Department of physics
Indian Institute of Technology, Kanpur

MKH/SMT/2016-17-II/Phy103 Assignment-6 Week of 13-2-17

Note: The problems assigned are from the text book *Introduction to Electrodynamics* (4^{th} edition) by David J. Griffiths

1. Problem 5.18
2. Problem 5.21
3. Problem 5.24
4. Problem 5.26
5. Problem 5.30
6. Problem 5.44
7. Problem 5.56
8. Problem 5.59
9. Problem 5.62

Hints:

5.18 Very close to the inner boundary, the wall of the solenoid can be considered to be flat. That gives the same field all over boundary.

5.21 Take divergence on both sides of the equation giving the curl of magnetic field and use continuity equation.

5.24 $\hat{\varphi}$ is not a constant unit vector.

5.26 Consider electrostatic analogy to find the relevant integrals.

5.30 Take the solid sphere to be made up of shells and use result of example 5.11

5.44 Consider what will be the magnetic field that will cause a force to act on a strip of current on the shell.

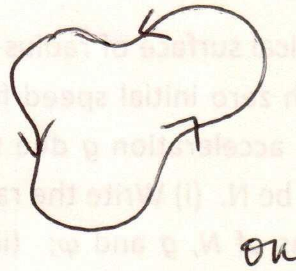
5.56 Use the vector identity for divergence of vector product of a vector $\vec{U}$ and $\vec{\nabla}\times\vec{V}$. Now assume that a given current distribution gives two different vector potentials $\vec{A}_1$and $\vec{A}_2$ and take $\vec{U} = \vec{V} = \vec{A}_1 - \vec{A}_2$.

5.59 Do not follow the hint given in the book. Instead, write the expression for the magnetic field using the Bio-Savart law and observe that the integral over $d\tau$ is nothing but proportional to the electric field due to a uniform charge distribution in the sphere.

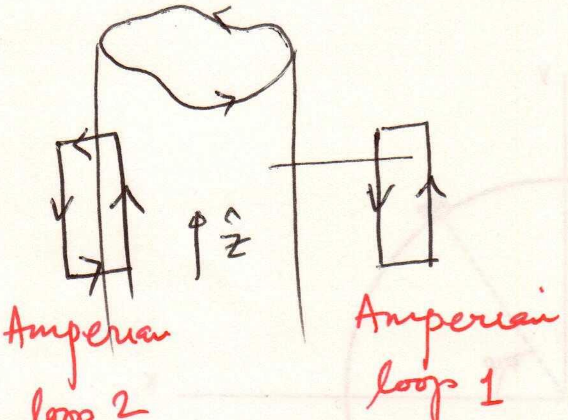
5.62 Follow the hint given in the problem.

SOLUTION

5.18

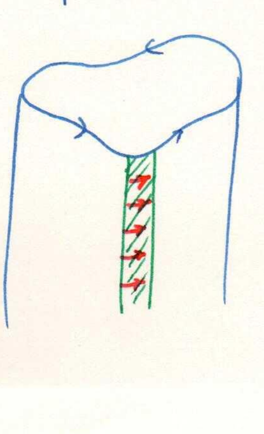


Cross section of a solenoid is shown on the left. As one goes far from the solenoid in lateral direction $\vec{B} \to 0$. Then if we choose an Amperian loop 1 as shown in the figure. It gives $\int \vec{B} \cdot d\vec{l} = 0$



$\Rightarrow B$ for points outside the solenoid vanishes. On the other hand, for Amperian loop 2, the field will come out to be $\mu_0 K \hat{z}$ independent of the cross-section of the loop.

Another way to look at the problem is as follows. The field outside is zero. Consider a thin strip as



shown where the current density is shown by red. Then at point very close to the strip, the field will be $\mu_0 K \hat{z}$. This is irrespective of the position of the strip anywhere on the periphery of the solenoid. Thus the field all over the boundary is $\mu_0 K\hat{z}$. Inside the solenoid, we have

$$\vec{\nabla}\times\vec{B}=0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{B}=0$$

The curl condition gives

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=0 \qquad \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}=0$$

$$\text{and} \quad \frac{\partial B_x}{\partial y}-\frac{\partial B_y}{\partial x}=0$$

But by symmetry there cannot be any z dependence of any component of B. Thus

$$\frac{\partial B_z}{\partial y}=\frac{\partial B_z}{\partial x}=0$$ so B_z remains unchanged inside the solenoid.

