Chapter 15

Rotation Dynamics: Definitions

Euler Angles

Rigid body and Euler angles

A rigid body is one in which the relative distance between any pair of points remains constant.

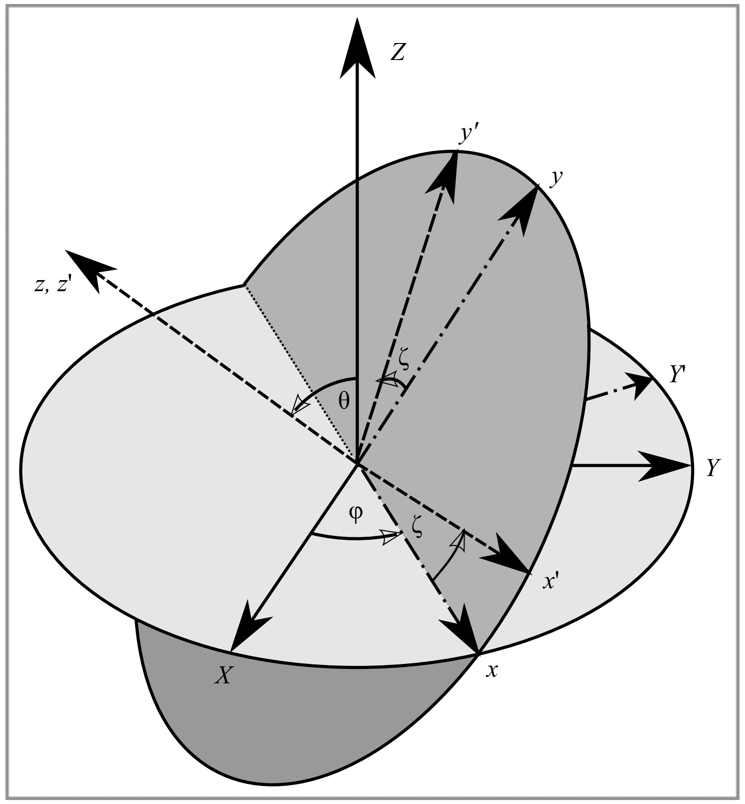
Applicable for moderate force.

Any motion of a rigid body can be split into two parts:

(a) translation of a given point on the rigid body: During the translation, all the points of the rigid body move by the same constant distance.

(b) rotation of the rigid body about the above point.

On many occasions, the CM of the rigid body is chosen as the reference point.



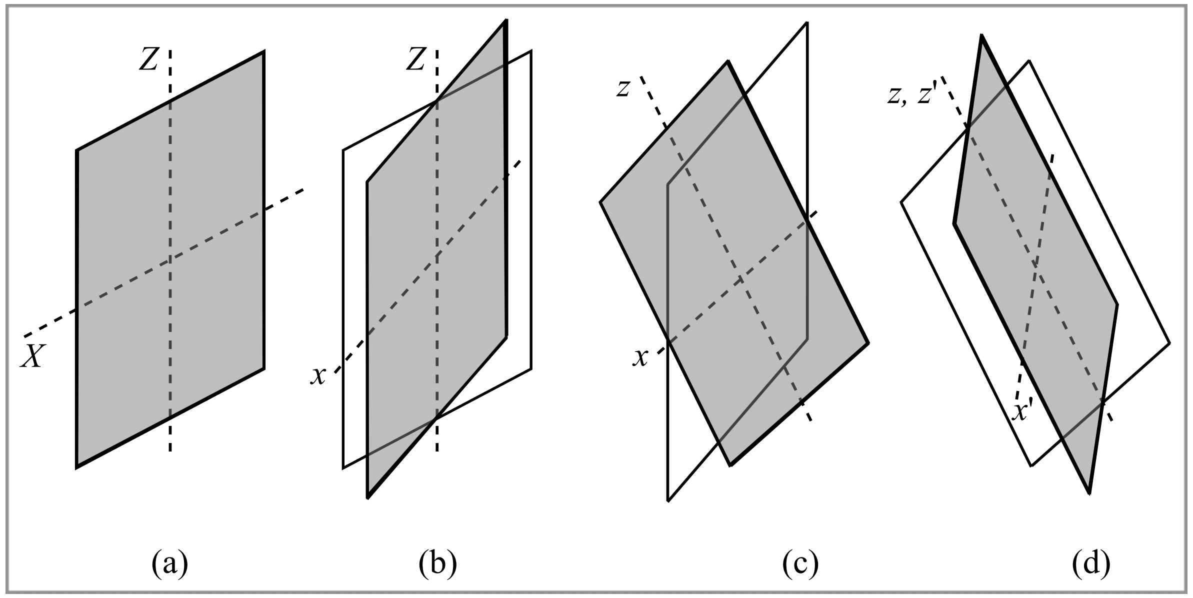
Rotation from the axes configuration (X, Y, Z) to another configuration (x', y', z'):

We use three Euler angles ϕ, θ, ζ:

1. A rotation by an angle ϕ about the Z axis, which shifts the X and Y axes to the x and Y' axes respectively (from (a) to (b)).

2. A rotation by an angle θ about the new x axis, which shifts the Y' and Z axes to the y and z axes respectively (from (b) to (c)).

3. A rotation by an angle ζ about the z axis, which shifts the x and y axes to the x' and y' axes respectively (from (c) to (d)).



Another way to look at these rotations is as follows: First locate the new axis of rotation of the rigid body, which requires two angles θ and ϕ relative to the original coordinate system. After the alignment of the rotation axis to (x, y, z) configuration, we rotate the rigid body by an angle ζ about the new z axis. These three angles are the aforementioned Euler angles. Note that the rotation about a fixed axis can be specified by one angle. However, when the rotation axis itself revolves, then the angles θ and ϕ provide the orientation of the rotation axis, and ζ provides the angle of rotation of the body about the new z axis.

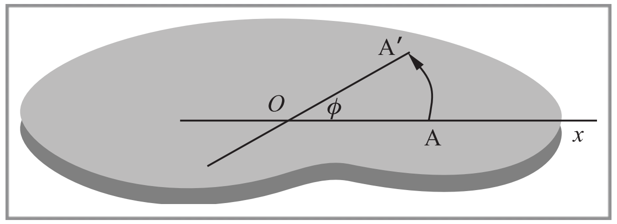
Section 2

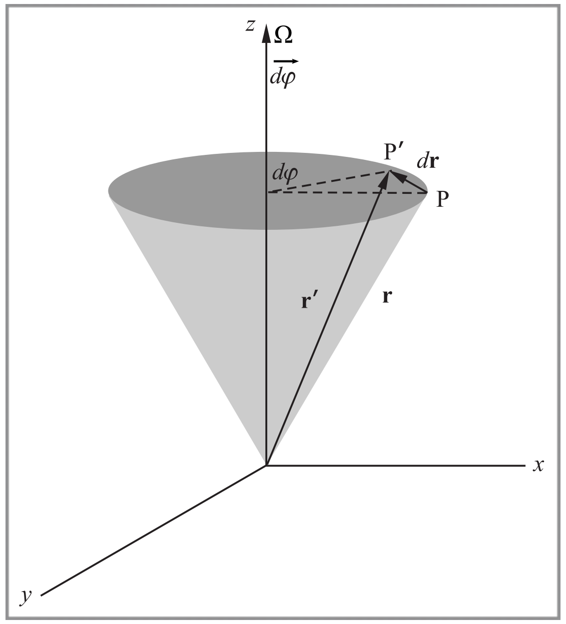
Angular Velocity

Angular Velocity

Rotation about a Single Axis

Rotation of aline wrt a reference axis (here x axis)





Displacement of a point P on a rigid body under rotation:

$d\mathbf{r} = (\boldsymbol{\Omega} \times \mathbf{r})dt$

The linear velocity of the point P is

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\Omega} \times \mathbf{r}. \qquad (1)$$

Theorem: *The angular velocity of a rigid body is the same for all points on the rigid body.*

Proof: Imagine that a disk is rotating about an axis passing through *O.* Let us assume that the angular velocity of the disk about *O* is $\boldsymbol{\Omega}_O$, while that about *O'* is $\boldsymbol{\Omega}_{O'}$. We need to prove that $\boldsymbol{\Omega}_O = \boldsymbol{\Omega}_{O'}$

The velocity of the point A is

$$\mathbf{V}_A = \boldsymbol{\Omega}_O \times \mathbf{r} = \boldsymbol{\Omega}_O \times (\mathbf{a} + \mathbf{r}') = \mathbf{V}_{O'} + \boldsymbol{\Omega}_O \times \mathbf{r}'. \qquad (2)$$

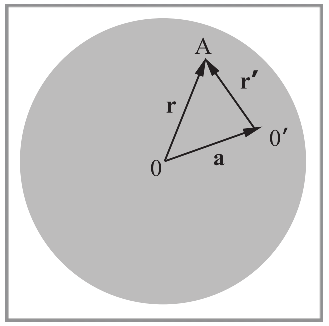


Fig. 1: Angular velocity of point *A* wrt *O* and *O'*.

However the velocity of the point A is the sum of the velocity of O' and that of the point A wrt *O'*, that is,

$$\mathbf{V}_A = \mathbf{V}_{O'} + \mathbf{\Omega}_{O'} \times \mathbf{r}'. \quad (3)$$

Comparing Eqs. (2) and (3) we can deduce that

$$\mathbf{\Omega}_O = \mathbf{\Omega}_{O'}$$

That is, the angular velocity measured at the points O and O' are the same. Physically, in a time dt, the lines OA and $O'A$ rotate by the same angle.

Rotation About a More than One Symmetric Axis

A general form of the angular velocity for rotation about multiple axes is

$$\mathbf{\Omega} = \Omega_x \hat{x} + \Omega_y \hat{y} + \Omega_z \hat{z}, \quad (4)$$

where $\hat{x}, \hat{y}$, and $\hat{z}$ are the directions of the rotation axes.

It is convenient to choose the axes of Euler rotation for the description of angular velocity. For a popular instrument named *gyroscope*, shown in Fig. 2. the angular velocity can be written as

$$\mathbf{\Omega} = \dot{\phi}\hat{Z} + \dot{\theta}\hat{x} + \dot{\zeta}\hat{z}. \quad (5)$$

The angular velocity $\dot{\phi}$ and $\dot{\zeta}$ are referred to as precession and spin angular velocities of the rigid body.

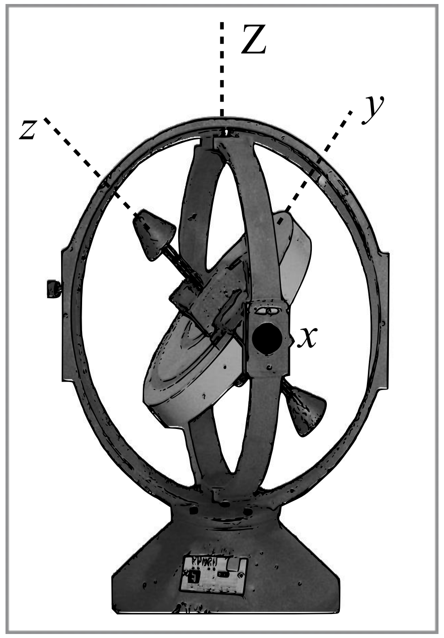


Fig 2: A Gyroscope

Figure 15.10 Gyroscope. The disk in the centre rotates about the three axes(x, y, z).

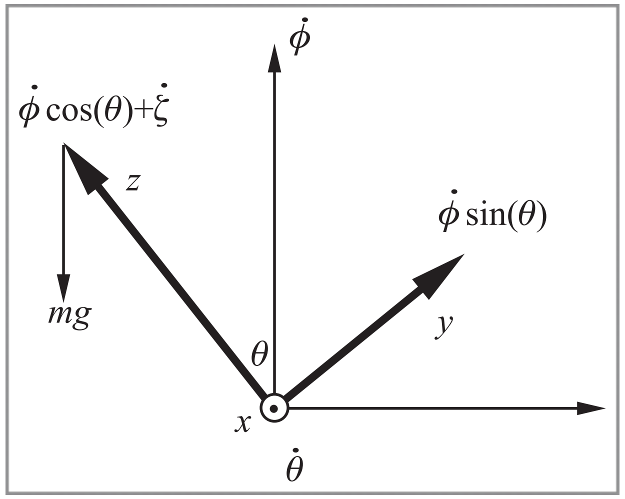


Figure 3: Angular velocity of the gyroscope in terms of the time derivatives of the Euler angles.

