Chapter 2

Constraints, Lagrange's equations

Section 1

Constraints

The position of the particle or system follows certain rules due to constraints:

Holonomic constraint: $f(\mathbf{r_1} . \mathbf{r_2}, \ldots \mathbf{r_n}, t) = 0$

Constraints that are not expressible as the above are called nonholonomic.

Examples:

Rigid body: $r_{a,b}$ = constant

Rolling without slipping: $V_{CM} = \omega R_{CM}$

particle moving on a circle.

Generalized coordinates: q_i with $i = 3N - C$, where *C* is the number of constraints.

$$\mathbf{r}_a = \mathbf{r}_a(q_i)$$

Note that $q_i, \dot{q}_i$ are independent variables.

1. Principle of Virtual work

System under equilibrium:

The total force on each particle $\mathbf{f}_a = 0$

Virtual displacement $\delta \mathbf{r}_a$: Arbitrary infinitesimal change in the position of the *a*-th particle keeping the constraints. This is called virtual displacement.

Therefore, the sum of virtual work is zero:

$$\sum_a \mathbf{f}_a \cdot \delta \mathbf{r}_a = 0$$

Note that $\mathbf{f}_a = \mathbf{f}_{a,\text{ext}} + \mathbf{f}_{a,\text{int}}$.

We choose $\delta \mathbf{r}_a$ such that

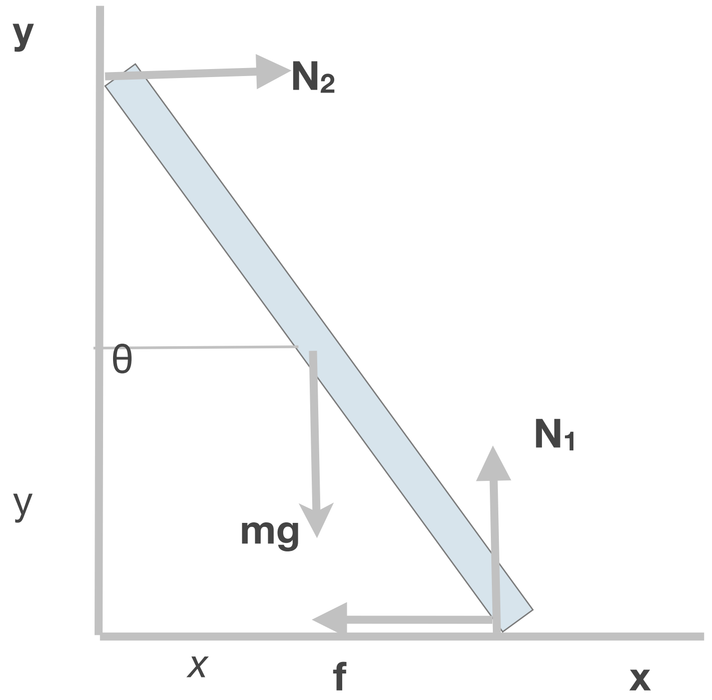
$$\sum_a \mathbf{f}_{a,int} \cdot \delta \mathbf{r}_a = 0,$$

then

$$\sum_a \mathbf{f}_{a,ext} \cdot \delta \mathbf{r}_a = 0.$$

This is the principle of virtual work.

Example: A plank resting agains at a wall. The bottom surface is frictional with the friction force = *f.*



Virtual displacement: δθ.

The internal forces between the molecules of the plank does not do any work under displacement δθ.

The normal forces do no work.

Work done by the frictional force:

$$W_1 = f dx = f l \delta(\sin\theta) = f l \cos\theta \delta\theta$$

Work done by mg:

$$W_2 = mg\delta y = mg(l/2)\delta(\cos\theta) = -mg(l/2)\sin\theta\delta\theta$$

Using principle of virtual work: $W_1 + W_2 = 0$.

Therefore $\tan\theta = \dfrac{2f}{mg}$.