Again by symmetry, there cannot be any B_x or B_y components. If B_x and $B_y \neq 0$ then by reversing the current direction, their direction will change. But changing current direction is equivalent to turning the solenoid upside-down ~~by~~ but that does not change the direction of B_x and B_y. Hence $B_x=B_y=0$

Note that these arguments apply only if the x-section of the solenoid is uniform.

5.21

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J}$$

$$0 = \mu_0 \vec{\nabla} \cdot \vec{J} = -\mu_0 \frac{\partial \rho}{\partial t}$$

for time-dependent situations. Thus if there is time-dependence, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ cannot be true.

5.24

$$\vec{A} = k \hat{\varphi}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{s} \frac{\partial k}{\partial \varphi} = 0 \quad \text{(Cylindrical coordinates)}$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \quad \text{and} \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\vec{B} = \frac{1}{s} \frac{\partial}{\partial s} (s A_\varphi) \hat{z} = \frac{k}{s} \hat{z}$$

$$\nabla^2 \vec{A} = \nabla^2 (-k \sin\varphi \, \hat{i} + k \cos\varphi \, \hat{j})$$

$$= \frac{1}{s^2} \times -k \frac{\partial^2 \sin\varphi}{\partial \varphi^2} \hat{i} + \frac{1}{s^2} \times k \frac{\partial^2 \cos\varphi}{\partial \varphi^2} \hat{j}$$

$$= -\frac{k}{s^2} \hat{\varphi}$$

$$\Rightarrow \quad \vec{J}(s) = \left(\frac{k}{\mu_0}\right) \frac{1}{s^2} \hat{\varphi}$$

Leave checking Ampere's law as an exercise

5.26

(a) A current carrying wire

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\tau'$$

$$= \frac{\mu_0 I \hat{k}}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{|\vec{r}\,'|}$$

from electrostatic analogy (or directly) this integral can be calculated easily. Except for a constant term (tending to ∞) we have for electrostatic potential of a line charge λ

$$\frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{|\vec{r}|} = -\frac{\lambda}{2\pi\epsilon_0} \ln s$$

(Cylindrical coordinates)

$$\Rightarrow \int \frac{dz'}{|\vec{r}|} = -2 \ln s$$

So $\vec{A} = -\frac{\mu_0 I}{2\pi} \ln(s) \hat{k}$

Now $\vec{\nabla} \cdot \vec{A} = 0$ and $\vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial s} \hat{\varphi}$

$$= \left(\frac{\mu_0 I}{2\pi s}\right) \hat{\varphi}$$

(b) Magnetic potential inside the wire

Let the radius of the wire be R

$$\vec{J} = \frac{I}{\pi R^2}\hat{k}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad \text{gives}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\hat{k}\int \frac{|\vec{J}(\vec{r}\,')|}{|\vec{r}-\vec{r}\,'|}\,d\tau' \qquad (s<R)$$

(s is in cylindrical coordinates)

$$= \frac{\mu_0}{4\pi}\hat{k}\left(\frac{I}{\pi R^2}\right)\int \frac{d\tau'}{|\vec{r}-\vec{r}\,'|}$$

Again by electrostatic analogy, one can find the integral. For a wire carrying a uniform charge density ρ (except for a constant) the potential is

$$-\left(\frac{\rho s^2}{4\epsilon_0}\right) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau'}{|\vec{r}-\vec{r}\,'|}$$

$(|s| < R|)$

$$\Rightarrow \int \frac{d\tau'}{|\vec{r}-\vec{r}\,'|} = -\pi s^2$$

$$\Rightarrow \vec{A} = -\frac{\mu_0 I s^2}{4\pi R^2}\hat{k}$$

5.30 Magnetic field inside a solid sphere of uniform charge density ρ and radius R rotating at constant angular velocity $\vec{\omega} = \omega\hat{z}$