Note that in the above decomposition, the angular velocity is decomposed along non-orthogonal (non-perpendicular) directions $(\hat{Z}, \hat{x}, \hat{z})$. Alternatively, $\mathbf{\Omega}$ can be decomposed along the three orthogonal axes $(\hat{x}, \hat{y}, \hat{z})$ shown in the Fig. 3. Note that the $\hat{Z}$ axis is fixed, but $\hat{x}, \hat{y}, \hat{z}$ axes rotate about $\hat{Z}$ with an instantaneous angular velocity of $\dot{\phi}$. The components of the angular velocity along the (x, y, z) axes are

$$\Omega_x = \dot{\theta}$$

$$\Omega_y = \dot{\phi}\sin\theta$$

$$\Omega_z = (\dot{\zeta} + \dot{\phi}\cos\theta). \qquad (6)$$

Another important point to note is that the angular velocity is the same for both laboratory frame and the rotating frame because the angle made by a line on the body wrt a reference line is the same in both the frames. Contrast this with the linear velocity; the velocities in the two frames differ by the relative velocity between the two frames.

The following examples illustrate how to write angular velocity of a rigid body.

Examples Illustrating Angular Velocity

1. A wheel rolling without slipping: The bottom-most point has zero velocity. Therefore,

   $V_{CM} = \omega R$

2. A COIN (C_2) ROLLING OVER ANOTHER FIXED COIN (C_1) OF THE SAME RADIUS

Focus on transition from α to β.

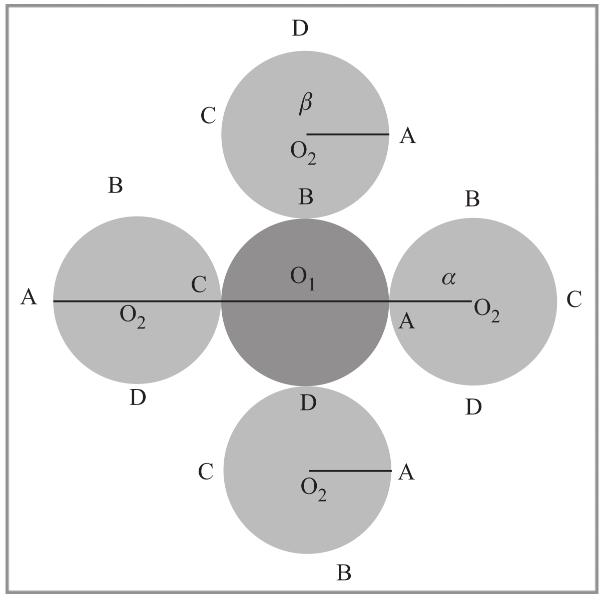


Figure 4: Coin C_2 rolling over the coin C_1 without slipping.

Orbital motion: The line joining the centres of C_1 and C_2 rotates by $\pi/2$. If the coin C_2 slides without rolling (the point A not losing contact), then the lines O_1-A-O_2 would move by an angle of $\pi/2$.

Spin: The line O_2A makes an additional rotation of $\pi/2$ due to the rolling of the C_2 coin.

The velocity of the contact point of the coin C_2 is a sum of the velocities due to orbital motion and due to spin, that is

$$V = \Omega_{\text{orbital}} R - \Omega_{\text{spin}} R,$$

where R is the radius of the coin. Since the net velocity of the contact point is zero,

$$\Omega_{\text{orbital}} = \Omega_{\text{spin}}$$

or $\theta_{\text{orbit}} = \theta_{\text{spin}}$

Hence the total angle traversed by the line O_2A is

$$\theta_{\text{net}} = \theta_{\text{orbit}} + \theta_{\text{spin}} = 2\theta_{\text{orbit}}.$$

When the coin C_2 returns to its original spot after $\theta_{\text{orbit}} = 2\pi$, the line O_2A would have covered $\theta_{\text{net}} = 4\pi$, which corresponds to two complete revolutions of the coin C_2.

3. EARTH-SUN SYSTEM

In one year (365.24 solar days), the centre of the Earth returns to its original position. Hence in one year, the Earth completes an orbital motion of 2π radian if the line *OP* of Fig. 5 is always facing the Sun. The Earth also spins by an angle of $2\pi \times 365.24$ radian.

Consider a point P on the surface of the Earth, which is closest to the Sun in configuration α (Fig. 5). A solar day is defined as a time interval after which the point P again cones closest to Sun (here configuration β). During this interval, the line OP has rotated by

$$\theta = 2\pi + \frac{2\pi}{365.24} = 2\pi \frac{366.24}{365.24} \text{ rad}. \qquad (15.5.11)$$

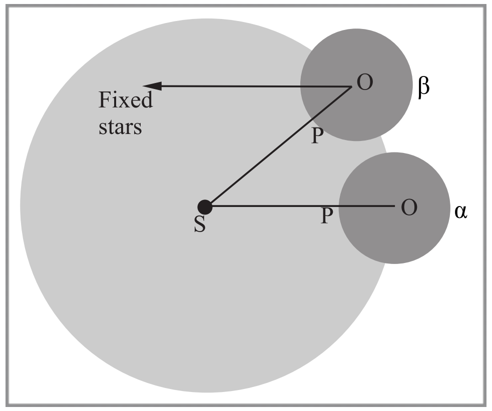


Figure 5: Motion of the Earth around the Sun.

Hence the angular velocity of the Earth is

$$\Omega = \frac{\theta}{24\text{ hours}} = \frac{2\pi \times 366.24}{365.25 \times 86400} = 7.29 \times 10^{-5}/\text{s} \ .$$

A *sidereal day* is the time interval in which the Earth makes one revolution about the fixed stars. In Fig. 15.13, it corresponds to the interval during which the line OP rotates by an angle of 2π. Using the ideas discussed above, we conclude that

$$T_{\text{sidereal}} = \frac{365.24}{366.24} \times 24\,\text{hours} \approx 23\,\text{hour}\ 56\,\text{min}\,.$$

4. A ROLLING CYCLE WHEEL

A cycle wheel of mass M and radius R is connected to a vertical rod through a horizontal shaft of length a, as shown in Fig. 15.14(a). The wheel rolls without slipping about the Z axis with an angular velocity of Ω.

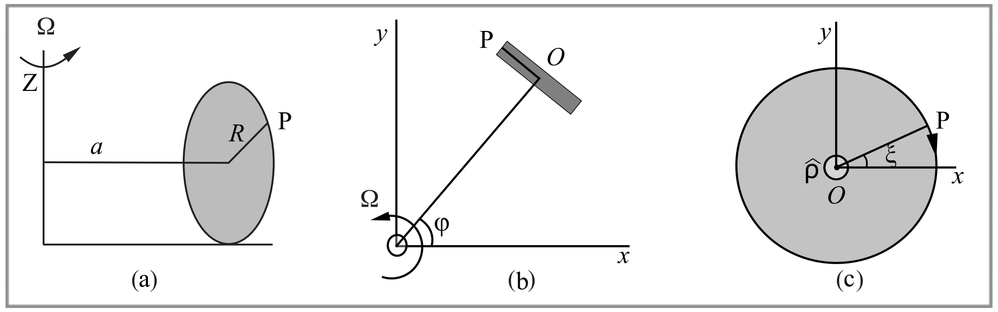


Figure 15.14 (a) Rolling motion of a cycle wheel connected to a horizontal shaft. (b) A view from the top. A line OP of the disk has rotated by an angle ϕ about X axis. (c) A side view of the disk (while facing straight to the disk). A line OP makes an angle ζ wrt a reference axis.

The cycle wheel spins about its own axis, as well as orbits around the vertical axis (Z). Hence its angular velocity is

$$\boldsymbol{\Omega} = \Omega\hat{Z} + \Omega_S\hat{\rho}, \qquad (15.5.14)$$

where Ω is the precession (about the Z axis) angular velocity, and Ω_S is spin angular velocities. We consider the counter clockwise direction as positive. Note that the linear velocity of the contact point of the wheel is $(\Omega a + \Omega_S R)\hat{\phi}$. Since the wheel rolls without slipping, that is, the linear velocity of the contact point is zero, we obtain

$$\Omega_S = -\,\Omega a/R. \qquad (15.5.15)$$

The negative sign of Ω_S implies that the wheel is spinning clockwise. Thus, the net angular velocity of the bicycle wheel is

$$\mathbf{\Omega} = \Omega\hat{Z} - \frac{a}{R}\Omega\hat{\rho}. \qquad (15.5.16)$$

Note that we can obtain the above equation from Eq. (15.5.5) if we substitute $\dot{\theta} = 0$ and $\hat{z} = -\hat{\rho}$.

Physically, a line OP on the wheel rotates about $\hat{Z}$ axis (angle ϕ) and about $\hat{\rho}$ axis (angle ζ). The spin angular velocity $\Omega_S = \dot{\zeta}$ and the precession angular velocity $\Omega = \dot{\phi}$ are illustrated in Figs. 15.14(b) and 15.14(c) respectively. Note that if the wheel slides without any spin, then $\Omega_S = 0$.

AN ORBITING CYLINDER MAKING AN ANGLE θ WITH THE ROTATING AXIS

A cylinder orbits around $AA' = Z$ axis with an angular velocity of $\mathbf{\Omega} = \Omega\hat{Z}$ (see Fig. 15.15(a)). The axis of the cylinder makes an angle θ with the Z axis. The cylinder does not spin about its axis. We can resolve $\mathbf{\Omega}$ along the cylindrical axis (z), and along its perpendicular direction (y) as

$$\mathbf{\Omega} = \Omega\hat{Z} = \Omega_z\hat{z} + \Omega_y\hat{y} = \Omega\cos\theta\hat{z} + \Omega\sin\theta\hat{y}, \qquad (15.5.18)$$

which can be interpreted as rotation of the cylinder about z and y axis with angular velocities Ω_z and Ω_y respectively. Note that $\Omega = \dot{\phi}$ of Eq. (15.5.5).

We focus on a line OP of the cylinder. In one rotation, the line OP covers an angle 2π, with the point P traversing via P_1, P_2, P_3, P_4 as shown in Fig. 15.15(b).

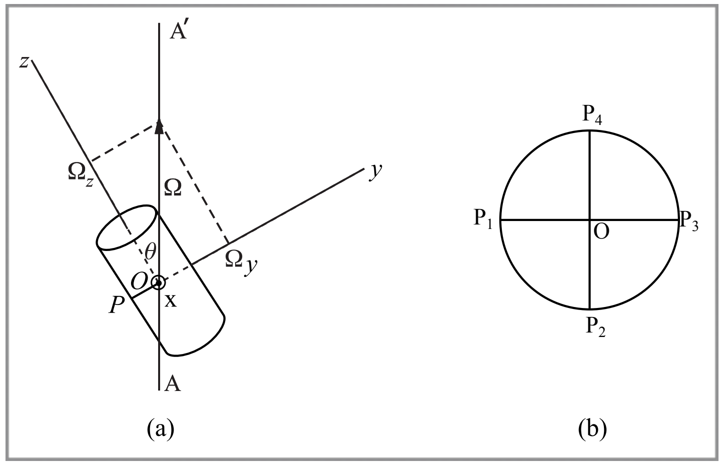


Figure 15.15 (a) A cylinder, which is inclined from the vertical axis by an angle θ, is orbits about a vertical axis with angular velocity $\mathbf{\Omega}$. (b) A line OP of the cylinder performs a circular path around O.

Section 3

Moment of Inertia, Kinetic Energy

Kinetic energy of a rigid body; Moment of inertia

The motion of a rigid body can be broken into two parts: a linear translation of all the point on the rigid body, and a rotation of the rigid body. The velocity of any point a of the rigid body is

$$\mathbf{v}_a = \mathbf{V}_P + \mathbf{\Omega} \times \mathbf{r}'_a, \qquad (1)$$

where $\mathbf{V}_P$ is the velocity of the reference point P, and $\mathbf{\Omega}$ is the angular velocity of the rigid body. The total kinetic energy of the rigid body is a sum of the kinetic energy of all the points, i.e.,

$$T = \sum_a \frac{1}{2} m_a v_a^2 = \frac{1}{2} M V_P^2 + \sum_a \frac{1}{2} m_a (\Omega \times \mathbf{r}'_a)^2 + \mathbf{V}_P \cdot (\sum_a m_a \Omega \times \mathbf{r}'_a) \qquad (2)$$

The third term of the RHS vanishes if

(a) The point P is the CM, i.e., $\sum_a m_a \mathbf{r}'_a = 0.$

(b) The reference point P is stationary, i.e., $\mathbf{V}_P = 0.$

For these case, the first term is the kinetic energy of the CM, while the second term is rotational kinetic energy wrt the CM.