2. D'Alembert's Principle

For dynamics

$$\mathbf{f}_a = \dot{\mathbf{p}}_a$$

Hence

$$\sum_a (\mathbf{f}_a - \dot{\mathbf{p}}_a) \cdot \delta \mathbf{r}_a = 0.$$

Again choose $\delta \mathbf{r}_a$ such that the virtual work done by the internal forces is zero. Hence

$$\sum_a (\mathbf{f}_{a,ext} - \dot{\mathbf{p}}_a) \cdot \delta \mathbf{r}_a = 0$$

Now some algebra:

[1.1] $$\sum_{a,i} f_{a,ext,i} \delta r_{a,i} = -\sum_{a,i,j} \frac{\partial U}{\partial r_{a,ext,i}} \frac{\partial r_{a,i}}{\partial q_j} \delta q_j = -\sum_j \frac{\partial U}{\partial q_j} \delta q_j$$

$$\sum_{a,i} m_a \dot{v}_{a,i} \delta r_{a,i} = \sum_{a,i,j} m_a \dot{v}_{a,i} \frac{\partial r_{a,i,j}}{\partial q_j} \delta q_j$$

$$= \sum_j \left\{ m_a \frac{d}{dt} \left[\sum_{a,i} v_{a,i} \frac{\partial r_{a,i}}{\partial q_j} \right] - m_a v_{a,i} \frac{d}{dt} \left[\frac{\partial r_{a,i}}{\partial q_j} \right] \right\} \delta q_j$$

Note: $\dot{r}_{a,i} = \dfrac{d r_{a,i}}{dt} = \sum_j \dfrac{\partial r_{a,i}}{\partial q_j} \dot{q}_j + \dfrac{\partial r_{a,i}}{\partial t}$

Hence, $\dfrac{\partial r_{a,i}}{\partial q_j} = \dfrac{\partial \dot{r}_{a,i}}{\partial \dot{q}_j}$

$$\frac{d}{dt} \left[\frac{\partial r_{a,i}}{\partial q_j} \right] = \sum_k \frac{\partial^2 r_{a,i}}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 r_{a,i}}{\partial q_j \partial t} = \frac{\partial \dot{r}_{a,i}}{\partial q_j}$$

substitution of which in the above yields

[1.2] $$\sum_{a,i} m \dot{v}_{a,i} \delta r_{a,i} = \sum_j \left\{ \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_j} \right] - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

where

$$T = \sum \frac{1}{2} m v_{a,i}^2$$

is the kinetic energy of the system. The displacement δq_j is arbitrary. Therefore, using Eqs. [1.1, 1.2] we obtain

$$\frac{d}{dt} \left[\frac{\partial T}{\partial q_j} \right] - \frac{\partial T}{\partial q_j} = \frac{\partial U}{\partial q_j}$$

Typically, $\partial U / \partial \dot{q}_j = 0$. Then

$$\frac{d}{dt} \left[\frac{\partial L}{\partial q_j} \right] - \frac{\partial L}{\partial q_j} = 0$$

where *L*=*T*-*U* is the Lagrangian of the system.

Advantages of the Lagrangian formalism

No need to worry about constraint forces, simpler

Analytical, For example, Mécanique analytique by Lagrange does not have a single figure.

Examples:

(1) a free particle

(2) a particle in 2D

(2) Consider the plank discussed before. Let us assume the ground surface to be frictionless.

Generalized coordinate = θ

The KE = $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{1}{12}ml^2\dot{\theta}^2 = \frac{1}{6}ml^2\dot{\theta}^2$

The potential energy $U = mgy = \frac{1}{2}mgl\sin\theta$

The Lagrangian $L = T - U$

The equation of motion yields

$$\ddot{\theta} = \frac{3}{2}\sin\theta$$

(3) Construct Lagrangian for a cylinder rolling down an incline.

Exercises:

(1) A particle is sliding on a uniformly rotating wire. Write down the Lagrangian of the particle. Derive its equation of motion.

(2) Verify D'Alembert's principle for a block of mass *M* sliding down a wedge with an angle of θ.