According to example 5.11, field inside a rotating shell is

$$\frac{2}{3}\mu_0 \sigma R \omega \hat{z}$$

$$= \frac{2}{3}\mu_0 \sigma R\omega \left[\hat{r}\cos\theta - \hat{\theta}\sin\theta \right]$$

And field outside is

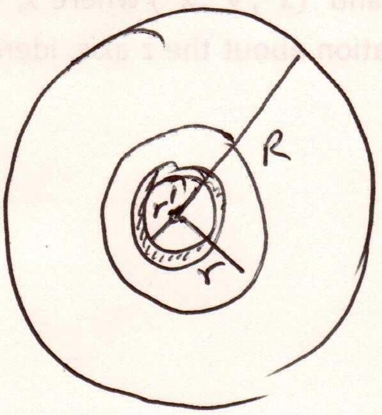
$$\vec{\nabla}\times\vec{A} = \frac{\mu_0 R^4 \omega\sigma}{3} \vec{\nabla}\times\left(\frac{\sin\theta}{r^2}\hat{\varphi}\right)$$

$$= \frac{\mu_0 R^4 \omega \sigma}{3}\left[\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\sin\theta}{r^2}\right) - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\sin\theta}{r^2}\right)\right]$$

$$= \frac{\mu_0 R^4 \omega\sigma}{3}\left[\hat{r}\,\frac{2\cos\theta}{r^3} + \hat{\theta}\,\frac{\sin\theta}{r^3}\right]$$

Now a rotating solid sphere, we will consider as made up of shell.

To calculate field at r, we add up field due to shells inside and outside. For a shell of radius r', the surface charge density is



$$\sigma = \rho\, dr'$$

where dr' is the thickness of the shell. This shell gives a field

$$\vec{B}(r) = \frac{\mu_0 r'^4 \omega \cdot \rho\, dr'}{3}\left[\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right]$$

Thus the net field due to shells inside is given by $\int_0^r$ and the answer is

$$\bar{B}(\bar{r}) = \frac{\mu_0 \omega \rho r^5}{15}\left[\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right]$$

$$= \frac{\mu_0 \omega \rho r^2}{15}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

Similarly, field due to shells outside is

$$\bar{B} = \frac{2}{3}\mu_0 \omega \rho \int_r^R r'\,dr' \left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$= \frac{2}{3} \mu_0 \omega \rho \frac{(R^2 - r^2)}{2} \left[\cos\theta \, \hat{r} - \sin\theta \, \hat{\theta} \right]$$

$$= \frac{\mu_0 \omega \rho}{3} (R^2 - r^2) \left[\cos\theta \, \hat{r} - \sin\theta \, \hat{\theta} \right]$$

Thus the total field is

$$\frac{\mu_0 \omega \rho}{3} R^2 \left[\cos\theta \, \hat{r} - \sin\theta \, \hat{\theta} \right]$$

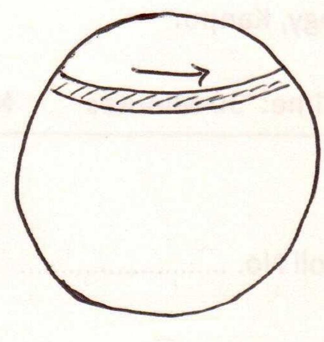
$$+ \mu_0 \omega \rho r^2 \left[\hat{r} \cos\theta \left(\frac{2}{15} - \frac{1}{3} \right) + \hat{\theta} \sin\theta \left(\frac{1}{15} + \frac{1}{3} \right) \right]$$

$$= \frac{\mu_0 \omega \rho}{3} R^2 \hat{z} + \mu_0 \omega \rho r^2 \left[\hat{r} \cos\theta \times -\frac{3}{15} + \hat{\theta} \sin\theta \times \frac{6}{15} \right]$$

$$= \frac{\mu_0 \omega \rho}{3} R^2 \hat{z} + \frac{\mu_0 \omega \rho r^2}{15} \left[-3 \hat{r} \cos\theta + 6 \hat{\theta} \sin\theta \right]$$

$$= \frac{\mu_0 \omega \rho}{3} R^2 \hat{z} + \frac{\mu_0 \omega \rho r^2}{5} \left[-\hat{r} \cos\theta + 2 \hat{\theta} \sin\theta \right]$$