In the following discussion, we will compute the second term of RHS, which is the rotational kinetic energy of the rigid body wrt P. For an arbitrary rotation with angular velocity $\mathbf{\Omega} = \Omega_x \hat{x} + \Omega_y \hat{y} + \Omega_z \hat{z}$, the rotational kinetic energy is

$$T_{\text{rot}} = \frac{1}{2} \sum_a m_a (\mathbf{\Omega} \times \mathbf{r}_a)^2$$

$$= \frac{1}{2} \sum_{a,i,\alpha,\beta,\gamma,\delta} m_a \epsilon_{i\alpha\beta} \Omega_\alpha r_{a,\beta} \epsilon_{i\gamma\eta} \Omega_\gamma r_{a,\eta}$$

$$= \frac{1}{2} \sum_{\alpha,\gamma} \left[\sum_a m_a (r_a^2 \delta_{\alpha\gamma} - r_{a,\alpha} r_{a,\gamma}) \right] \Omega_\alpha \Omega_\gamma$$

$$= \frac{1}{2} \sum_i \sum_j I_{ij} \Omega_i \Omega_j \qquad (3)$$

where

$$I_{il} = \sum m_a (r_a^2 \delta_{il} - r_{a,i} r_{a,l})$$

is the *moment of inertia* of the rigid body measured from a reference point P. A different choice of P will yield a new set of I_{ij}'s.

Moment of inertia is a second rank tensor with $I_{ij} = I_{ji}$ (symmetric). In a matrix form

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \quad (4)$$

Theorem: The above symmetric matrix can always be transformed to the following diagonal form using a coordinate transformation from xyz to a new system $x'y'z'$:

$$I' = \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix} \quad (5)$$

The new axes are called the *symmetry axes* or the *principal axes* of the rigid body. Because of the above simplification, the symmetry axes of a rigid body are very important and convenient for calculations.

Example 1 Compute the moment of inertia of a thin square plate of size a and mass M about its centre of mass.

Solution We choose the xy axis shown in Fig. 15.19 for our computation.

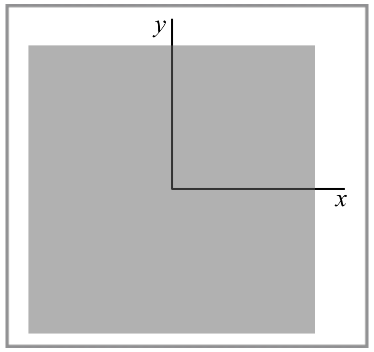


Figure 1:

Using the formulas of MI

$$I_{xx} = \sigma \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy (y^2)$$

$$= \sigma a \frac{a^3}{12} = M \frac{a^2}{12},$$

where σ is the two-dimensional mass density of the plate, and $M = \sigma a^2$. Similarly,

$$I_{yy} = \sigma \int_{-a/2}^{a/2} dx (x^2) \int_{-a/2}^{a/2} dy = I_{xx};$$

$$I_{zz} = \sigma \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy (x^2 + y^2) = I_{xx} + I_{yy} = M \frac{a^2}{6};$$

Due to symmetry we can conclude that $I_{xy} = I_{yx} = 0$, hence, x and y are the principal axes of the plate. Also, since $z = 0$ for all the points on the thin plate.

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0;$$

It turns out that any rotated axes ($x'y'$) on the xy plane yields the same moment of inertia. You can obtain this result using somewhat complex integration. This result shows that any orthogonal $x'y'z$ axes are principal axes for the square plate. In matrix form,

$$I = \begin{pmatrix} \frac{Ma^2}{12} & 0 & 0 \\ 0 & \frac{Ma^2}{12} & 0 \\ 0 & 0 & \frac{Ma^2}{6} \end{pmatrix}$$

In the above example, the principal axes pass through the CM. However it is not necessary for the principal axes to pass through the CM of the rigid body, as we illustrate in the following example.

Example 2 Compute the moment of inertia of a thin square plate of size a and mass M about one of its corners. Find the principal axes of the plate passing through the aforementioned corner.

Solution We compute the moment of inertia of the square about the corner point P shown in Fig. 15.20.

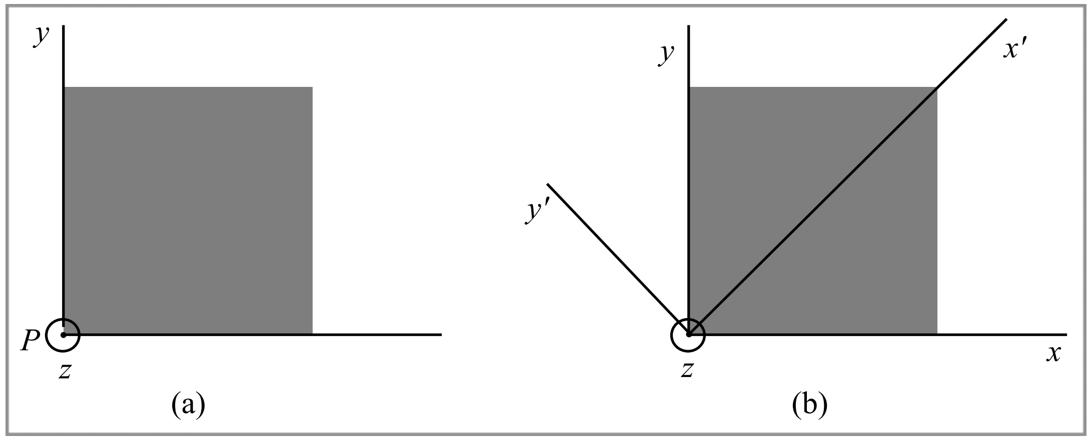


Figure 2:

Using the formulas for the moment of inertia, we obtain

$$I_{xx} = \sigma \int_0^a dx \int_0^a dy(y^2)$$

$$= \sigma \frac{a^3}{3} a = M \frac{a^2}{3};$$

$$I_{yy} = \sigma \int_0^a dx(x^2) \int_0^a dy = M \frac{a^2}{3} = I_{xx};$$

$$I_{zz} = \sigma \int_0^a dx \int_0^a dy(x^2 + y^2) = I_{xx} + I_{yy} = 2M \frac{a^2}{3};$$

$$I_{xy} = I_{yx} = -\sigma \int_0^a dx \int_0^a dy(xy)$$

$$= -\sigma \frac{a^2}{2}\frac{a^2}{2} = -M\frac{a^2}{4};$$

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0;$$

where σ is the constant mass density of the thin plate. The above values yield the following moment of inertia matrix:

$$I = \begin{pmatrix} \frac{Ma^2}{3} & -\frac{Ma^2}{4} & 0 \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} & 0 \\ 0 & 0 & \frac{2Ma^2}{3} \end{pmatrix} \quad (1)$$

Since the matrix is not in a diagonal form, the axes xyz are not principal axes.

We diagonalize the above matrix, which is

$$I = \begin{pmatrix} \frac{Ma^2}{12} & 0 & 0 \\ 0 & \frac{7Ma^2}{12} & 0 \\ 0 & 0 & \frac{2Ma^2}{3} \end{pmatrix}$$

with the eigenvectors being (1,1,0), (1,-1,0), (0,0,1). The first two vectors are along the *x'* and *y'* axes respectively shown in Fig. 2(b).

This is an example where the principal axes do not pass through the CM of the rigid body.

Example 3: A plank of length l and mass m is standing vertically on a frictionless surface. The plank starts to fall at $t = 0$. Assuming conservation of energy (to be proved in the next chapter), compute the angular velocity of the plank as a function of time.

Solution We solve the above problem by applying conservation of energy. At $t = 0$, the total energy of the plank is $mgl/2$. Since no horizontal force acts on the plank, the CM of the plank will fall down vertically. Consider a configuration when the CM of the plank is at y.

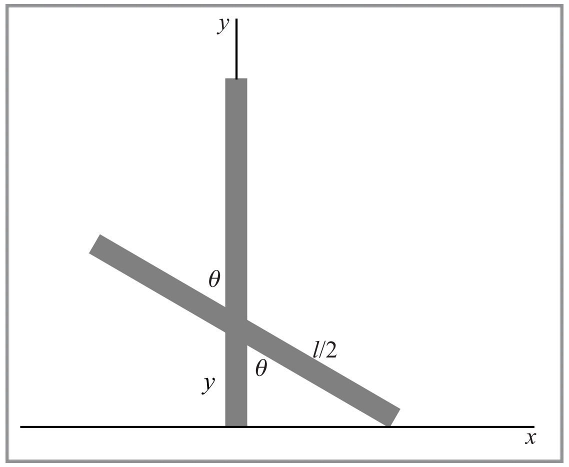


Figure 3: A plank falling under gravity.

The total kinetic energy of the plank is a sum of the kinetic energy of CM and the rotational kinetic energy, that is

$$\mathrm{KE} = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2$$

The potential energy PE = mgy

The Lagrangian of the system is

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2 - mgy.$$

The conservation of energy yields

$$mg\frac{l}{2} = mgy + \frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2 + \frac{1}{2}m\dot{y}^2 .$$

Using Eq. (1) we obtain

$$\frac{1}{24}l^2\dot{\theta}^2(1 + 3\sin^2\theta) = g(l/2 - y)$$

or

$$\dot{\theta}^2 = \frac{24g}{l}\frac{\sin^2(\theta/2)}{(1 + 3\sin^2\theta)}. \qquad (2)$$

For small θ, the above equation yields

$$\dot{\theta} = \sqrt{\frac{6g}{l}}\theta,$$

which is analogous to the motion of an inverted pendulum with small θ. The angle θ grows exponentially. We can compute θ and the vertical velocity of the CM of the plank.

Parallel Axis Theorem

The moment of inertia about an axis (z') *that is a distance* a *away and parallel to the symmetry axis (say* z*-axis) is*

$$I'_{zz} = I_{zz,\mathrm{CM}} + Ma^2, \qquad (15.6.8)$$

where $I_{zz,\mathrm{CM}}$ is the moment of inertia about the axis passing through the CM, and I'_{zz} is the moment of inertia about the new axis.

Section 4

Angular momentum

Angular momentum of a rigid body

The angular momentum of a system of particles about a reference point is

$$\mathbf{L} = \sum_a \mathbf{r}_a \times \mathbf{p}_a,$$

where $\mathbf{r}_a$ denotes the position vector of the ath particle from the reference point, and $\mathbf{p}_a$ its linear momentum. In terms of the CM coordinates:

$$\mathbf{L} = \mathbf{R}_{\text{CM}} \times \mathbf{P}_{\text{CM}} + \sum \mathbf{r}'_a \times \mathbf{p}'_a$$

Equation of motion:

$$\frac{d\mathbf{L}_P}{dt} = \mathbf{N}_{\text{ext}} - M(\mathbf{R}_{\text{CM}} - \mathbf{R}_P) \times \ddot{\mathbf{R}}_P$$

Angular momentum of a rigid body

The angular momentum of a rigid body about a reference point P is

$$\mathbf{L} = \sum \mathbf{r}_a \times \mathbf{p}_a$$

In component form:

$$L_i = \left[\sum \mathbf{r}_a \times m_a(\boldsymbol{\Omega} \times \mathbf{r}_a)\right]_i$$

$$= \sum_a \sum_{jklm} \epsilon_{ijk} r_{a,j} \epsilon_{klm} m_a \boldsymbol{\Omega}_l r_{a,m}$$

$$= \left[\sum_a m_a(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r_{a,j} r_{a,m}\right] \boldsymbol{\Omega}_l$$

$$= \left[\sum_a m_a(r_a^2 \delta_{il} - r_{a,i} r_{a,l})\right] \boldsymbol{\Omega}_l$$

$$= I_{il} \boldsymbol{\Omega}_l$$

The above formula is valid for any reference point.

