Patterns and bifurcations in low–Prandtl-number Rayleigh-Bénard convection

P. K. Mishra1, P. Wah1(a) and M. K. Verma1

1 Department of Physics, Indian Institute of Technology - Kanpur, India
2 Department of Mechanical Engineering, Indian Institute of Technology - Kanpur, India

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Abstract – We present a detailed bifurcation structure and associated flow patterns for low–Prandtl-number (low-P) Rayleigh-Bénard convection near its onset. We use both direct numerical simulations and a 30-mode low-dimensional model for this study. We observe that low–Prandtl-number (low-P) convection exhibits similar patterns and chaos as zero-P convection (Pal P. et al., EPL, 87 (2009) 04003) namely squares, asymmetric squares, oscillating asymmetric squares, relaxation oscillations, and chaos. At the onset of convection, low-P convective flows have stationary 2D rolls and associated stationary and oscillatory asymmetric squares in contrast to zero-P convection where chaos appears at the onset itself. The range of Rayleigh number for which stationary 2D rolls exist decreases rapidly with decreasing Prandtl number. Our results are in qualitative agreement with results reported earlier.

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Thermal convection is an important problem of classical physics, engineering, geophysics, atmospheric physics, and astrophysics. As a consequence of its inherent nonlinearity, good understanding of convection eludes us even after sustained efforts for more than a century. Convection in an arbitrary geometry is quite complex, so researchers have focussed on convection between two conducting parallel plates. This idealized problem is called Rayleigh-Bénard convection (RBC).

Convective flow in RBC is characterized by two non-dimensional parameters: the Rayleigh number $R$ which is the ratio of the destabilizing force to the stabilizing force, and the Prandtl number $P$ which is the ratio of the kinematic viscosity to the thermal diffusivity. RBC shows a wide range of phenomena including instabilities, patterns, chaos, and turbulence for various ranges of $R$ and $P$ [1]. For low–Prandtl-number (low-P) and zero–Prandtl-number (zero-P) convection, the inertial term $u \cdot \nabla u$ becomes quite important and generates vertical vorticity. As a result, the flow pattern becomes three-dimensional, and oscillatory waves along the horizontal axes are generated just near the onset of convection. On the contrary, for large–Prandtl-number (large-P) convection, vertical vorticity is absent near the onset and the two-dimensional (2D) rolls survive till large Rayleigh numbers [2].

In this letter we investigate the bifurcation scenario for low-P convection and compare it with the recent zero-P convection results of Pal et al. [3]. We will also explore the origin of chaos near the onset in low-P convection. The relevant equations (under Boussinesq approximation) in nondimensionalized form are

$$
\partial_t (\nabla^2 v) \alpha = \nabla^4 v + R \nabla^2 \theta \\
- \epsilon_3 \cdot \nabla \times ([\omega \cdot \nabla] v - (v \cdot \nabla) \omega),
$$

(1)

$$
\partial_t \omega = \nabla^2 \omega_3 + [(\omega \cdot \nabla) v_3 - (v \cdot \nabla) \omega_3],
$$

(2)

$$
P(\partial_t \theta + (v \cdot \nabla) \theta) = v_3 + \nabla^2 \theta,
$$

(3)

$$
\nabla \cdot v = 0,
$$

(4)

where $v \equiv (v_1, v_2, v_3)$ is the velocity field, $\theta$ is the deviation in the temperature field from the steady conduction profile, $\omega = \nabla \times v$ is the vorticity field, $R = \alpha g (\Delta T) d^3 / \nu \kappa$ is the Rayleigh number, $P = \nu / \kappa$ is the Prandtl number, $\epsilon_3$ is the unit vector along the vertical (buoyancy) direction, and $\nabla^2 = \partial_{xx} + \partial_{yy}$ is the horizontal Laplacian. Here, $\nu$ and $\kappa$ are fluid’s kinematic viscosity and thermal diffusivity, respectively, $d$ is the vertical distance between the two plates, and $\Delta T$ is the temperature difference between

(a)E-mail: wahi@iitk.ac.in
the horizontal plates. For the above nondimensionalization, we used $d$ as the length scale, $d^2/\nu$ as the time scale, and $\nu \Delta T/\kappa$ as the temperature scale. We consider perfectly conducting and free-slip boundary conditions at the top and bottom plates and periodic boundary conditions along the horizontal directions. We will mostly use the reduced Rayleigh number $r = R/R_c$, where $R_c$ is the critical Rayleigh number, as the control parameter for our bifurcation analysis.

