Chapter 2

Constraints, Lagrange’s equations
Section 1

Constraints

The position of the particle or system follows certain rules due to constraints:

Holonomic constraint: \( f(r_1, r_2, \ldots, r_n, t) = 0 \)

Constraints that are not expressible as the above are called nonholonomic.

Examples:

Rigid body: \( r_{a,b} = \text{constant} \)

Rolling without slipping: \( V_{CM} = \omega R_{CM} \)

particle moving on a circle.

Generalized coordinates: \( q_i \) with \( i = 3N - C \), where \( C \) is the number of constraints.

\[ r_a = r_a(q_i) \]

Note that \( q_i, \dot{q}_i \) are independent variables.

1. Principle of Virtual work

System under equilibrium:

The total force on each particle \( f_a = 0 \)

Virtual displacement \( \delta r_a \) : Arbitrary infinitesimal change in the position of the \( a \)-th particle keeping the constraints. This is called virtual displacement.

Therefore, the sum of virtual work is zero:

\[ \sum_a f_a \cdot \delta r_a = 0 \]

Note that \( f_a = f_{a,\text{ext}} + f_{a,\text{int}} \).

We choose \( \delta r_a \) such that

\[ \sum_a f_{a,\text{int}} \cdot \delta r_a = 0, \]
then
\[ \sum_{a} f_{a,ext} \cdot \delta r_a = 0. \]

This is the principle of virtual work.

Example: A plank resting against a wall. The bottom surface is frictional with the friction force = \( f \).

Virtual displacement: \( \delta \theta \).

The internal forces between the molecules of the plank does not do any work under displacement \( \delta \theta \).

The normal forces do no work.

Work done by the frictional force:
\[ W_1 = f dx = fl \delta (\sin \theta) = f l \cos \theta \delta \theta \]

Work done by \( mg \):
\[ W_2 = mg \delta y = mg(\frac{l}{2}) \delta (\cos \theta) = -mg(\frac{l}{2}) \sin \theta \delta \theta \]

Using principle of virtual work: \( W_1 + W_2 = 0 \).

Therefore \( \tan \theta = \frac{2f}{mg} \).

2. D’Alembert’s Principle

For dynamics
\[ f_a = \dot{p}_a \]

Hence
\[ \sum_{a} (f_a - \dot{p}_a) \cdot \delta r_a = 0. \]
Again choose $\delta \mathbf{r}_a$ such that the virtual work done by the internal forces is zero. Hence

$$\sum_a (\mathbf{f}_{a,ext} - \mathbf{p}_a) \cdot \delta \mathbf{r}_a = 0$$

Now some algebra:

$$\sum_{a,i} f_{a,ext,i} \delta r_{a,i} = - \sum_{a,i,j} \frac{\partial U}{\partial r_{a,i}} \frac{\partial r_{a,i}}{\partial q_j} \delta q_j = - \sum_j \frac{\partial U}{\partial q_j} \delta q_j$$

$$\sum_{a,i} m_a \dot{v}_{a,i} \delta r_{a,i} = \sum_{a,i,j} m_a \dot{v}_{a,i} \frac{\partial r_{a,i,j}}{\partial q_j} \delta q_j$$

$$= \sum_j \left\{ m_a \frac{d}{dt} \left[ \sum_{a,i} \dot{v}_{a,i} \frac{\partial r_{a,i}}{\partial q_j} \right] - m_a v_{a,i} \frac{d}{dt} \left[ \frac{\partial r_{a,i}}{\partial q_j} \right] \right\} \delta q_j$$

Hence, $\dot{r}_{a,i} = \frac{d r_{a,i}}{dt} = \sum_j \frac{\partial r_{a,i}}{\partial q_j} \dot{q}_j + \frac{\partial r_{a,i}}{\partial t}$