Chapter 3

Principle of Least Action

Section 1

Variational Calculus

Function of functions

$$L = L(q, \dot{q}, t)$$

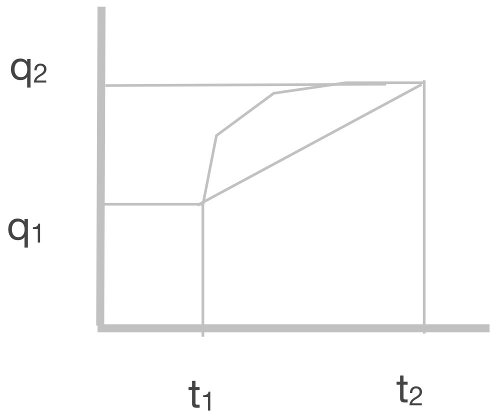
L is a function of *q(t)*, which itself is a function of *t*.

Objective:

Extremize *action*

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

with the ends fixed at (t_1, q_1) and (t_2, q_2).



We will derive an equation for the required function $q(t)$ that extremizes the action. We will compute action for another function

$$q(t, \alpha) = q(t, 0) + \alpha\eta$$

where $\alpha\eta$ is the deviation from the required function. Here α is a number and $\eta(q, \dot{q}, t)$. The change in action due to the above is

$$\delta S = \int_{t_1}^{t_2} [\delta L(q, \dot{q}, t)] dt = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \alpha\eta + \frac{\partial L}{\partial \dot{q}} \alpha\dot{\eta} + HOT \right] dt$$

where *HOT* stands for the higher order terms. For extremization, we take the limit $\alpha \to 0$ (ignore HOT). An integration by parts yields

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \dot{q}} \alpha\dot{\eta} \right] dt = \left[\frac{\partial L}{\partial \dot{q}} \alpha\eta \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \alpha\eta dt$$

The variation of *q* at the ends must vanish, that is η=0 at the ends. Hence, the boundary term vanishes. Therefore,

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] \alpha \eta dt$$

Since η is arbitrary,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Note: The following Lagrangian

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

yields the same equation of motion.

Proof (a): $S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t) dt = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt + \int_{t_1}^{t_2} \frac{df}{dt} dt$

$$= S + f(q_2, t_2) - f(q_1, t_1)$$

Hence, $\delta S' = \delta S$. QED

Proof (b):

$$\frac{d}{dt} f = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}$$

Hence

$$\frac{\partial}{\partial \dot{q}}\left(\frac{df}{dt}\right) = \frac{\partial f}{\partial q}$$

Therefore,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}} \frac{df}{dt}\right) = \frac{\partial^2 f}{\partial q^2}\dot{q} + \frac{\partial^2 f}{\partial q \partial t}$$

and

$$\frac{\partial}{\partial q}\left(\frac{df}{dt}\right) = \frac{\partial^2 f}{\partial q^2}\dot{q} + \frac{\partial^2 f}{\partial q \partial t}$$

Hence the additional terms cancel each other. Q.E.D.

NOTE: On many occasions, the dependent variable is *x* rather than time. On those cases, we replace $\dot{q}$ by q'.

For Multi Variables

Here the generalized variables are q_i's. Hence

$$L = L(q_i, \dot{q}_i, t)$$

For this case, $q_i(t, \alpha) = q_i(t,0) + \alpha \eta_i$. Hence Eq. () becomes

$$\delta S = \int_{t_1}^{t_2} [\delta L(q_i, \dot{q}_i, t)] dt = \int_{t_1}^{t_2} \left[\sum_i \left\{ \frac{\partial L}{\partial q_i}\alpha \eta_i + \frac{\partial L}{\partial \dot{q}_i}\alpha \dot{\eta}_i \right\} + HOT \right] dt$$

$$\delta S = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \alpha \eta_i dt$$

Since it is valid for arbitrary η_i, we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

Beltrami Identity

If *L* is not an explicit function of time *t*, then

$$L - \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \text{const}$$

Proof:

$$\text{LHS} = \frac{dL}{dt} - \sum_i \frac{d}{dt}\left(q_i \frac{\partial L}{\partial \dot{q}_i} \right)$$

$$= \frac{\partial L}{\partial t} + \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i - \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

$$= 0$$

Here we have used the equation of motion.