Instabilities and chaos in low-$P$ and zero-$P$ convection have been widely studied using analytical tools (e.g., approximation techniques), experiments, and numerical simulations. Busse [4] showed using perturbative analysis that for small Prandtl numbers, the 2D rolls become unstable when the amplitude of the convective motion exceeds a critical value. Busse and Bolton [5] argued that under free-slip boundary conditions, the stability of the 2D rolls exists only for Prandtl numbers above a critical value $P_c$, which is around 0.543. Clever and Busse [6] extended the oscillatory instability analysis to no-slip boundary conditions and showed that the convective rolls are unstable for Prandtl numbers less than about 5.

Siggia and Zippelius [7] showed the importance of vertical vorticity in convective instabilities using amplitude equations. Fauve et al. [8] investigated the origin of instabilities in low-$P$ convection using the phase dynamical equations and argued that the instability always saturates into travelling waves as observed in experiments and numerical simulations. Jenkins and Proctor [9] studied the transition from 2D rolls to square patterns in RBC for different Prandtl numbers using analytical tools. Kumar et al. [10] investigated zero-$P$ convection using a 6-mode model and showed the existence of wavy instability at the onset.

Interesting experiments have been performed to explore the instabilities and chaos in convective flows near the onset. Willis and Beardorff [11] carried out experiments on air ($P = 0.7$) and observed time dependence as a result of wavy instability. Rossby [12] demonstrated the existence of wavy instability in mercury ($P = 0.02$). Krishnamurti [13] performed extensive convection experiments on a variety of fluids ($P = 0.1-50$) and observed stationary and time-dependent three-dimensional patterns in her experiments; time-dependent rolls appear at much lower Rayleigh numbers for low-$P$ convection compared to that for large-$P$ convection.

Numerical simulations complement experiments in the investigation of instabilities and chaos in convection. Lipps [14] simulated convective flows in air ($P \approx 0.7$) and investigated various states including 2D rolls, time-periodic and aperiodic convection as a function of Rayleigh number and aspect ratio. Bolton and Busse [15] studied the stability of the steady convection rolls with respect to arbitrary three-dimensional infinitesimal disturbances and observed stability of steady solutions for only a small regime of Rayleigh numbers and wave numbers. Clever and Busse [16] simulated convective flows for $P = 0.02$ with rigid boundary conditions using the Galerkin technique and observed asymmetric travelling wave patterns. Meneguzzi et al. [17] performed three-dimensional convective simulations for $P = 0.2$ fluid under stress-free boundary conditions, and for $P = 0.025$ for no-slip boundary conditions. For $P = 0.2$ they observed stationary, periodic, biperiodic and chaotic regimes as the Rayleigh number is increased. For the latter, they found stationary and time-periodic solutions only. Thual [18] studied zero-$P$ and low-$P$ convection using numerical simulations for both no-slip and free-slip boundary conditions. In his simulations he observed supercritical oscillatory instabilities, competition between two-dimensional rolls, square and hexagonal patterns, travelling and standing waves, and chaos for both zero-$P$ and low-$P$ convection. Ozoe and Hara [19] performed numerical simulations for Prandtl numbers $P = 0.001-0.1$ in a two-dimensional rectangular enclosure of aspect ratio 4 and observed that the time-dependent behaviour sets in at the onset of convection for $P = 0.01$, thus decreasing the value of $P_c$ to much lower limit than that predicted by Busse and Bolton [5].

Recently Pal et al. [3] performed numerical simulations and bifurcation analysis using a low-dimensional model of zero-$P$ convection under free-slip boundary conditions. They could explain the origin of squares (SQ), asymmetric squares (ASQ), oscillating asymmetric squares (OASQ), relaxation oscillations with an intermediate square pattern (SQOR), and chaos using a series of bifurcations. Pal et al. [3] start their analysis from square patterns at $r = 1.4$. Under reduction of $r$, SQ bifurcates into asymmetric squares through a pitchfork bifurcation, which in turn transforms into OASQ through a Hopf bifurcation. OASQ transforms to SQOR that subsequently becomes chaotic. The chaotic attractor continues till the onset of convection. There have been other attempts to study RBC using low-dimensional models for other ranges of Prandtl numbers [20].

In this letter we perform numerical simulations and bifurcation analysis of low-$P$ convection along similar lines as Pal et al. [3] to investigate the origin of patterns and chaos. We also carry out a comparative study of low-$P$ convection with zero-$P$ convection and show how the limiting behavior of zero-$P$ convection is obtained as $P$ approaches zero.