5-44 (9)



We will Calculate

$$\bar{B}'_{average} = \frac{\bar{B}_{in} + \bar{B}_{out}}{2}$$

which will represent the field due to all the current except that due to the strip shown and then calculate the force as $I \, d\bar{l} \times \bar{B}$

$$I = 2\pi R^2 \sigma \sin\theta \, d\theta \cdot \frac{\omega}{2\pi} = \omega \sigma R^2 \, d(\cos\theta)$$

$$\bar{B}' = \frac{\bar{B}_{in} + \bar{B}_{out}}{2}$$

$$= \frac{1}{2}\Big[\frac{2}{3}\mu_0 \sigma R \omega \, \hat{r} \cos\theta - \frac{2}{3}\mu_0 \sigma R \omega \, \hat{\theta} \sin\theta$$

$$+ \frac{2}{3}\mu_0 \sigma R \omega \, \hat{r} \cos\theta + \frac{1}{3}\mu_0 \sigma R \omega \, \hat{\theta} \sin\theta \Big]$$

(From page 6)

$$= \frac{2}{3}\mu_0 R \omega \sigma \cos\theta \, \hat{r} - \frac{1}{6}\mu_0 R \omega \sigma \sin\theta \, \hat{\theta}$$

$d\bar{l}$ is in $\hat{\phi}$ direction and the length of the strip is $2\pi R \sin\theta$ so

Force on the strip (in the $\hat{z}$ direction)

$$\hat{z}\cdot d\vec{F} = \omega\sigma R^2\, d(\cos\theta) \times 2\pi R\sin\theta \times$$

$$\frac{\mu_0 R\omega\sigma}{3}\left\{\hat{\varphi}\times\left[2\cos\theta\,\hat{r} - \frac{\sin\theta}{2}\hat{\theta}\right]\right\}\cdot\hat{z}$$

$$= \frac{\mu_0\,\omega^2\sigma^2 R^3}{3} \times 2\pi R\; d(\cos\theta)\cdot\sin\theta \times$$

$$\times\left[2\cos\theta\,\hat{\theta} + \frac{\sin\theta}{2}\hat{r}\right]\cdot\hat{z}$$

$$= \frac{2\pi\mu_0\omega^2\sigma^2 R^4}{3}\, d(\cos\theta)\,\sin\theta \times$$

$$\left[-2\cos\theta\sin\theta + \frac{\cos\theta\sin\theta}{2}\right]$$

$$= -\frac{2\pi\mu_0\omega^2\sigma^2R^4}{3}\times\frac{3}{2}\times\cos\theta\sin^2\theta\, d(\cos\theta)$$

So net force

$$F = -\pi\mu_0\omega^2\sigma^2R^4\underbrace{\int_0^1 d(\cos\theta)\,(\cos\theta-\cos^3\theta)}_{1/4}$$

$$= -\frac{\pi\mu_0\omega^2\sigma^2R^4}{4}$$

(− sign shows force in $-\hat{z}$ direction ⇒ Attraction)

5.56 We know (11)

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

Take $\vec{A}' = \vec{\nabla}\times\vec{U}$. Then taking $\vec{B}=\vec{V}$

$$\vec{\nabla}\cdot\left[(\vec{\nabla}\times\vec{U})\times\vec{V}\right] = \vec{V}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{U})) - (\vec{\nabla}\times\vec{U})\cdot(\vec{\nabla}\times\vec{V})$$