Note: $\dot{r}_{a,i} = \frac{d}{dt} \left[ \frac{\partial r_{a,i}}{\partial q_j} \right] = \sum_k \frac{\partial^2 r_{a,i}}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 r_{a,i}}{\partial q_j \partial t} = \frac{\partial r_{a,i}}{\partial q_j}$

$$\frac{d}{dt} \left[ \frac{\partial r_{a,i}}{\partial q_j} \right] = \sum_k \frac{\partial^2 r_{a,i}}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 r_{a,i}}{\partial q_j \partial t}$$

substitution of which in the above yields

$$\sum_{a,i} m \dot{v}_{a,i} \delta r_{a,i} = \sum_j \left\{ \frac{d}{dt} \left[ \frac{\partial T}{\partial q_j} \right] - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

where

$$T = \sum_{a,i} \frac{1}{2} m v_{a,i}^2$$

is the kinetic energy of the system. The displacement $\delta q_j$ is arbitrary. Therefore, using Eqs. [1.1, 1.2] we obtain

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial q_j} \right] - \frac{\partial L}{\partial q_j} = 0$$

Typically, $\partial U / \partial q_j = 0$. Then

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial q_j} \right] = 0$$

where $L=T-U$ is the Lagrangian of the system.

**Advantages of the Lagrangian formalism**

No need to worry about constraint forces, simpler

Analytical, For example, Mécanique analytique by Lagrange does not have a single figure.
Examples:

(1) a free particle

(2) a particle in 2D

(2) Consider the plank discussed before. Let us assume the ground surface to be frictionless.

Generalized coordinate = $\theta$

The KE = $T = \frac{1}{2} m(x^2 + y^2) + \frac{1}{2} \frac{1}{12} ml^2 \dot{\theta}^2 = \frac{1}{6} ml^2 \dot{\theta}^2$

The potential energy $U = mg y = \frac{1}{2} mgl \sin \theta$

The Lagrangian $L = T - U$

The equation of motion yields

$$\ddot{\theta} = \frac{3}{2} \sin \theta$$

(3) Construct Lagrangian for a cylinder rolling down an incline.

Exercises:

(1) A particle is sliding on a uniformly rotating wire. Write down the Lagrangian of the particle. Derive its equation of motion.

(2) Verify D’Alembert’s principle for a block of mass $M$ sliding down a wedge with an angle of $\theta$. 


Chapter 3

Principle of Least Action
Function of functions

\[ L = L(q, \dot{q}, t) \]

\( L \) is a function of \( q(t) \), which itself is a function of \( t \).

Objective:

Extremize action

\[ S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \]

with the ends fixed at \((t_1, q_1)\) and \((t_2, q_2)\).

We will derive an equation for the required function \( q(t) \) that extremizes the action. We will compute action for another function

\[ q(t, \alpha) = q(t, 0) + \alpha \eta \]

where \( \alpha \eta \) is the deviation from the required function. Here \( \alpha \) is a number and \( \eta(q, \dot{q}, t) \). The change in action due to the above is

\[ \delta S = \int_{t_1}^{t_2} \left[ \delta L(q, \dot{q}, t) \right] dt = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} \alpha \eta + \frac{\partial L}{\partial \dot{q}} \alpha \dot{\eta} + \text{HOT} \right] dt \]

where \( \text{HOT} \) stands for the higher order terms. For extremization, we take the limit \( \alpha \to 0 \) (ignore HOT). An integration by parts yields

\[ \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} \alpha \dot{\eta} \right] dt = \left[ \frac{\partial L}{\partial q} \alpha \eta \right]_{t_2}^{t_1} - \int_{t_1}^{t_2} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right] \alpha \eta dt \]

The variation of \( q \) at the ends must vanish, that is \( \eta = 0 \) at the ends. Hence, the boundary term vanishes. Therefore,
\[ \delta S = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right] \alpha \eta dt \]

Since \( \eta \) is arbitrary,

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} \]

Note: The following Lagrangian

\[ L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \]

yields the same equation of motion.

Proof (a):

\[ \int_{t_1}^{t_2} L'(q, \dot{q}, t) dt = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt + \int_{t_1}^{t_2} \frac{df}{dt} dt \]

\[ = S + f(q_2, t_2) - f(q_1, t_1) \]

Hence, \( \delta S' = \delta S \). Q.E.D

Proof (b):

\[ \frac{d}{dt} f = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial t} \]

Hence

\[ \frac{\partial}{\partial \dot{q}} \left( \frac{df}{dt} \right) = \frac{\partial f}{\partial q} \]

Therefore,

\[ \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{q}} \right) = \frac{\partial^2 f}{\partial q^2} \dot{q} + \frac{\partial^2 f}{\partial q \partial t} \]

and

\[ \frac{\partial}{\partial q} \left( \frac{df}{dt} \right) = \frac{\partial^2 f}{\partial q^2} \dot{q} + \frac{\partial^2 f}{\partial q \partial t} \]

Hence the additional terms cancel each other. Q.E.D.