We perform direct numerical simulations (DNS) of convective flows for $P = 0.02, 0.005, 0.002$ and 0.0002 for $r$ ranging from 1 to 1.25 on $64^3$ rectangular grid. We use a pseudo-spectral procedure with fourth-order Runge-Kutta as the time-stepping scheme. The aspect ratio of our simulations is $\Gamma_x = 2\sqrt{2}$, $\Gamma_y = 2\sqrt{2}$. Equations (1) and (3) provide us an estimate of $dt$ for the DNS ($dt \sim P_0/\nu = \sim P/R$). For $P = 0.0002, dt \sim 10^{-7}$ that makes numerical simulations very demanding. In DNS we observe stationary 2D rolls, ASQ, wavy rolls, OASQ, SQOR, and SQ (acronyms defined earlier) for $P = 0.02$.

For a detailed bifurcation analysis of this problem, we construct a 30-mode low-dimensional model using the
energetic modes of the DNS [3]. These modes have 1% or more of the total energy. We pick 11 large-scale vertical velocity modes: \( W_{101}, W_{011}, W_{112}, W_{211}, W_{121}, W_{301}, W_{103}, W_{013}, W_{031}, W_{202}, W_{022} \); 12 large-scale \( \theta \) modes: \( \theta_{101}, \theta_{011}, \theta_{112}, \theta_{211}, \theta_{121}, \theta_{103}, \theta_{013}, \theta_{031}, \theta_{202}, \theta_{022}, \theta_{002} \); and 7 large-scale vertical vorticity modes: \( Z_{110}, Z_{112}, Z_{211}, Z_{121}, Z_{220}, Z_{310}, Z_{130} \). The three subscripts here are the indices of the wave numbers along the \( x \), \( y \), and \( z \) directions. All the Fourier modes, 30 in total, are taken to be real. This model is inspired by the 13-mode model for zero-\( P \) convection [3]. We investigate the origin of the convective flow patterns using a bifurcation diagram of this low-dimensional model following the approach of Pal et al. [3]. A MATLAB package named MATCONT has been used for some of the analysis.

The phase space of our low-dimensional model is 30-dimensional. Among all the modes, \( W_{101}, W_{011}, \theta_{101}, \) and \( \theta_{011} \) are the most important ones. In fig. 1 we plot the numerical values of the fixed points \( W_{101} \) and \( W_{011} \) for \( P = 0.02 \) as a function of \( r \). The stable fixed points are shown as solid lines, while the unstable ones are shown as dashed lines. At \( r = 1 \), the conductive state (cyan curve) becomes unstable and bifurcates into four stable 2D rolls (shown as purple curves) and four unstable SQ (shown as black dashed lines) through a codimension-2 pitchfork bifurcation associated with double zero eigenvalues. With an increase of \( r \), the stable 2D rolls bifurcate into stable patterns ASQ shown as the solid blue lines in fig. 1. These ASQ patterns lose stability at some point through a Hopf bifurcation. With a further increase in \( r \), the unstable branch of ASQ solutions (dashed blue curves) regains stability through an inverse Hopf bifurcation resulting in stable ASQ solutions (solid blue curves). These stable ASQ solutions subsequently meet the unstable SQ patterns (represented by dashed black curves originating from \( r = 1 \)) in yet another inverse bifurcation, and stable SQ patterns (solid black curves) are formed. These stable SQ patterns subsequently lose stability beyond \( r = 1.25 \) and complex chaotic attractors are generated, however an analysis in this regime is beyond the scope of this letter. This sequence of fixed points remains the same for all finite but small values of \( P \). However, the range of \( r \) corresponding to various states changes with \( P \).

We will describe the above-mentioned fixed points and associated time-dependent patterns using a bifurcation diagram in the range of \( 0.95 \leq r \leq 1.25 \). Figure 2 illustrates the bifurcation diagram for \( P = 0.02 \), where we plot the positive value of \( (W_{101})_{\text{extremum}} \) as a function of \( r \). At \( r = 1 \) the conduction state bifurcates to stationary 2D rolls (purple curve) through a codimension-2 pitchfork bifurcation. As described above unstable stationary SQ (shown as dashed black curve) with \( W_{101} = \pm W_{011} \) are also generated as a result of this bifurcation. As \( r \) is increased further, at \( r \approx 1.0035 \), the stationary 2D rolls bifurcate to ASQ patterns (solid blue curves) through another pitchfork bifurcation. Subsequently, at \( r \approx 1.0114 \), ASQ patterns bifurcate to limit cycles (red curves) through a Hopf bifurcation. These limit cycles represent OASQ. Note that the limit cycles are quite close to the axes. The oscillatory patterns corresponding to these limit cycles form standing waves along the roll axis that have been discussed earlier by Thual [18]. The time-period and amplitude of these limit cycles increase as \( r \) is increased. At \( r \approx 1.0184 \), these limit cycles form homoclinic orbits.
of the unstable SQ fixed points (saddles) originating from \( r = 1 \).