Note that if $\boldsymbol{\Omega} = \Omega_z \hat{z}$, but the rotation axis is not one of the principal axes, then

$$\mathbf{L} = I_{xz}\Omega_z\hat{x} + I_{yz}\Omega_z\hat{y} + I_{zz}\Omega_z\hat{z}.$$

When the axes of rotation are the principle axes:

$$\mathbf{L} = I_{xx}\Omega_x\hat{x} + I_{yy}\Omega_y\hat{y} + I_{zz}\Omega_z\hat{z},$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors along the principal axes.

NOTE: The angular momentum $\mathbf{L}$ is in general not parallel to $\boldsymbol{\Omega}$. They are parallel only when

(a) $I_{xx} = I_{yy} = I_{zz} = I$, which is valid for a sphere or a cube when CM is chosen as a reference point. For this case $\mathbf{L} = I\boldsymbol{\Omega}$.

(b) $\boldsymbol{\Omega}$ is along one of the principal axis. For example, if $\Omega_x = \Omega_y = 0$ and $\Omega_z \neq 0$, then $\mathbf{L} = I_{zz}\boldsymbol{\Omega}$.

Contrast rotation and translation:

(a) The proportionality constant between **L** and **Ω** is the moment of inertia (a tensor), while the proportionality constant between **P** and **V** is the mass (a scalar). **P** and the **V** are always parallel, but **L** and **Ω** are not necessarily parallel.

(b) **L** depends on the reference point, but **P** does not.

(c) The linear velocities in the laboratory and rotating frame differ by the relative velocity between the two frames. The angular velocity in the two frames however is the same.

Examples of Angular Momentum

(1) A ROLLING WHEEL

The angular momentum of the wheel about its CM is

$$L_{\mathrm{CM}} = I\Omega = \frac{1}{2}MR^2\Omega\,.$$

$$\mathbf{L}_A = \mathbf{R}_{\mathrm{CM}} \times \mathbf{P}_{\mathrm{CM}} + \mathbf{L}_{\mathrm{CM}} \quad \text{or}$$

$$\mathbf{L}_A = \left(MR^2\Omega + \frac{1}{2}MR^2\Omega\right)\hat{z} = \frac{3}{2}MR^2\Omega\hat{z},$$

The angular momentum of the wheel about the bottom-most point of the wheel is

$$\mathbf{L}_B = I_B\Omega\hat{z} = (MR^2 + \frac{1}{2}MR^2)\Omega\hat{z} = \frac{3}{2}MR^2\Omega\hat{z}\,.$$

Clear $\mathbf{L}_A = \mathbf{L}_B$.

(2) A COIN ROLLING OVER ANOTHER COIN OF THE SAME RADIUS

The angular momentum of C2 is

$$\mathbf{L} = \mathbf{R}_{\mathrm{CM}} \times \mathbf{P}_{\mathrm{CM}} + \mathbf{L}_{\mathrm{CM}}$$

$$= (2R \times M \times 2R_E\Omega_{\mathrm{orbit}})\hat{z} + \frac{1}{2}MR^2(\Omega_{\mathrm{spin}} + \Omega_{\mathrm{orbit}})\hat{z},$$

$$= 5MR^2\Omega_{\mathrm{spin}}\hat{z}\,.$$

(3) THE EARTH-SUN SYSTEM

The angular momentum of the Earth wrt the centre of the Sun is

$$\mathbf{L} = mR_{ES}^2\Omega\hat{z} + \frac{2}{5}mR_E^2\Omega_S\hat{s} + \frac{2}{5}mR_E^2\Omega\hat{z}$$

$$\Omega = \frac{2\pi}{365.25\times 86400}\text{rad/s}$$

$$\Omega_S = \frac{2\pi}{86400}\text{rad/s}.$$

(4) A ROLLING CYCLE WHEEL

The angular momentum of the wheel about the hinge is

$$\mathbf{L} = \mathbf{L}_{\mathrm{CM}} + \mathbf{L}_{\mathrm{aboutCM}}$$

$$= MaV_{\mathrm{CM}}\hat{z} + \left(\frac{1}{2}MR^2\Omega_S\hat{s} + \frac{1}{4}MR^2\,\Omega\hat{z}\right)$$

$$= MaV_{\mathrm{CM}}\hat{z} + \left(-\frac{1}{2}MR^2\,\Omega\frac{a}{R}\hat{\rho} + \frac{1}{4}MR^2\,\Omega\hat{z}\right)$$

$$= M\left(a^2 + \frac{1}{4}R^2\right)\Omega\hat{z} - \frac{1}{2}MRa\hat{\rho}$$

5 A CYLINDER MAKING AN ANGLE θ WITH THE ROTATING AXIS

The angular momentum of the cylinder along the principal axes is

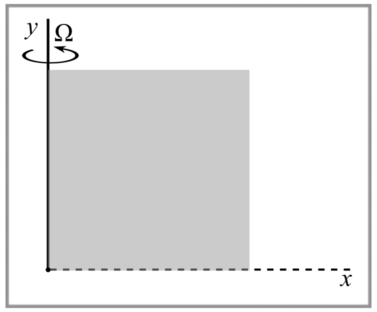
$$\mathbf{\Omega} = \Omega\cos\theta\hat{z} + \Omega\sin\theta\hat{y}.$$

Therefore the angular momentum of the cylinder is

$$\mathbf{L} = I_{zz}\Omega\cos\theta\hat{z} + I_{yy}\Omega\sin\theta\hat{y};$$

here $I_{zz} = MR^2/2$ and $I_{yy} = MR^2/4 + MH^2/12$. Clearly $\mathbf{L}$ and $\mathbf{\Omega}$ are *not* in the same direction.

Example 1: A square plate of size a rotates about the y axis with an angular velocity $\mathbf{\Omega} = \Omega\hat{y}$, as shown in Fig. 15.23. Compute the angular velocity of the plate about its corner.



Solution:

$$\mathbf{L} = I_{xy}\Omega\hat{y} + I_{yy}\Omega\hat{y} = -\frac{Ma^2}{4}\Omega\hat{x} + \frac{Ma^2}{3}\Omega\hat{y}. \qquad (1)$$

Note that the angular momentum and the angular velocity are not parallel.

When we resolve along the principal axes:

$$\mathbf{\Omega} = \frac{\Omega}{\sqrt{2}}\hat{x}' + \frac{\Omega}{\sqrt{2}}\hat{y}'.$$

and the angular momentum is

$$\mathbf{L} = \frac{Ma^2}{12}\frac{\Omega}{\sqrt{2}}\hat{x}' + \frac{7Ma^2}{12}\frac{\Omega}{\sqrt{2}}\hat{y}'. \tag{2}$$

It can be easily shown that the angular momentum of Eq. (1) and (2) are identical.

Section 5

Torque-free Rotation

Torque-free Precession of Symmetric Tops

Symmetric top: $I_{xx} = I_{yy} \neq I_{zz}$.

Thin tops: $I_{xx} = I_{yy} > I_{zz}$ (thin cylinder)

Flat tops: $I_{xx} < I_{zz}$ (flat disks, frisbee, Earth)

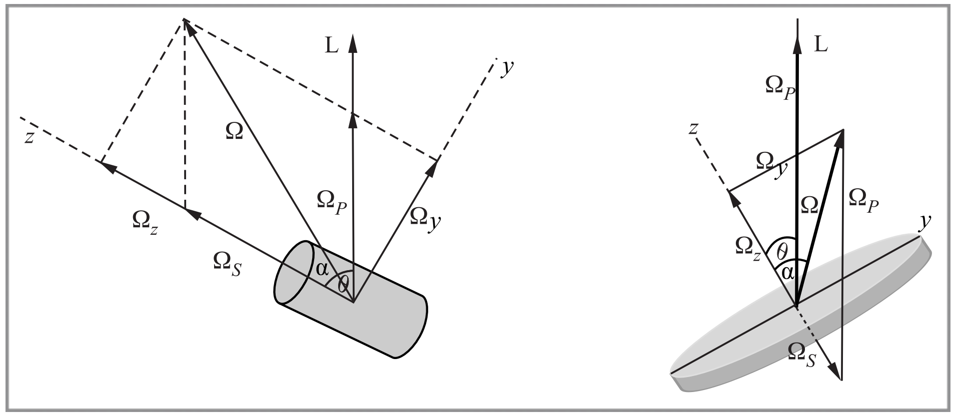


Figure 1 (a) Torque-free precession of a thin top. (b) Torque-free precession of a flat top.

$$\boldsymbol{\Omega} = \Omega_z \hat{z} + \Omega_y \hat{y}\,. \tag{1}$$

It is also useful to resolve the angular velocity along the z and Z (along $\mathbf{L}$) axes as

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_S + \boldsymbol{\Omega}_P, \tag{2}$$

where $\boldsymbol{\Omega}_S$, $\boldsymbol{\Omega}_P$ are called the spin and precession angular velocities respectively, and they are not orthogonal. Note that spin axis is along the z direction.

The components of the angular velocity along the two coordinate systems are related:

$$\Omega_z = \Omega_S + \Omega_P \cos\theta, \tag{3}$$

$$\Omega_y = \Omega_P \sin\theta, \tag{4}$$

where θ is the angle between $\mathbf{L}$ and the z axis of the rigid body. The net angular momentum can be resolved along the principal axes as

$$L_y = L\sin\theta = I_{yy}\Omega_y \tag{5}$$

$$L_z = L\cos\theta = I_{zz}\Omega_z \tag{6}$$

The above equations yield

$$\Omega_y = \frac{L_y}{I_{yy}} = \frac{L\sin\theta}{I_{yy}}, \tag{7}$$

$$\Omega_z = \frac{L_z}{I_{zz}} = \frac{L\cos\theta}{I_{zz}} \tag{8}$$

Also from Eq. (15.8.4) and (15.8.5)

$$\Omega_P = \frac{\Omega_y}{\sin\theta} = \frac{L}{I_{yy}}, \tag{9}$$

$$\Omega_S = \Omega_z - \Omega_P\cos\theta = \Omega_z\left[1 - \frac{I_{zz}}{I_{yy}}\right], \tag{10}$$

$$\Omega_S = \Omega_P\cos\theta\left[\frac{I_{yy}}{I_{zz}} - 1\right], \tag{11}$$

$$\Omega_P = \frac{\Omega_z}{\cos\theta}\frac{I_{zz}}{I_{yy}}. \tag{12}$$

According to the Eq. (15.8.11), Ω_P and Ω_S have same signs for the thin tops (type (a) with $I_{zz} < I_{yy}$), but different signs for the flat tops (type (b) with $I_{zz} > I_{yy}$). Note that $\Omega_S \neq \Omega_z$. The above computation also reveals that the system can be uniquely specified using (Ω_y, Ω_z) or (Ω_S, Ω_P).

Another important quantity in this problem is the angle α between the z axis and the angular velocity vector $\mathbf{\Omega}$ (see Fig. 15.24). Let us compute a relationship between the angles θ and α. Since

$$\tan\alpha = \frac{\Omega_y}{\Omega_z}. \tag{13}$$

Using Eq. (7) and (8) we can deduce that

$$\frac{\Omega_y}{\Omega_z} = \tan\theta\frac{I_{zz}}{I_{yy}} \tag{14}$$

which implies that

$$\tan\alpha = \tan\theta\frac{I_{zz}}{I_{yy}} \tag{15}$$

The torque-free motion has an interesting physical interpretation. The top is spinning, as well as precessing so as to maintain zero torque. Note that the net angular velocity $\mathbf{\Omega}$ precesses about $\mathbf{L}$, not about the spin axis.

It is instructive to analyse the motion of a cylinder (a thin top) and a thin disk (a flat top) separately. We take a cylinder with $I_{xx} = I_{yy} = 2I_{zz}$ as an example for which Eq. (15.8.11) yields $\Omega_S \approx \Omega_P$ for small θ. That is, precession and spin angular velocities have approximately same magnitudes. Hence the total angular velocity of the cylinder is

$$\mathbf{\Omega} \approx (\Omega_P + \Omega_S)\hat{z} \approx 2\Omega_P \hat{z}. \qquad (16)$$

In Fig. 1 we illustrate the motion of the line OP of a torque-free cylinder. By the time the z axis completes half rotation (from (a) to (c)), the line OP makes a full circle. Thus, a line on the cylinder rotates twice as fast as Ω_P.

