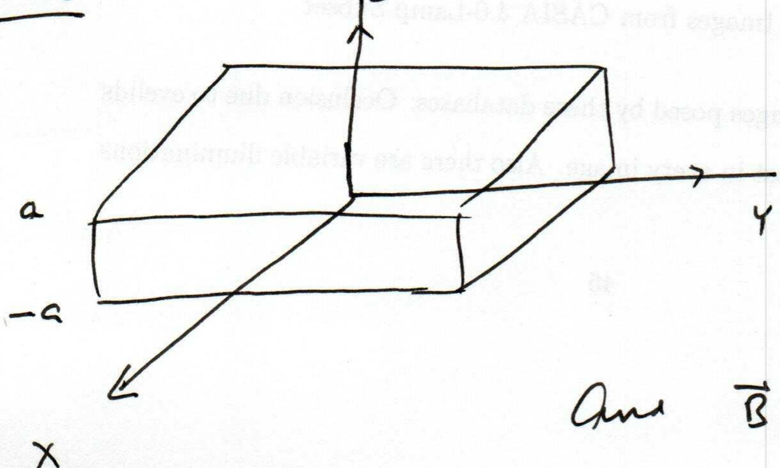
$$= \mu_0 \omega \sigma R \sin^2\theta \, d(\cos\theta) \, \hat{z}$$

Thus

$$\vec{B}' = \mu_0 \omega \sigma R \hat{z} \int_0^1 (1 - \cos^2\theta) \, d(\cos\theta)$$

$$= \frac{2\mu_0 \omega \sigma R}{3} \hat{z}$$

5.15



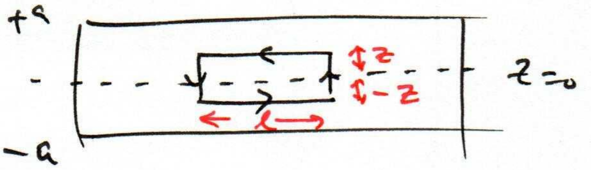
The magnetic field will be $\vec{B}' = -B(z)\hat{y}$ in the upper half

And $\vec{B} = +B(z)\hat{y}$

in the lower half as is easily seen by translational symmetry in x & y direction

Also $\quad B(+z) = B(-z)$.

for $\quad |z| \le a$, take an Amperean loop as shown



Then $\quad \int \vec{B}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{B})\cdot ds$

gives

$$\cancel{2} B \cancel{l} = \mu_0 \cancel{2} z \cancel{l} . J$$

$$\Rightarrow \quad B = \mu_0 z J$$

And for $\quad |z| \geq a$

$$\cancel{2} B \cancel{l} = \mu_0 . \cancel{2} a \cancel{l} \, J$$

$$B = \mu_0 a J$$

So as a function of z the field looks like