Beyond \( r \approx 1.0184 \), there is a smooth transition from homoclinic orbits to regular limit cycles corresponding to SQOR, illustrated as green curves in the bifurcation diagram. The SQOR patterns and all subsequent patterns generated at higher \( r \) values are similar to that observed by Pal \textit{et al.} [3] for zero-\( P \) convection. In brief, the limit cycles corresponding to SQOR form another set of homoclinic orbits at \( r \approx 1.1034 \) after which each of the homoclinic orbits bifurcate into two separate limit cycles. These limit cycles (OASQ shown as red curves) diminish in size as \( r \) is increased until they transform to stable fixed points (ASQ shown as blue curves) through an inverse Hopf bifurcation. As \( r \) is increased further, at \( r \approx 1.183 \), the stable ASQ branches meet the unstable SQ branch (the dashed black curve) resulting in stabilization of the SQ pattern (solid black curves) as a consequence of an inverse pitchfork bifurcation. Thus, convective flow for \( P = 0.02 \) exhibits SQ, ASQ, OASQ, and SQOR patterns that are common to zero-\( P \) convection [3]. In addition, stationary 2D rolls and associated stationary and oscillating ASQ are observed near the onset of convection. We performed DNS for \( P = 0.02 \) and observed similar behaviour as the low-dimensional model. The chaotic attractors \( \text{Ch1, Ch2, and Ch3} \) of zero-\( P \) convection [3] are absent for \( P = 0.02 \). In the following discussion we will consider the bifurcation diagrams for lower Prandtl numbers (\( P = 0.005, 0.002, 0.0002 \)) for which chaotic attractors are observed.

The bifurcation diagram for \( P = 0.005 \), shown in fig. 3, is similar to the corresponding figure for \( P = 0.02 \) with a major difference that for \( P = 0.005 \), the chaotic attractors \( \text{Ch1, Ch2, and Ch3} \) of zero-\( P \) convection [3] appear near the onset in the band of \( r \approx 1.00685-1.0068 \). As we increase \( r \) from 1, we observe 2D straight rolls, ASQ, and OASQ just like \( P = 0.02 \), however, at lower \( r \) values (see table 1). The phase space projection on the \((W_{101}, W_{011})\)-plane of two of the limit cycles corresponding to OASQ are shown in fig. 4(a). Subsequently these limit cycles appear to approach their basin boundary (the horizontal axis of the figure), and the system becomes chaotic. This phenomenon is due to a "gluing bifurcation" [21], and it could be related to the "attractor merging crisis" [22]. Gluing bifurcation and attractor merging crisis refer to the phenomena wherein two or more distinct attractors
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Simultaneously hit their common basin boundaries to give rise to a single large attractor. The former is used when the pre-bifurcation attractors are regular, e.g., limit cycles, while the latter involves chaotic attractors. The resulting chaotic attractor, whose phase space projection is shown in fig. 4(b), is same as the Ch3 attractor of zero-P convection [3]. As r is increased further, the various Ch3 attractors merge together in yet another crisis to form a single large chaotic attractor Ch3. At a larger r value, the Ch2 attractor breaks into four separate chaotic attractors Ch1. Phase space projections of the chaotic attractors Ch2 and Ch1 for P = 0.005 are shown in figs. 4(c) and (d), respectively. The Ch1 chaotic attractors become regular for r ≥ 1.0068 giving rise to the SQOR limit cycles. The subsequent patterns and the bifurcation diagram are same as those for P = 0.02. Note that the Ch1 chaotic attractors are generated as a result of “homoclinic chaos” [3], where the stable and unstable manifolds of a saddle intersect each other rather than merge smoothly to form a homoclinic orbit.

The patterns and chaotic attractors observed in the low-dimensional model for P = 0.005 are also observed in DNS. Phase space projections of the chaotic attractors (Ch3 and Ch1) obtained in DNS are shown in figs. 5(a) (for r = 1.0023) and 5(b) (for r = 1.0068), respectively. The chaotic attractors obtained from the DNS data are not thick, however an enlarged view of the attractors in the insets show that the attractors are chaotic.