**NOTE**: On many occasions, the dependent variable is \( x \) rather than time. On those cases, we replace \( \dot{q} \) by \( q' \).

**For Multi Variables**

Here the generalized variables are \( q_i \)'s. Hence

\[ L = L(q_i, \dot{q}_i, t) \]

For this case, \( q_i(t, \alpha) = q_i(t, 0) + \alpha \eta_i \). Hence Eq. (a) becomes

\[ \delta S = \int_{t_1}^{t_2} [\delta L(q_i, \dot{q}_i, t)] dt = \int_{t_1}^{t_2} \left[ \sum_i \left( \frac{\partial L}{\partial q_i} \alpha \eta_i + \frac{\partial L}{\partial \dot{q}_i} \alpha \dot{\eta}_i \right) \right] dt + \text{HOT} \]
\[ \delta S = \int_{t_1}^{t_2} \sum_i \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \alpha \eta_i dt \]

Since it is valid for arbitrary \( \eta_i \), we obtain

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \]

**Beltrami Identity**

If \( L \) is not an explicit function of time \( t \), then

\[ L - \sum_i q_i \frac{\partial L}{\partial \dot{q}_i} = \text{const} \]

Proof:

\[
\text{LHS} = \frac{dL}{dt} - \sum_i \frac{d}{dt} \left( q_i \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial t} + \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i - \dot{q}_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0
\]

Here we have used the equation of motion.

**Examples:**

(1) Minimize the distance between two points in 3D:

\[ \|\| = \int_1^2 \sqrt{1 + \left( \frac{dy}{dx} \right)^2 + \left( \frac{dz}{dx} \right)^2} \, dx \]

\( x \): independent variable

\( y, z \): Dependent variables

\[ L = \sqrt{1 + \left( \frac{dy}{dx} \right)^2 + \left( \frac{dz}{dx} \right)^2} \]

Since \( \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0 \),

\[ \frac{\partial L}{\partial y'} = C_1 \text{ and } \frac{\partial L}{\partial z'} = C_2 \]

Therefore,

\[ \frac{y^2}{1 + y^2 + z^2} = C_1^2 \text{ and } \frac{z^2}{1 + y^2 + z^2} = C_2^2 \]

Hence, \( y^2 + z^2 = \) constant. Therefore, \( y' \) and \( z' \) are constants.

Hence, the particle moves on a straight line.

(2) Minimize the time of descent between two points in a gravitational field:
Hence, the Lagrangian is

\[ L = \sqrt{\frac{1+y'^2}{x}} \]

Since \( \partial L / \partial y = 0, \partial L / \partial y' = \sqrt{C} \), a constant, which yields

\[ \frac{y^2}{x(1+y^2)} = C. \]

Hence

\[ T = \int_{A}^{B} \frac{ds}{v} = \int_{A}^{B} \frac{\sqrt{1+y'^2}}{\sqrt{2gx}} \, dx. \]

\[ y' = \sqrt{\frac{Cx}{1-Cx}}. \]

A substitution of \( Cx = \sin^2 \theta \) yields \( y' = \tan \theta \). Therefore,

\[ \frac{dx}{d\theta} = \frac{\sin 2\theta}{C} \quad \text{and} \quad \frac{dy}{d\theta} = \frac{1 - \cos 2\theta}{C}, \]

whose parametric solution with initial condition \((x=0, y=0)\) is

\[ x = A(1 - \cos \phi) \quad \text{and} \quad y = A(\phi - \sin \phi) \]

where \( \phi = 2\theta \). The above is an equation of cycloid.