Examples:

(1) Minimize the distance between two points in 3D:

$$D = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

x: independent variable

y,z: Dependent variables

$$L = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}$$

Since $\frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$,

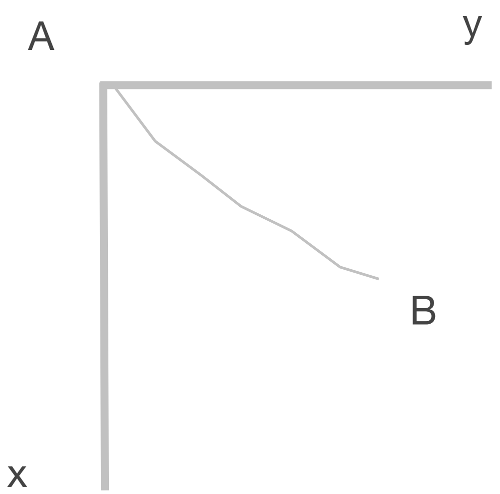
$$\frac{\partial L}{\partial y'} = C_1 \text{ and } \frac{\partial L}{\partial z'} = C_2$$

Therefore,

$$\frac{y'^2}{1 + y'^2 + z'^2} = C_1^2 \quad \text{and} \quad \frac{z'^2}{1 + y'^2 + z'^2} = C_2^2$$

Hence, $y'^2 + z'^2 =$ constant. Therefore, y' and z' are constants. Hence, the particle moves on a straight line.

(2) Minimize the time of descent between two points in a gravitational field:



$$T = \int_A^B \frac{ds}{v} = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gx}} dx$$

Hence, the Lagrangian is

$$L = \sqrt{\frac{1+y'^2}{x}}$$

Since $\partial L/\partial y = 0$, $\partial L/\partial y' = \sqrt{C}$, a constant, which yields

$$\frac{y'^2}{x(1+y'^2)} = C.$$

Hence

$$y' = \sqrt{\frac{Cx}{1-Cx}}.$$

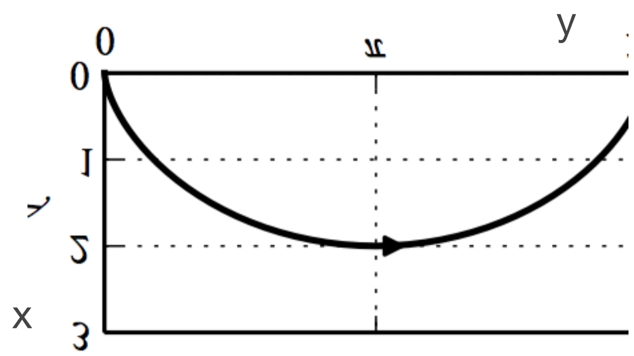
A substitution of $Cx = \sin^2\theta$ yields $y' = \tan\theta$. Therefore,

$$\frac{dx}{d\theta} = \frac{\sin 2\theta}{C} \text{ and } \frac{dy}{d\theta} = \frac{1-\cos 2\theta}{C},$$

whose parametric solution with initial condition (x=0, y=0) is

$$x = A(1-\cos\phi) \text{ and } y = A(\phi - \sin\phi)$$

where $\phi = 2\theta$. The above is an equation of cycloid.



Note: We chose the vertical axis as *x*, so that *L* is independent of *y.* It helps simplify the solution. If we interchanged the axes, the time will be

$$T = \int_A^B \frac{ds}{v} = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx.$$

Hence the Lagrangian will be

$$L = \sqrt{\frac{1 + y'^2}{y}}$$

Using Beltrami identity, we obtain

$$L - \dot{q}\frac{\partial L}{\partial \dot{q}} = \frac{1}{\sqrt{y(1+y'^2)}} = C$$