For a thin disk ($I_{yy} = I_{zz}/2$), Eq. (11) yields $\Omega_S \approx -\Omega_P/2$ for small θ. Hence,

$$\mathbf{\Omega} \approx (\Omega_P + \Omega_S)\hat{z} \approx \frac{1}{2}\Omega_P \hat{z}. \qquad (17)$$

Therefore, the net angular velocity of a line on the disk is around half of the precession velocity. In Fig. 15.26 we illustrate that the line OP completes a half revolution by the time the z axis completes one revolution.

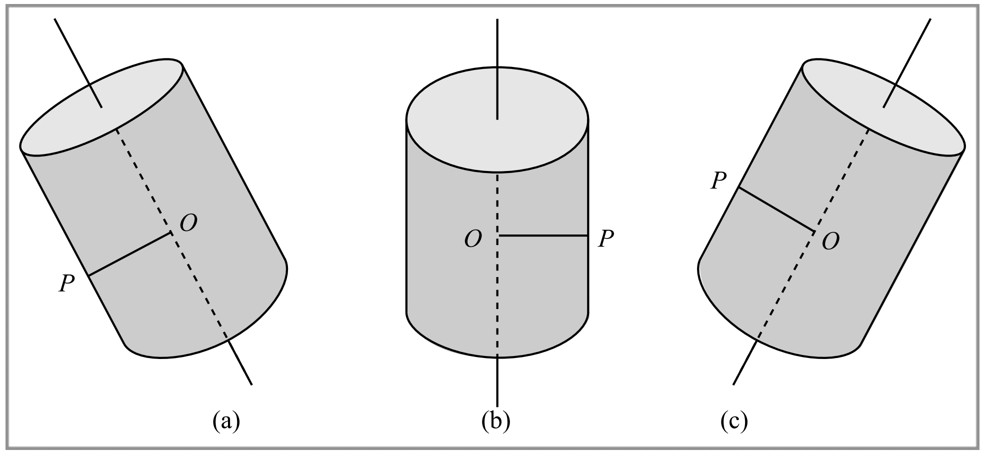


Figure 2 Motion of the line OP of a cylinder during its torque-free precession. It makes a full rotation when the cylinder has made only half precession.

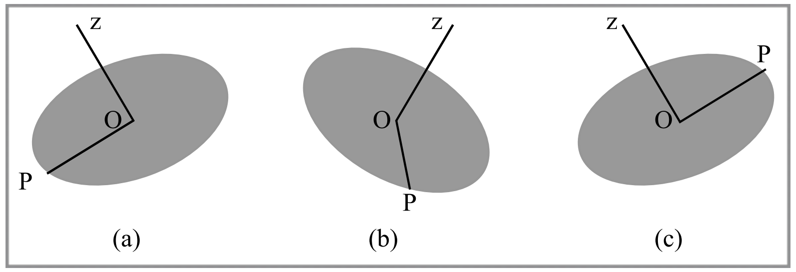


Figure 3 Motion of the line OP of a thin disk during its torque-free precession. It makes half a rotation when the disk has made a full round of precession.

Torque-free Precession of the Earth

In 1891, Chandler observed that the spin axis of the Earth precesses around the polar region with a period of approximately 435 days. As shown in Fig. 4(a), the variation of the location of the spin (z) axis is of the order of 10 meters, which is negligible compared to the radius of the Earth, as well as it is somewhat irregular. The aforementioned precession is largely attributed to the torque-free precession of the Earth. We will estimate this effect in the following discussion.

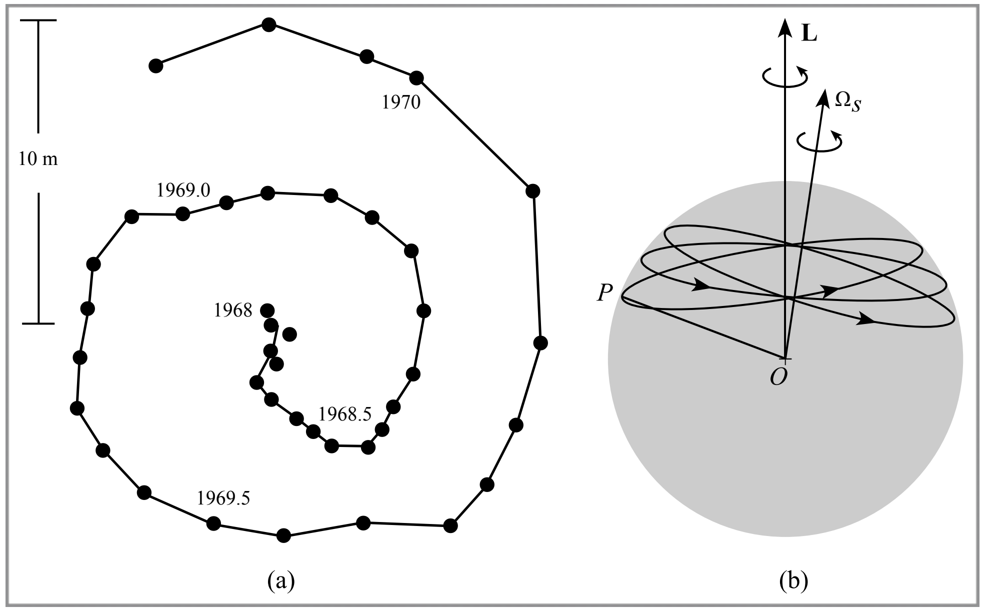


Fig: 4: Chandler wobble

The range of motion of the spin axis of the Earth is approximately 10 meters. Using this data and Eq. (15.8.13), we estimate the angles α of Fig. 15.24 using

$$\tan\alpha \approx \frac{10 \text{ meters}}{R_E} = \frac{10}{6\times 10^6} = 1.6\times 10^{-6},$$

$$\alpha \approx 1.6\times 10^{-6}\times 3600\times \frac{180}{\pi} \approx 0.3 \text{ arc second.}$$

Since θ is of the same order as α, the spin axis and the precession axis of the Earth make an angle of the order of a few tenths of a second.

Now let us compute a relationship between Ω_S and Ω_z. For the Earth

$I_{xx} = I_{yy} = 0.329591MR^2$, and $I_{zz} = 0.330675MR^2$, substitution of which in Eq. (15.8.10) yields

$$\frac{\Omega_S}{\Omega_z} = \frac{I_{yy} - I_{zz}}{I_{yy}} \approx -\frac{1}{304}.$$

Since the angle between the $\mathbf{\Omega}$ and $\mathbf{L}$ is quite small, we conclude that

$$\mathbf{\Omega} \approx \mathbf{\Omega}_P + \mathbf{\Omega}_S \approx (\Omega_z - \Omega_S)\hat{z} \approx \left(1 - \frac{1}{304}\right)\Omega_z\hat{z},$$

where $|\mathbf{\Omega}| \approx \Omega_z \approx 2\pi/(1 \text{ solar day})$. Hence, the time period of the precession of the axis is 304 days.

A physical interoperation of the above calculations is as follows: A line joining the centre of the Earth to its surface, for example the line OP of Fig. 4(b), would cover $(1 - 1/304)2\pi$ angle in one revolution of the spin axis. That is, the line OP lags behind its starting orientation by an angle of$(1/304)2\pi$ in every revolution of the spin axis. Therefore, the line OP would return to its original position only after 304 solar days, during which time OP would have covered an angle of $303 \times 2\pi$. These calculations yield time period of Earth's precession to be 304 days. Also note that the above computations predict a circular orbit for the spin axis, which is not what is observed (see Fig. 4(a)).

The actual measurement by Chandler (1891) and others indicate that the aforementioned precession time period is approximately 435 days. Also, the observed precession of the spin axis is somewhat irregular. The difference between the observed time period and the computed one, as well as the irregular motion of the spin axis, is attributed to the fact that the Earth is not a rigid body; the fluid motion inside the Earth yields corrections to the time period. For details refer to Goldstein et al. (2002).

How is the Chandler wobble measured? If the spin $\mathbf{\Omega}_S$ and the angular momentum $\mathbf{L}$ are aligned, then the fixed stars would go around the pole star in a circular orbit every 24 hours, that is, the line OP of Fig. 4(b) would follow a circular orbit. A precession of the spin axis around $\mathbf{L}$ however causes a wobble in the trajectories of the fixed stars, as shown in Fig. 15.27(b). We measure the precession of the spin axis using the observed the trajectories of fixed stars.

It is important to note that the Earth's motion has another precession, which occurs due to the tidal effects induced by the moon and the Sun. Because of the oblate nature of the Earth, and the pull by the Sun and the moon cause a torque on the Earth that leads to a precession of Earth's spin axis along a cone whose half-angle is 23.5 degrees.

The time period of this precession is around 26000 years. At present Earth's axis points towards the pole star, however it will point to a different direction at a later time.

Example 1 Consider a uniform thin rod of mass M and length l lying on a horizontal plane. The rod can can rotate freely about a hinge at the mid-point O, as shown in Fig. 15.28. A point particle of mass M moving with a velocity v collides inelastically with the rod at its bottom. Compute the angular velocity of the rod after the collision.

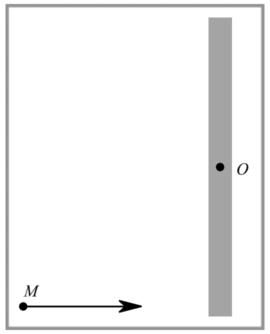


Figure 5 Example 1: A point mass M collides inelastically with a rod of same mass.

Solution Since the rod is hinged, the linear momentum of the system (particle + rod) is not conserved due to the external forces exerted on the rod by the hinge. The total kinetic energy is also not conserved in this example. Hence, we need to use the conservation of angular momentum for solving this problem.

We use the hinge O as the reference point for computing the angular momentum of the system. We assume that the radius of the hinge is very small so that we can neglect the torque on the rod due to the contact forces at the hinge. Therefore, we can apply conservation of angular momentum to the system.

Before the collision, the angular momentum of the point mass about the hinge is $\mathbf{L} = (Mvl/2)\hat{z}$. The mass sticks to the rod after the collision, leading to a rotation of the rod+mass system about O. Let us denote the post-impact angular velocity of the combined system with Ω. Hence, after the collision, the angular momentum of the rod+mass system about the hinge is $I_O\mathbf{\Omega}$, where

$$I_0 = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}. \qquad (1)$$

is the moment of inertia of the rod+mass about the hinge. An application of the conservation of angular momentum about the hinge yields

$$I_0\mathbf{\Omega} = \frac{Ml}{2}v\hat{z} \qquad (2)$$

or

$$\Rightarrow \frac{\cancel{M}L^{\cancel{2}}}{3}\omega = \frac{\cancel{M}\cancel{L}}{2}v$$

$$\mathbf{\Omega} = \frac{3}{2}\frac{v}{l}\hat{z}. \qquad (3)$$

Exercises

1. Two discs of moment of inertia I_1 and I_2 are rotating about a common vertical axis with angular frequencies Ω_1 and Ω_2 respectively. The top disc falls onto the bottom disc, and the two discs move with a common angular velocity after a while Compute the common angular velocity of the discs. Compute the kinetic energy lost in the process. Where does lost kinetic energy go?

2. A fixed gear A, whose moment of inertia is I_1 and radius is R_1, is rotating with angular frequency Ω_1. A second gear B touches gear A and rotates without slipping about an axis parallel to the axis of gear A. Assume that the gear A maintains its angular velocity. The radius and moment of inertia of the second gear are R_2 and I_2 respectively. Compute the angular momentum of the whole system about the axis of gear A.

3. Consider the two-coin problem done in the class. Redo the angular velocity and angular momentum computations when the radii of the inner and outer coins are R_1 and R_2 respectively.

4. Consider the setup of Example 1 of Section 5, but assume that the rod is without a hinge.

   a. Compute the velocity of the CM of rod+mass before and after the collision.

   b. Compute the angular momentum of the CM of rod+mass before and after the collision. Choose appropriate reference point.

   c. Compute the angular velocity of the rod.

   d. Describe the motion of the system.

5. In cricket, every batsman likes to hit the ball such that the reaction force on the batsman's hand is as small as possible. Where should the point of impact of the bat and ball be to achieve this objective? For simplicity, assume the bat has a uniform cross-section.