Note: We chose the vertical axis as \( x \), so that \( L \) is independent of \( y \). It helps simplify the solution. If we interchanged the axes, the time will be

\[ T = \int_{A}^{B} \frac{ds}{v} = \int_{A}^{B} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \, dx. \]
Hence the Lagrangian will be

\[ L = \sqrt{\frac{1 + y^2}{y}} \]

Using Beltrami identity, we obtain

\[ L - \dot{q} \frac{\partial L}{\partial \dot{q}} = \frac{1}{y(1 + y^2)} = C \]

Therefore,

\[ y' = \sqrt{\frac{C - y}{y}}, \]

whose solution is same as before except the change of axis.

Extremization under constraints
Detour to Lagrange multiplier

We illustrate using an example. Suppose we want to extremize \( f(x, y) \) under the constraint that \( g(x, y) = c \). The constraint would make \( f(x, y) \) a function of single variable (say \( x \)) that can be maximized using the standard method. However solving a constraint equation could be tricky. Also, this method is not convenient when we have more constraints and variables.

Lagrange proposed an alternative. He suggests that the variables \( x, y \) and a new variable \( \lambda \) be made independent. Idea is to look for a contour of \( f(x, y) \) that is tangent to the \( g(x, y) = c \) curve. See figure below.

From Wikipedia

The intersection point is the desired extremum point. Here

\[ \nabla f(x, y) = -\lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = c. \]

The above equations can be derived by extremizing

\[ F(x, y, \lambda) = f(x, y) + \lambda[g(x, y) - c] \]

wrt \( x, y, \lambda \) that yields

\[ f_x = \lambda g_x \]

\[ f_y = \lambda g_y \]
\[ g(x, y) = c \]

**Example:**

(1) Find minimum of the function \( x^2 + y^2 \) under the constraint that \( y - x - 1 = 0 \).

**Solution:** We minimize the function

\[ F = (x^2 + y^2) + \lambda (y - x - 1) \]

By taking derivatives wrt \( x, y, \lambda \) we obtain

\[ 2x - \lambda = 0; 2y + \lambda = 0; y = x + 1 \]

whose solution is \( y = 1/2, x = -1/2, \lambda = -1/2 \).

**Application to variation calculus**

Extremize functions under constraints variationally.

We illustrate using an example.

(1) Parametric curve

\[ x = x(t), y = y(t) \]

We return to the original point, but with a constant perimeter. That is, \( x(t_1) = x(t_2) = x_0 \) and \( y(t_1) = y(t_2) = y_0 \). We want to know a function that yields maximum area.

\[ S = \frac{1}{2} \int_{t_1}^{t_2} (\dot{x}\dot{y} - \dot{y}\dot{x})dt \]

under the constraint that

\[ I = \int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)dt \]

Hence we extremize

\[ L = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda \sqrt{\dot{x}^2 + \dot{y}^2} \]

which yields

\[ \frac{1}{2} \ddot{y} - \frac{d}{dt} \left( -\frac{1}{2} \dot{y} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0 \]

\[ -\frac{1}{2} \ddot{x} - \frac{d}{dt} \left( \frac{1}{2} \dot{x} + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0 \]

that yields

\[ y - \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1; x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2 \]

which yields

\[ (x - C_1)^2 + (y - C_2)^2 = \lambda^2 \]
which is an equation of a circle. The parameter $\lambda$ is determined by the perimeter of the circle.

Exercises:

(1) A rope of linear density $\gamma$ and length $L$ is hanging by two supports that are located horizontally $2a$ apart. Assuming equilibrium position for the rope, compute its equation.

(2) On a sphere, the great arc is defined as the curves that minimizes the distance travelled between the given two points. Compute the equation of a great arc.

(3) Analyze the variational problems corresponding to the following functionals. In each case take $y(0) = 0$ and $y(1) = 1$.

(a) $\int_{0}^{1} y'^2 dx$

(b) $\int_{0}^{1} yy' dx$

(c) $\int_{0}^{1} xyy' dx$

(4) Consider the functional

$$S[y] = \int_{a}^{b} (Py^2 + Qy^2) dx$$

Find the extrema of the above subject to the condition that

$$\int_{a}^{b} y^2 dx = 1$$

The resulting equation is called *Sturm-Liouville problem*. Relate this equation to the Schrodinger’s equation.