Taking volume integral and using divergence theorem, we get

$$\int\left[(\vec{\nabla}\times\vec{U})\times\vec{V}\right]\cdot d\vec{S} = \int\vec{V}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{U}))\,d\tau - \int(\vec{\nabla}\times\vec{U})\cdot(\vec{\nabla}\times\vec{V})\,d\tau$$

which is the same as the given form in the book

Now consider a current distribution $\vec{J}(\vec{r})$ and either $\vec{A}'$ or $\vec{B}$ specified on the boundary. To prove uniqueness assume two different solutions $\vec{A}_1'$ and $\vec{A}_2'$ (and corresponding $\vec{B}_1$ and $\vec{B}_2$) such that

$$\vec{\nabla}\times\vec{B}_1' = \vec{\nabla}\times\vec{B}_2 = \mu_0\vec{J}$$

And $\vec{\nabla} \times \vec{A}_1 = \vec{B}_1$, $\vec{\nabla} \times \vec{A}_2 = \vec{B}_2$

Take $\vec{A}_3 = \vec{A}_1 - \vec{A}_2$ and $\vec{B}_3 = \vec{B}_1 - \vec{B}_2$

Then $\vec{\nabla} \times \vec{B}_3 = 0$

In the divergence theorem based result,

take $\vec{U} = \vec{V} = \vec{A}_3$ so that we have

$$\int \left[(\vec{\nabla} \times \vec{A}_3) \times \vec{A}_3 \right] \cdot d\vec{S} = \int \vec{A}_3 \cdot (\vec{\nabla} \times \vec{B}_3)\, d\tau - \int (\vec{\nabla} \times \vec{A}_3) \cdot (\vec{\nabla} \times \vec{A}_3)\, d\tau$$

$$= 0 - \int (\vec{B}_3)^2\, d\tau$$

Since $\vec{A}$ or $\vec{B}$ is specified on the boundary, ~~that~~ value of $\vec{A}_3$ or $(\vec{\nabla} \times \vec{A}_3)$ vanishes on the boundary. This means

$$\int (\vec{B}_3)^2\, d\tau = 0 \quad \text{or} \quad \vec{B}_3 = 0$$

$$\Rightarrow \vec{B}_1 = \vec{B}_2$$

5.59

$$\vec{B}_{av} = \frac{1}{\frac{4\pi}{3}R^3}\int \vec{B}\, d\tau$$

$$= \frac{1}{\frac{4\pi}{3}R^3}\int \frac{\mu_0}{4\pi}\frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau'\, d\tau$$

$$= \frac{1}{\frac{4\pi}{3}R^3}\left(\frac{\mu_0}{4\pi}\right)\int d\tau'\, \vec{J}(\vec{r}')\times\int\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau$$

Now $\int \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau$ is like the negative of electric field at $\vec{r}'$ due to a sphere of radius R carrying uniform charge density $\rho=1$

$$\frac{1}{4\pi\epsilon_0}\int\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau = -\frac{\vec{r}'}{3\epsilon_0}$$

$$\Rightarrow \int\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau = -\frac{4\pi}{3}$$

So

$$\vec{B}_{av} = -\frac{\mu_0}{4\pi R^3}\int d\tau'\, \vec{J}(\vec{r}')\times\vec{r}'$$

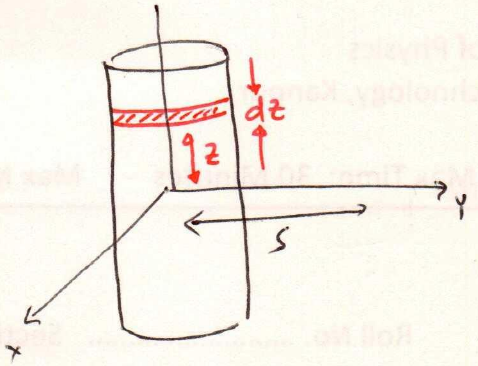
$$= \frac{\mu_0}{4\pi R^3}\int d\tau'\, \vec{r}'\times\vec{J}(\vec{r}')$$

$$\boxed{\vec{m} = \frac{1}{2}\int \vec{r}\times \vec{J}(\vec{r})\, d\tau}$$

$$= \frac{\mu_0}{4\pi R^3}\cdot 2\vec{m}$$

Proved

5.62



Consider a strip of height dz at z distance

Its magnetic moment

$$d\vec{m} = I\cdot \pi R^2 \hat{z}$$

$$= (2\pi\sigma R\,dz)\frac{\omega}{2\pi}\pi R^2\hat{z}$$

$$= \pi\sigma\omega R^3\hat{z}\,dz$$

For $s \gg R$, the field due to this strip is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{3(d\vec{m}\cdot\hat{r})\hat{r} - d\vec{m}}{r^3}$$

$$\hat{r} = \frac{s\hat{s} - z\hat{z}}{\sqrt{s^2+z^2}}, \quad r = \sqrt{s^2+z^2}$$