6. A solid cone of length h and half angle α is rolling on a plane about its vortex with an angular velocity of $\Omega\hat{z}$, as shown in Fig. 6(a). Compute the angular velocity, angular momentum, and kinetic energy of the cone.

7.The vertex of the aforementioned cone is fixed on the z axis at a height equal to the radius of the cone. The cone rotates an angular velocity of $\Omega\hat{z}$ about the vertical axis as shown in Fig. 6(b). Compute the angular velocity, angular momentum, and kinetic energy of the cone.

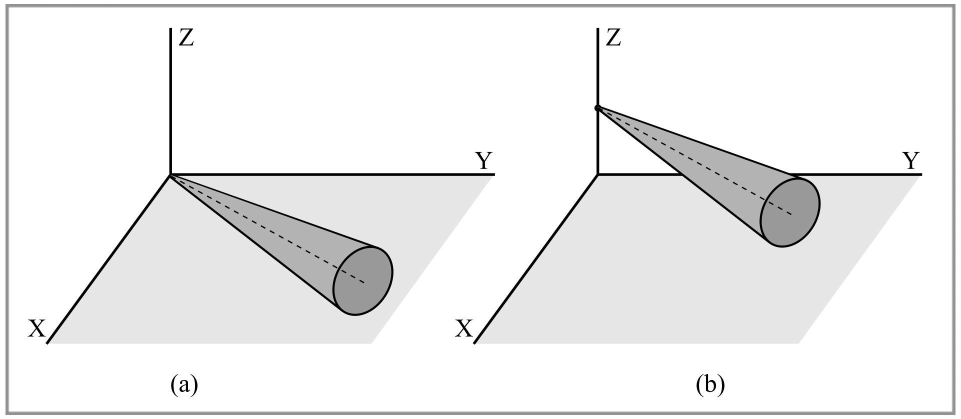


Figure 6: (a) Exerice 6 (b) Exercise 7

8. A particle of mass m, connected to one end of a string, is rotating around in a circle of radius r_0 with speed u on a frictionless table, as shown in Fig. 15P.2(a). For $t > 0$, the other end of the string is pulled through the hole in the middle with a force such that the radius of the circle decreases at a constant rate.

   (a) What is the force on the string?

(b) What are the linear and angular velocities of the mass?

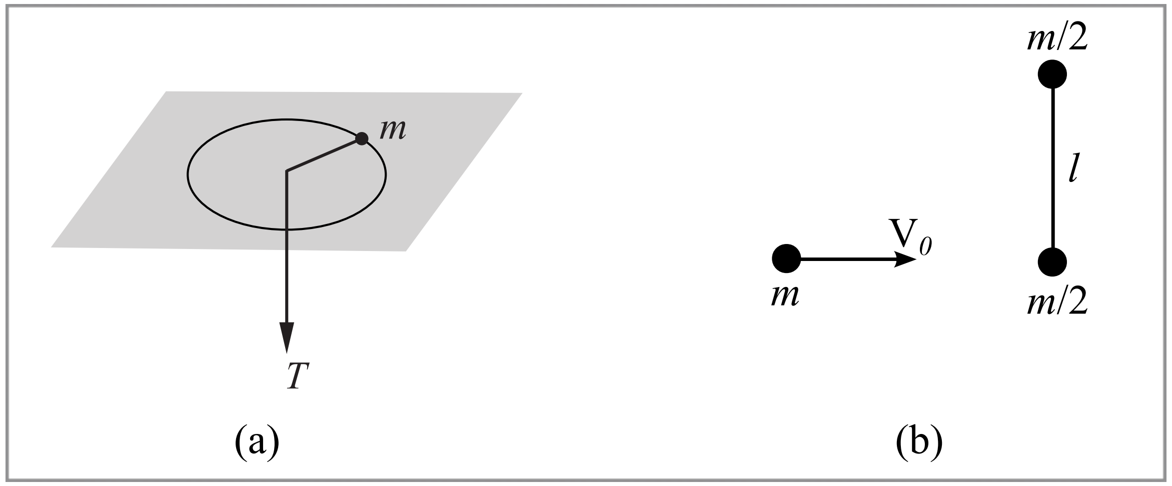
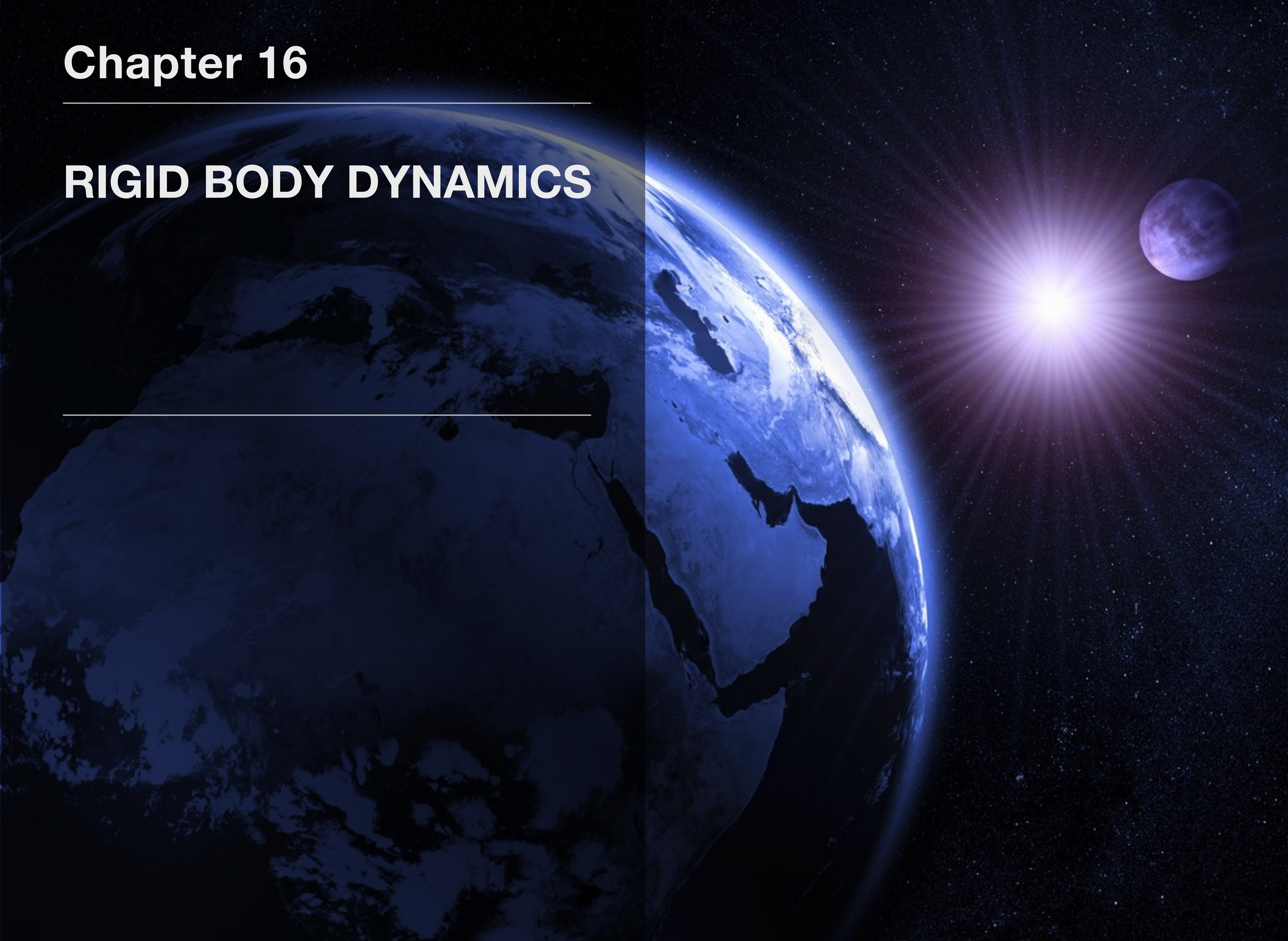


Figure 7 (a) Exerice 8 (b) Exercise 9

9. ball of mass m moving with velocity v_0 collides head-on with the lower mass of the rigid dumbbell, as shown in Fig. 15P.2(b). The dumbbell consists of two masses $m/2$ each, and a stick of length l separating the two masses. Assuming that the collision is elastic and instantaneous, describe the motion of the system after the collision.

Chapter 16

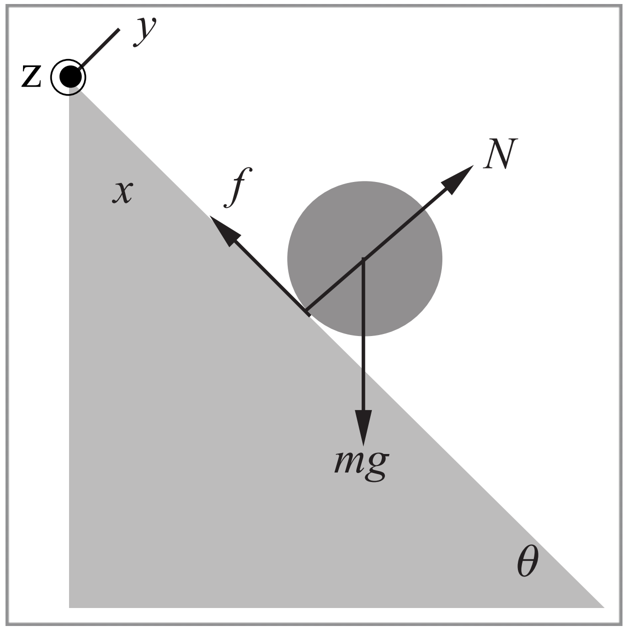
RIGID BODY DYNAMICS



Singleaxis Rotation

Example 16.1: A cylinder rolls down an incline without slipping. Describe the motion of the cylinder.

Solution: A cylinder is rolling down the inclined plane without slipping.



Lagrangian of the cylinder is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 - mgx\cos\theta$$

The constraint that the cylinder rolls down without slipping yields

$$\dot{x} = \dot{\phi}R$$

Therefore,

$$L = \frac{1}{2}m(1+k)\dot{x}^2 - mgx\cos\theta$$

where $k = I/(mR^2) = 1/2$. The equation of motion of the cylinder is

$$(1+k)\ddot{x} = -g\sin\theta$$

Hence the acceleration of the cylinder is $-(2/3)g\sin\theta$. The above acceleration works for both ascent and descent of the cylinder.

Multiaxis Rotation, Gyroscope

Rotation About Multiple Principal Axes

(1) **Dynamics of the precessing cylinder**

The angular velocity of the cylinder is

$$\mathbf{\Omega} = \Omega_z \hat{}z + \Omega_y \hat{}y = \dot{\phi}\cos\theta \hat{}z + \dot{\phi}\sin\theta \hat{}y$$

Therefore the Lagrangian of the cylinder is

$$L = \frac{1}{2}I_3(\dot{\phi}\cos\theta)^2 + \frac{1}{2}I_1(\dot{\phi}\sin\theta)^2$$

We also have constraint that $\theta = \theta_0$= constant. Therefore we use Lagrange multipliers to solve this problem.

$$L = \frac{1}{2}I_3(\dot{\phi}\cos\theta)^2 + \frac{1}{2}I_1(\dot{\phi}\sin\theta)^2 + \lambda(\theta - \theta_0)$$

The equations of motion are

$$\frac{\partial L}{\partial \dot{\phi}} = (I_3\cos^2\theta + I_1\sin^2\theta)\dot{\phi} = L_Z = \text{const}$$

$$(I_1 - I_3)\dot{\phi}\sin\theta\cos\theta + \lambda = 0$$

Here λ is the constraint force or torque acting on the cylinder due to the hinge.