Therefore,

$$y' = \sqrt{\frac{C - y}{y}},$$

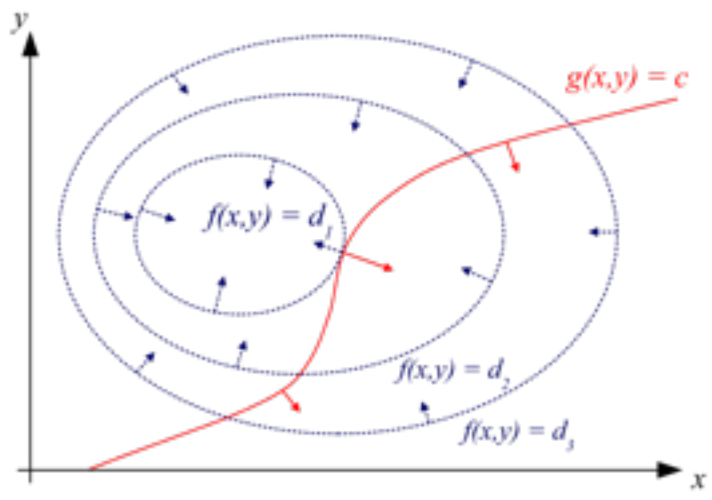
whose solution is same as before except the change of axis.

Extremization under constraints

Detour to Lagrange multiplier

We illustrate using an example. Suppose we want to Extremize $f(x, y)$ under the constraint that $g(x, y) = c$. The constraint would make $f(x, y)$ a function of single variable (say x) that can be maximized using the standard method. However solving a constraint equation could be tricky. Also, this method is not convenient when we have more constraints and variables.

Lagrange proposed an alternative. He suggests that the variables x, y and a new variable λ be made independent. Idea is to look for a contour of $f(x, y)$ that is tangent to the $g(x, y) = c$ curve. See figure below.



From Wikipedia

The intersection point is the desired extremum point. Here

$\nabla f(x, y) = -\lambda \nabla g(x, y)$ and $g(x, y) = c$.

The above equations can be derived by extremizing

$$F(x, y, \lambda) = f(x, y) + \lambda[g(x, y) - c]$$

wrt x, y, λ that yields

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$g(x, y) = c$$

Example:

(1) Find minimum of the function $x^2 + y^2$ under the constraint that $y - x - 1 = 0$.

Solution: We minimize the function

$$F = (x^2 + y^2) + \lambda(y - x - 1)$$

By taking derivatives wrt x, y, λ we obtain

$$2x - \lambda = 0;\ 2y + \lambda = 0;\ y = x + 1$$

whose solution is $y = 1/2, x = -1/2, \lambda = -1/2$.

Application to variation calculus

Extremize functions under constraints variationally.

We illustrate using an example.

(1) Parametric curve

$$x = x(t), y = y(t)$$

We return to the original point, but with a constant perimeter. That is, $x(t_1) = x(t_2) = x_0$ and $y(t_1) = y(t_2) = y_0$. We want to know a function that yields maximum area.

$$S = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})dt$$

under the constraint that

$$I = \int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)dt$$

Hence we extremize

$$L = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$$

which yields

$$\frac{1}{2}\dot{y} - \frac{d}{dt}\left(-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0$$

$$-\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0$$

that yields

$$y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1;\ x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2$$

which yields

$$(x - C_1)^2 + (y - C_2)^2 = \lambda^2$$

which is an equation of a circle. The parameter λ is determined by the perimeter of the circle.

Exercises:

(1) A rope of linear density γ and length L is hanging by two supports that are located horizontally 2a apart. Assuming equilibrium position for the rope, compute its equation.

(2) On a sphere, the great arc is defined as the curves that minimizes the distance travelled between the given two points. Compute the equation of a great arc.

(3) Analyze the variational problems corresponding to the following functionals. In each case take $y(0) = 0$ and $y(1) = 1$.

(a) $\int_0^1 y'^2 dx$

(b) $\int_0^1 yy' dx$

(c) $\int_0^1 xyy' dx$

(4) Consider the functional

$$S[y] = \int_a^b (Py'^2 + Qy^2) dx$$

Find the extrema of the above subject to the condition that

$$\int_a^b y^2 dx = 1$$

The resulting equation is called *Sturm-Liouville problem*. Relate this equation to the Schrodinger's equation.