So

$$d\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{-3\pi\sigma\omega R^3 z\,dz\,(s\hat{s} - z\hat{z})}{(s^2+z^2)^{5/2}} - \frac{\pi\sigma\omega R^3\hat{z}\,dz}{(s^2+z^2)^{3/2}}\right]$$

So the net field is (Integral of first term = 0 since it's odd in z)

$$\vec{B}(s) = \hat{z}\frac{\mu_0\sigma\omega R^3}{4}\left[\int_{-L/2}^{L/2}\frac{3z^2\,dz}{(s^2+z^2)^{5/2}} - \int_{-L/2}^{L/2}\frac{dz}{(s^2+z^2)^{3/2}}\right]$$

$$\vec{B}(s) = \hat{z}\,\frac{\mu_0\sigma\omega R^3}{4}\left[3\int_{-L/2}^{L/2}\frac{z^2\,dz}{(s^2+z^2)^{5/2}} - \int_{-L/2}^{L/2}\frac{dz}{(s^2+z^2)^{3/2}}\right]$$

$$\int_{-L/2}^{L/2}\frac{z^2\,dz}{(s^2+z^2)^{5/2}} = \int_{z=-L/2}^{L/2}\frac{s^2\tan^2\theta\; s\cdot\sec^2\theta\, d\theta}{s^5\sec^5\theta}$$

$$= \frac{1}{s^2}\int_{z=-L/2}^{z=L/2}\sin^2\theta\; d(\sin\theta)$$

$$= \frac{1}{3s^2}\,\sin^3\theta\,\Big|_{z=-L/2}^{L/2}$$

$$= \frac{1}{3s^2}\;\frac{L^3}{4s^3}\;\frac{1}{\left(1+\left(\frac{L}{2s}\right)^2\right)^{3/2}}$$

$$= \frac{1}{3}\;\frac{L^3}{4s^2}\cdot\frac{1}{\left(s^2+\left(\frac{L}{2}\right)^2\right)^{3/2}}$$

And

$$\int_{-L/2}^{L/2}\frac{dz}{(s^2+z^2)^{3/2}} = \int_{z=-L/2}^{L/2}\frac{s\sec^2\theta\, d\theta}{s^3\sec^3\theta}$$

$$= \frac{1}{s^2}\cdot\sin\theta\,\Big|_{z=-L/2}^{L/2}$$

$$= \frac{1}{s^3}\;\frac{L}{\left(1+\left(L/2s\right)^2\right)^{1/2}}$$

$$\frac{L}{s^2}\frac{1}{\sqrt{s^2+(L/2)^2}}$$

So

$$\vec{B}(s) = \hat{z}\,\frac{\mu_0\sigma\omega R^3}{4}\left[\frac{L^3}{4s^2}\frac{1}{\{s^2+(L/2)^2\}^{3/2}} - \frac{L}{s^2}\frac{1}{\sqrt{s^2+(L/2)^2}}\right]$$

$$= \hat{z}\,\frac{\mu_0\sigma\omega R^3}{4}\,\frac{L}{4s^2}\left[\frac{L^2}{(s^2+L^2/4)^{3/2}} - \frac{4}{\sqrt{s^2+L^2/4}}\right]$$

$$= \hat{z}\,\frac{\mu_0\sigma\omega R^3 L}{16s^2}\,\frac{1}{(s^2+L^2/4)^{3/2}}\left[\cancel{L^2} - 4s^2 - \cancel{L^2}\right]$$

$$= -\hat{z}\,\frac{\mu_0\sigma\omega R^3 L}{4\left(s^2+L^2/4\right)^{3/2}}$$

Answer.