(2) **Torque-free precession**

Here $\mathbf{\Omega} = \dot{\theta}\hat{}x + \dot{\phi}\sin\theta\hat{}y + (\dot{\zeta} + \dot{\phi}\cos\theta)\hat{}z$

and $L = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1(\dot{\phi}\sin\theta)^2 + \frac{1}{2}I_3(\dot{\zeta} + \dot{\phi}\cos\theta)^2$

Since $\partial L/\partial\phi = 0$ and $\partial L/\partial\zeta = 0$, we obtain

$$\frac{\partial L}{\partial \dot{\zeta}} = I_3(\dot{\zeta} + \cos\theta) = L_z = \text{const}$$

$$\frac{\partial L}{\partial \dot{\phi}} = I_3(\dot{\zeta} + \dot{\phi}\cos\theta)\cos\theta + I_1\dot{\phi}\sin^2\theta = L_Z = \text{const}$$

$$I_1\ddot{\theta} - I_1\dot{\phi}^2\sin\theta\cos\theta + I_3(\dot{\zeta} + \cos\theta)\dot{\phi}\sin\theta = 0$$

In the last chapter, we consider θ=const. If we substitute $\ddot{\theta} = 0$ in the above equation, we recover the formula derived earlier.

(3) Gyroscope

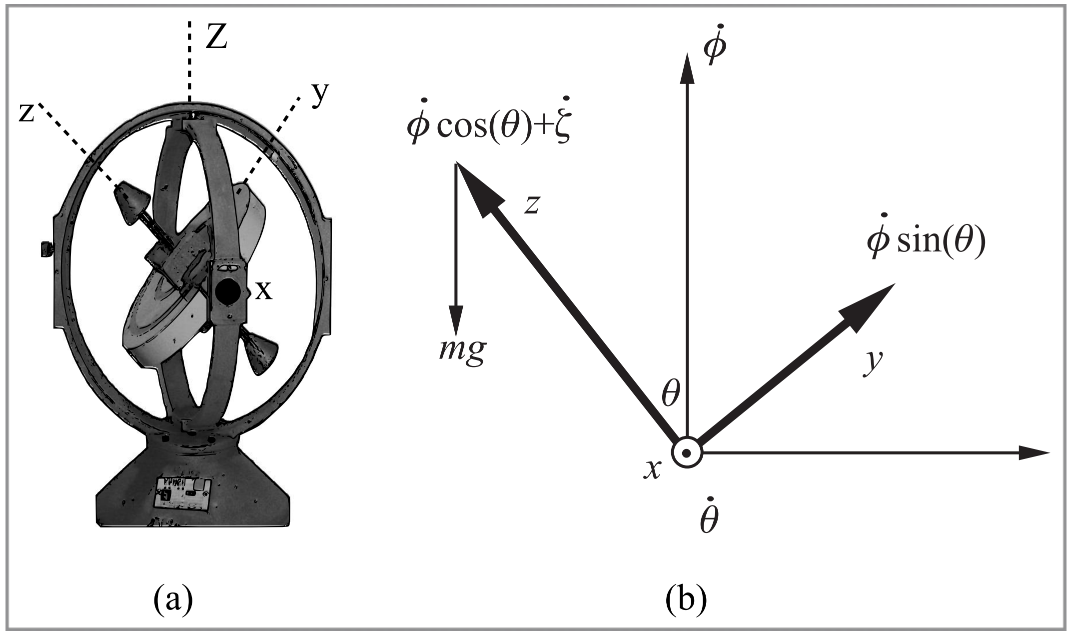


Figure 1: Gyroscope

We focus on the heavy disc in the middle of the gyroscope. The disc spins about an axis normal to the disc. Note that the CM of the gyroscope is hinged, hence it does not move. The setup can also rotate freely about the vertical axis $\hat{Z}$, as well as about the $\hat{x}$ axis. The components of the angular velocity along the orthogonal axes $(\hat{x}, \hat{y}, \hat{z})$ are

$$\Omega_x = \dot{\theta}$$

$$\Omega_y = \dot{\phi}\sin\theta$$

$$\Omega_z = (\dot{\zeta} + \dot{\phi}\cos\theta)$$

The potential energy of the mass located at a distance *l* from the center along the *z* axis is $mgl\cos\theta$

The Lagrangian of the gyroscope is

$$L = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1(\dot{\phi}\sin\theta)^2 + \frac{1}{2}I_3(\dot{\zeta} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta$$

Note that $\partial L/\partial\phi = 0$ and $\partial L/\partial\zeta = 0$. Hence,

$$\frac{\partial L}{\partial\dot{\zeta}} = I_3(\dot{\zeta} + \cos\theta) = L_z = \text{const} \quad (1)$$

$$\frac{\partial L}{\partial\dot{\phi}} = I_3(\dot{\zeta} + \dot{\phi}\cos\theta)\cos\theta + I_1\dot{\phi}\sin^2\theta = L_Z = \text{const}$$

which implies that

$$\dot{\phi} = \frac{L_Z - L_z\cos\theta}{I_1\sin^2\theta} \qquad (2)$$

The equation of θ is

$$I_1\ddot{\theta} - I_1\dot{\phi}^2\sin\theta\cos\theta + I_3(\dot{\zeta} + \cos\theta)\dot{\phi}\sin\theta = mgl\sin\theta$$

or

$$I_1\ddot{\theta} + \left[L_z \sin\theta - \frac{(L_Z - L_z\cos\theta)}{\sin\theta}\cos\theta\right]\left[\frac{L_Z - L_z\cos\theta}{I_1\sin^2\theta}\right] = mgl\sin\theta \quad (3)$$

A reformulation of the above problem in terms of energy provides a first order differential equation, which is easier to solve. The energy of the disk of the gyroscope is

$$\frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{L_z^2}{2I_3} + mgl\cos\theta = E \quad (4)$$

$$\frac{1}{2}I_1\dot{\theta}^2 + U_{\text{eff}}(\theta) = E'$$

where,

$$U_{\text{eff}}(\theta) = \frac{(L_Z - L_z\cos\theta)^2}{2I_1\sin^2\theta} - mgl(1-\cos\theta) \text{ and}$$

$$E' = E - mgl - \frac{L_z^2}{2I_3}.$$

Note that E' is a constant. It is easy to verify that the time derivative of the above equation yields the second-order equation for θ.

We non-dimensionalise the above equation by choosing $\sqrt{I_1 I_3}/L_z$ as the time scale, i.e.,

$$t = \frac{\sqrt{I_1 I_3}}{L_z}t'.$$

In Eq. (16.6.14) we substitute the above form of t, and divide the equation by L_z^2/I_3, which yields

$$\frac{1}{2}\left(\frac{d\theta}{dt'}\right)^2 + \frac{a}{2}\left(\frac{b-\cos\theta}{\sin\theta}\right)^2 - c(1-\cos\theta) = \tilde{E},$$

where

$$a = \frac{I_3}{I_1},\ b = \frac{L_Z}{L_z},\ c = \frac{mgl}{L_z^2/I_3},\ \tilde{E} = \frac{E'}{L_z^2/I_3},$$

and

$$U_{\text{eff}}(\theta) = \frac{a}{2}\left(\frac{b-\cos\theta}{\sin\theta}\right)^2 - c(1-\cos\theta).$$

We can solve the above equation given $\theta(t=0)$ as an initial condition. However, this computation involves square-root function ($\dot{\theta} = \pm\sqrt{f(\theta)}$), which causes difficulty at the turning points where the sign of $\dot{\theta}$ changes. Hence we solve the dimensionless form of Eq. (1), which is

$$\frac{d^2\theta}{dt'^2} + \frac{a}{\sin^3\theta}(b-\cos\theta)(1-b\cos\theta) = c\sin\theta. \quad (4)$$

We solve the above nonlinear equation numerically for a set of convenient parameters. We also solve for $\phi(t)$ and $\zeta(t)$ using

$$\frac{d\phi}{dt'} = \sqrt{a}\frac{b - \cos\theta}{\sin^2\theta} \tag{5}$$

$$\frac{d\zeta}{dt'} = \frac{1}{\sqrt{a}} - \frac{d\phi}{dt'}\cos\theta \tag{6}$$

In the following discussion, we will consider two cases when $c > 0$ (nonzero torque).

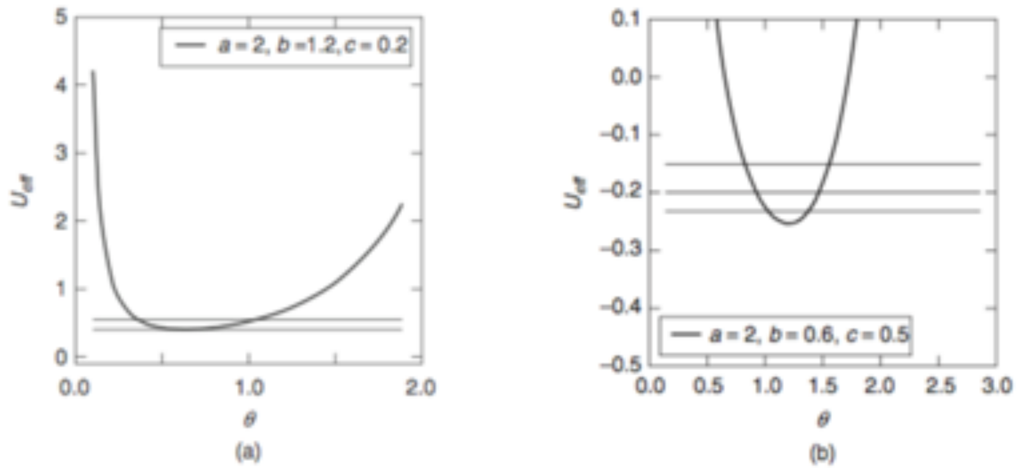


Figure 2: Plot of U_{eff} vs. θ for (a) $a = 2$, $b = 1.2$, and $c = 0.2$; (b) $a = 2$, $b = 0.6$, and $c = 0.2$.

$b > 1$**:**

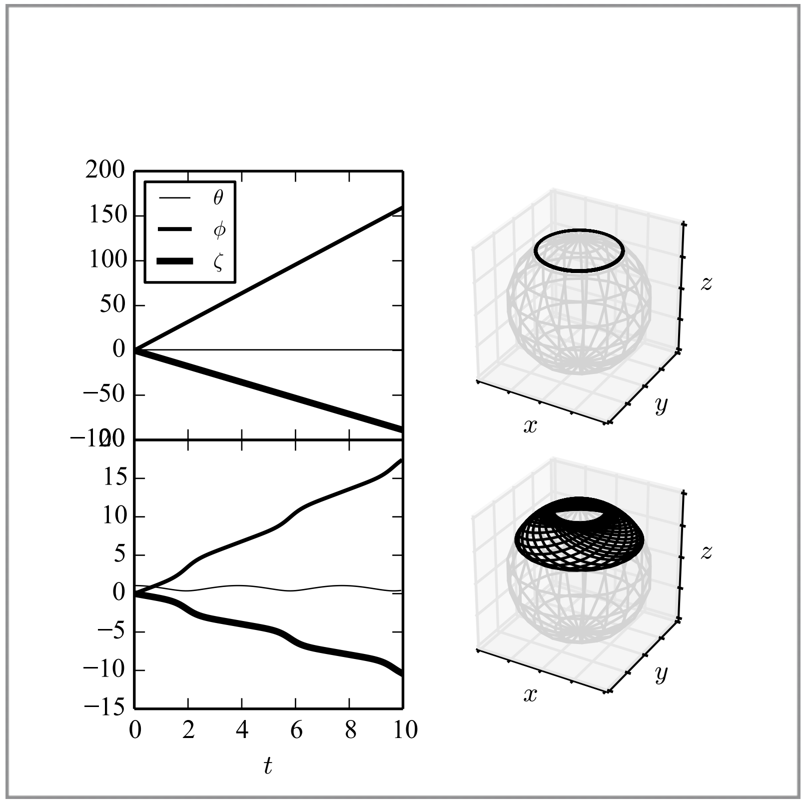


Figure 3: Precession with $a = 2$, $b = 1.2$, and $c = 0.2$:   Time series of $\theta(t), \phi(t), \zeta(t)$ (left panel) and motion of the top of the z axis of the gyroscope (right panel) for (a) top panel: $\tilde{E} = U_{\text{min}} = 0.40$, and (b) bottom panel: $\tilde{E} = U_{\text{min}} = 0.55$ (see Fig. 16.12(a) for U_{eff} plots).