When we lower the value of the Prandtl number even further from P = 0.005, the stationary 2D rolls and the associated ASQ patterns occur even nearer to r = 1 as evident from the entries of table 1. It can be noted from fig. 6 that the windows of 2D rolls and ASQ for P = 0.0002 is very small. The 2D rolls and ASQ occur for 1 < r < 1 + 2.5 × 10^{-7} and 1 + 2.5 × 10^{-7} < r < 1 + 1.8 × 10^{-5}, respectively. Consequently it would be very difficult to observe these 2D rolls and the related ASQ patterns in experiments and in DNS; a small experimental or numerical noise will be sufficient to push the system out of this narrow ordered region to chaos Ch1-Ch3.

This feature is reminiscent of zero-P convection where Ch3-Ch1 occur at the onset itself. Thus our study indicates that zero-P convection is an appropriate limiting case of low-P convection in terms of bifurcations near the onset. For practical purposes, the bifurcation diagram for very low-P convection is topologically equivalent to zero-P convection.

It can be seen from table 1 that the branch points corresponding to ASQ (rP) and OASQ (rP) asymptotically approach r = 1 as P → 0. In table 1 we also list the imaginary part of the eigenvalue (ω) of the stability matrix at the Hopf bifurcation (r = rP). We observe that rP, rP − 1, and ω appear to vary approximately as P^2 with prefactors around 10, 30 and 3, respectively. The P^2 dependence is consistent with the theoretical predictions of Busse [4] that rP − 1 ≥ 0.31P^2 for free-slip boundary conditions. However the multiplying factor of Busse and our model differ by an order of magnitude. Also, Busse and Bolton [5] predict an absence of stable 2D rolls for P < P_b = 0.543. Our analysis however indicates stable 2D rolls for nonzero P, with the range of rP − 1 decreasing rapidly with the lowering of P. Earlier Krishnamurti [13] had found stable 2D rolls for mercury (P = 0.025) near the onset (R ≈ (2.3 ± 0.1) × 10^5) in her experiments with no-slip boundary conditions, thus indicating that P_b predicted by Busse and Bolton [5] is an overestimate. Regarding ω, Busse [4] predicts that ω ∼ P in contrast to our finding that ω ∼ P^2 near the onset of convection. In a related work on convection, Mercader et al. [23] have shown that the relationship between ω and P depends critically on the aspect ratio and they have reported both linear and quadratic dependence. These issues require a more careful and detailed theoretical investigation.

A comparison of the above results with the bifurcation diagram of zero-P convection of Pal et al. [3] reveal strong similarities. The bifurcation scenario from SQ to SQOR for
the larger $r$ values are topologically equivalent. We observe two major differences between the bifurcation diagrams of zero-$P$ convection and low-$P$ convection: a) no chaos for $P \geq 0.02$ near the onset; b) existence of stable 2D rolls and associated ASQ and OASQ patterns near the onset. For $P = 0.02$ we find only stationary and time-periodic solutions near the onset which is in agreement with the results of Meneguzzi et al. [17] for $P = 0.025$ and Thual [18] for $P = 0.02$ and 0.2. However, for $P \leq 0.005$ we observe chaotic attractors $Ch_1$, $Ch_2$ and $Ch_3$ between OASQ (closer to $r = 1$) and SQOR. The stationary 2D rolls and associated ASQ and OASQ patterns are observed in a narrow window between the conduction state and the chaotic states.

In conclusion, we perform a bifurcation analysis for low-$P$ convection near the onset using DNS and a related low-dimensional model. The results of the low-dimensional model are in good agreement with those of DNS since the model is derived using DNS by choosing most of the relevant modes. We observe that the bifurcation diagram for low-$P$ convection is very similar to the zero-$P$ convection reported by Pal et al. [3], except near the onset of convection where 2D stationary rolls, and stationary and oscillatory asymmetric squares are observed for nonzero Prandtl numbers. The range of Rayleigh numbers for which 2D rolls and associated ASQ and OASQ are observed shrinks rapidly ($\sim P^2$) as $P$ is decreased. This result is in qualitative agreement with the results of Busse [4]. For $P \leq 0.0002$, the range of reduced Rayleigh numbers for which the stationary 2D rolls could be observed is too narrow ($< 10^{-7}$) to be observed in experiments or in DNS. Our results predict the critical Prandtl number $P_c$ to be much lower than that predicted by Busse and Bolton [5].

Our bifurcation analysis provides very useful insights into the origin of patterns and chaos for low-$P$ convection. An extension of the present work to different aspect ratios, and to low-dimensional models with a larger set of modes is in progress.

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REFERENCES