$b < 1:$

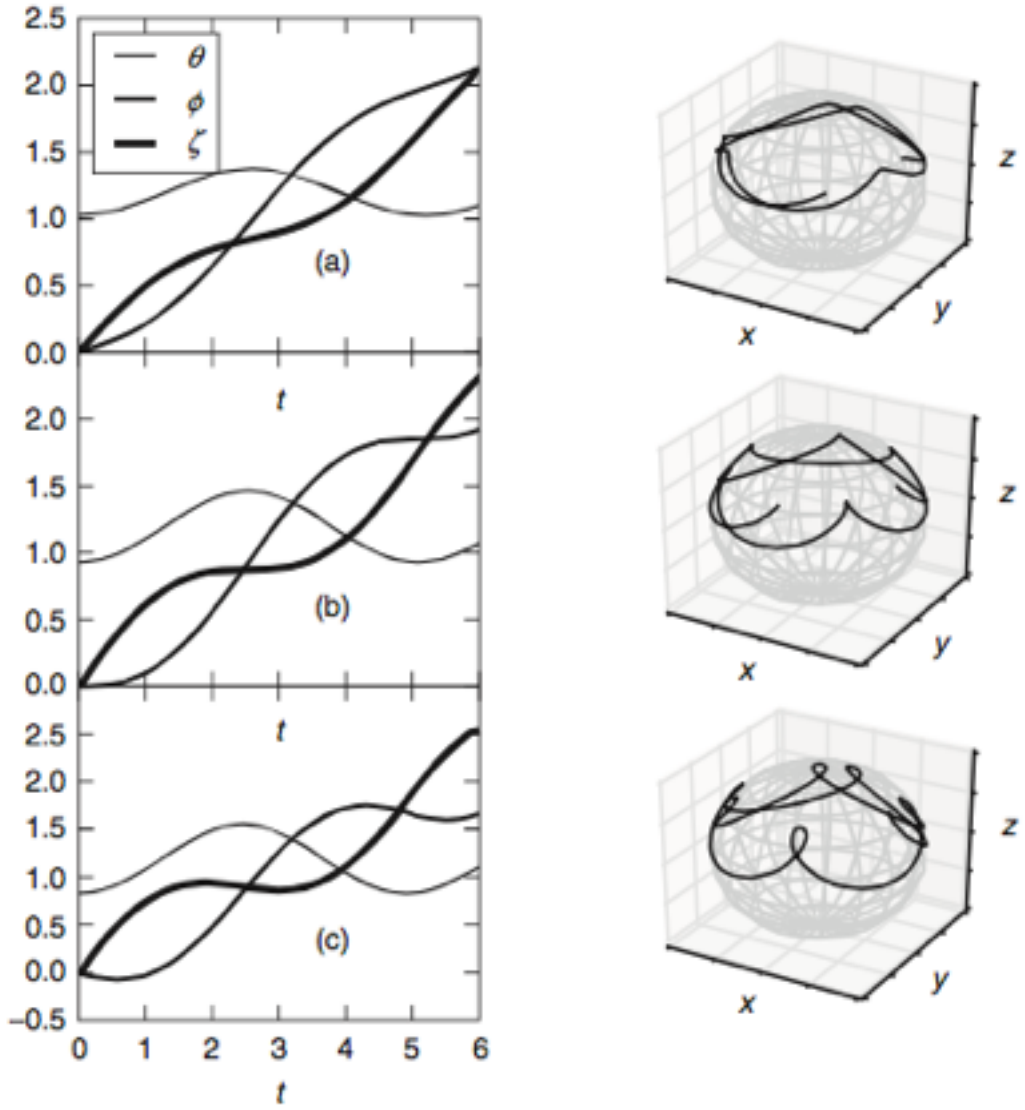


Figure 4: Precession with $a = 2$, $b = 0.6$, and $c = 0.2$: Time series of $\theta(t), \phi(t), \zeta(t)$ (left panel) and motion of the top of the z axis of the gyroscope (right panel) for (a) top panel: $\tilde{E} = 0.11$;

In summary, motion of a gyroscope consists of a spin about the spin axis (z-axis), a precession about the Z-axis, and a *nutation* about x-axis (periodic variation in θ). In the absence of frictional force, θ variation is periodic. However, in an actual gyroscope the motion get damped due to the frictional torque.

A major application of the gyroscope is in navigation. Note that a torque-free gyroscope (set $m = 0$ in the above example) whose angular momentum and angular velocity are parallel will maintain its direction of spin irrespective of the orientations or position of its base. Hence the direction of the spin can be used as a reference direction or initial direction for navigation.

Other Types of Gyroscopes: Top and Bicycle Wheel

The derivation is same as that of the earlier section. The Lagrangian however is

$$L = \frac{1}{2}I_1'\dot{\theta}^2 + \frac{1}{2}I_1'(\dot{\phi}\sin\theta)^2 + \frac{1}{2}I_3(\dot{\zeta} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta$$

where $I_1' = I_1 + ml^2$ is the moment of inertia about the hinge. The derivation is essentially the same as derived in the earlier section.

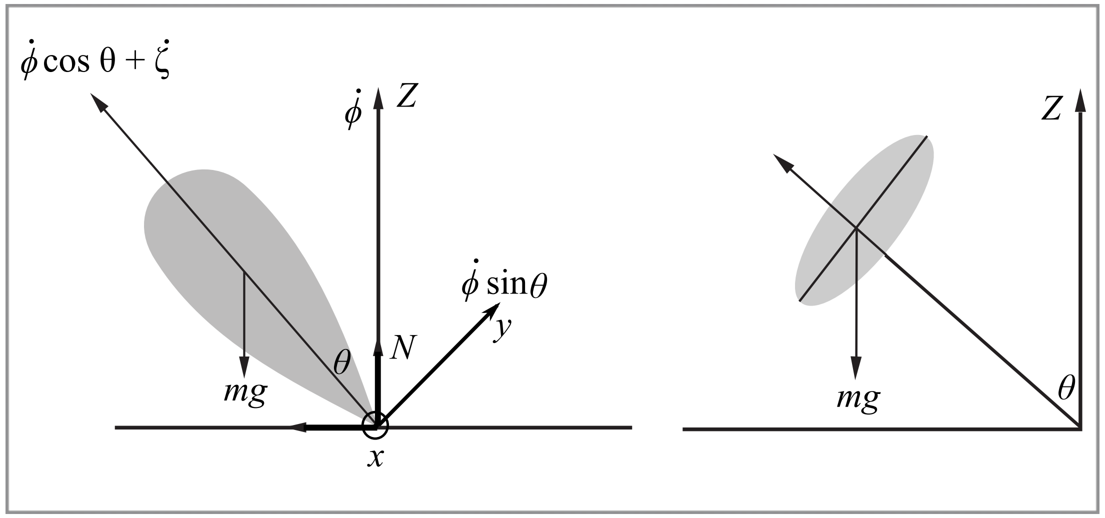

$\dot{\phi}\cos\theta + \dot{\zeta}$
$\dot{\phi}$
Z
$\dot{\phi}\sin\theta$
y
N
θ
mg
x
Z
mg
θ

Figure 5:

Exercises

1. Two masses m_1 and m_2 are hanging on the two sides of a pulley that has moment of inertia I. Assume the string to be massless and inextensible. Compute the acceleration of the masses. Compute the tension of the string.

2. A pendulum consisting of a massless inextensible string length I and bob of mass M is revolving with a constant angular velocity ω about a point on the ceiling. The bob describes a conical surface under steady state. Compute the angle of deviation of the rod from the vertical, and the reaction force at the support.

3. A uniform cylinder of radius R and mass M is spinning with an angular velocity of Ω about its axis. The cylinder is brought in contact with two walls as shown in Fig. 1(a). The coefficient of friction between the wall and the cylinder is μ. Compute the angular velocity of the cylinder as a function of time.

4. A disk of radius R and mass M hangs from a roof by a string, as shown in Fig. 1(b). The disk starts to falls under gravity at $t = 0$. Compute the linear and angular velocity of the disk as a function of time.

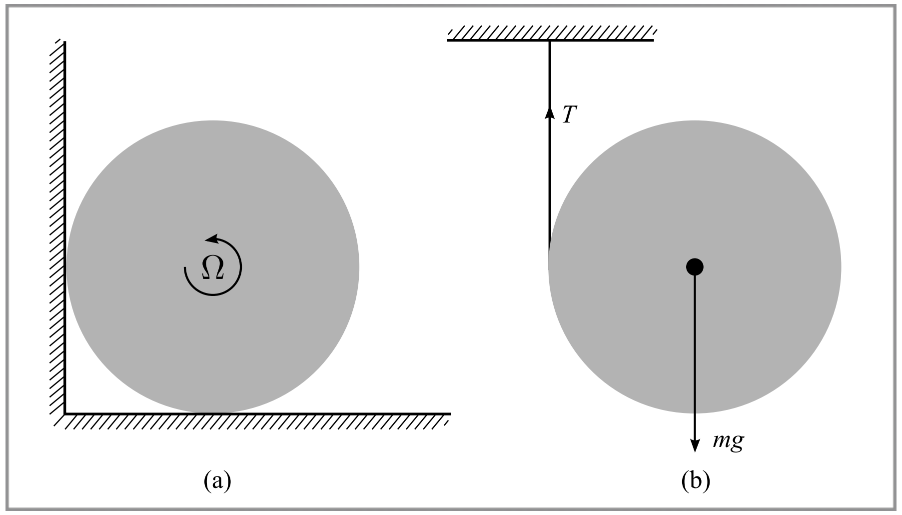


Figure 1 (a) Exercise 3; (b) Exercise 4.

5. Under an application of electric field $E\hat{}x$, a charged ball of mass m, radius R, and charge q is rolling without slipping on a horizontal slab (motion along *x* axis). Describe the motion of the ball.

6. A ladder is leaning against a frictionless wall and the ground, which is also frictionless. The ladder starts to slip downward.

   (a) Obtain an expression for the angular velocity of the ladder as a function of time.

   (b) Show that the top of the plank loses contact with the wall when it is at two-thirds of its initial height.

7. A disk of mass M and radius R is *rigidly* attached at the end of a rod of length l and mass m, as shown in Fig 16P.4(b). Compute

the time period of oscillations for the system. Repeat the calculation if the disk is *freely* attached, i.e., the disk rotates freely about the hinge.

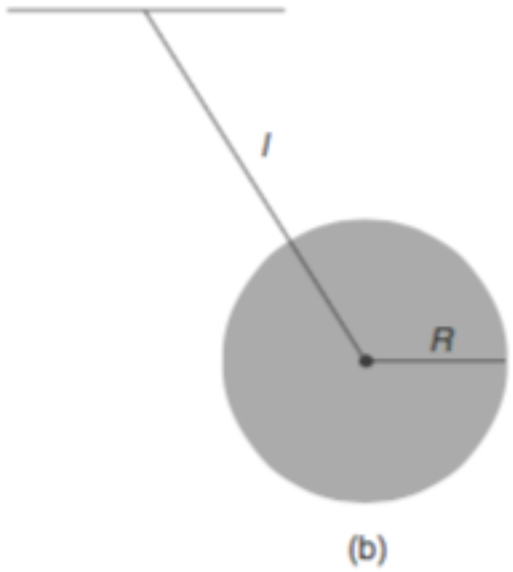


Figure 2: Exercise 7

8. A wooden plank is supported by two rotating rollers that are separated by distance a, as shown in Fig. 16P.5. The direction of rotation of the rollers are reversed in the two cases. For both cases:

(a) Write down the equations of motion and solve them.

(b) Obtain the equilibrium configuration of the systems.

(c) Determine whether the equilibrium configuration is stable or unstable. Compute the period of oscillations for stable system (when the displacement is small).

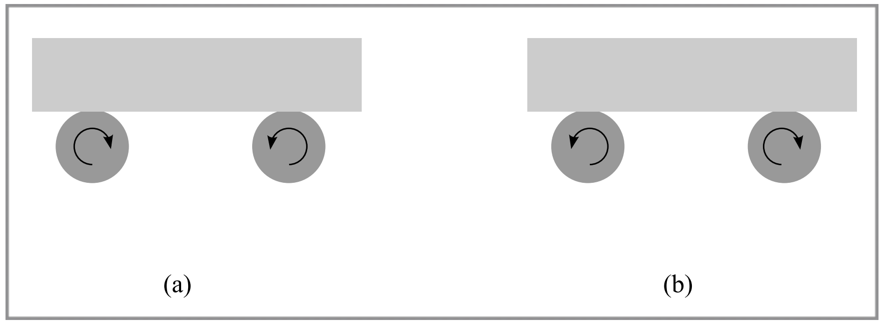


Figure 3 Exercise 8.