Chapter 4
Symmetries and Conservation Laws
**Legendre transform**

A system is described by a function $f$ with $x, y$ as independent variables.

$$f = f(x, y)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = udx + vdy$$

Now we want to describe the system in terms of $x, v$.

Since $d(vy) = vdy + ydv$,

$$d(vy - f) = -udx + ydv$$

The new function is $g = g(x, v) = vy - f$.

$$\frac{\partial g}{\partial x} = -u; \quad \frac{\partial g}{\partial v} = y$$

**Hamiltonian**

Lagrangian $L = L(q_\mu, \dot{q}_\mu, t)$

Legendre's transform to go from $\dot{q}_\mu$ to $p_\mu$ as a independent variable. The Hamiltonian

$$H = \sum p_\mu \dot{q}_\mu - L,$$

Hence

$$dH = \sum (p_\mu \delta \dot{q}_\mu + \dot{q}_\mu \delta p_\mu - \frac{\partial L}{\partial q_\mu} \delta q^\mu - \frac{\partial L}{\partial \dot{q}_\mu} \delta \dot{q}_\mu) \delta t$$

$$dH = \sum (p_\mu \delta \dot{q}_\mu + \dot{q}_\mu \delta p_\mu - \dot{p}_\mu \delta q_\mu - p_\mu \delta \dot{q}_\mu) \delta t$$

Hence

$$\dot{q}_\mu = \frac{\partial H}{\partial p_\mu}; \quad \dot{p}_\mu = -\frac{\partial H}{\partial q_\mu}; \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$
Note: \[
\frac{dL}{dt} = \frac{\partial L}{\partial q^\mu} \dot{q}^\mu + \frac{\partial L}{\partial \dot{q}^\mu} \ddot{q}^\mu + \frac{\partial L}{\partial t} \delta t = \dot{p}_\mu \dot{q}^\mu + p_\mu \ddot{q}^\mu + \frac{\partial L}{\partial t}
\]

\[
\frac{dH}{dt} = -\frac{\partial L}{\partial t}
\]

Homogeneity of time implies that for an isolated system, the potential function is not an explicit function of time. For a pair charge particles, \( U \) depends only the distance between the charged particles. Hence, for an isolated system, \( \frac{\partial L}{\partial t} = 0 \).

Hence

\[
\frac{dH}{dt} = 0,
\]

or the total energy of the system is a constant. This is the statement of conservation of energy, which is derived using the homogeneity property of the space.

**Exercise**

1. Argue that the energy is conserved for spring-mass system and planetary system.

2. An oscillator of mass \( m \) and spring constant \( k \) is forced with \( F_0 \cos(\omega_0 t) \). Write down the Lagrangian for the same. Is the energy conserved for this system?
Conservation of Linear Momentum

Homogeneity of space: If each particle of an isolated system is shifted by a constant distance \( \epsilon \), then the system's Lagrangian should not change. Again think of a collection of charged particle. Interaction potential is function only of the distances between the particles. Here

\[
\delta L = \sum_a \frac{\partial L}{\partial r_a} \cdot \epsilon = 0
\]

Since \( \epsilon \) is arbitrary,

\[
\sum_a \frac{\partial L}{\partial r_a} = 0.
\]

Since

\[
\frac{\partial L}{\partial r_a} = \frac{d}{dt} \frac{\partial L}{\partial v_a},
\]

we obtain

\[
\sum_a \frac{\partial L}{\partial r_a} = \sum_a \frac{d}{dt} \frac{\partial L}{\partial v_a} = \frac{d}{dt} \sum_a \frac{\partial L}{\partial v_a} = 0.
\]

Therefore, the total linear momentum \( P = \sum a \frac{\partial L}{\partial v_a} \) of the system is conserved. It is derived from the homogeneity of space.

On many occasions, a system may be symmetric about some translations, but not arbitrary translations. For example for a line charge aligned along the z axis, the system is invariant under any translation along z. Therefore only \( P_z = \sum a \frac{\partial L}{\partial v_{az}} \) is conserved for such system.

If the Lagrangian is not an explicit function of \( q_i \), then the generalized force \( F_i = \partial L / \partial q_i = 0 \), and the generalized momenta

\[
p_i = \frac{\partial L}{\partial \dot{q}_i}
\]

is conserved.
Conservation of Angular Momentum

Isotropy of space: The system or Lagrangian of an isolated system is invariant under the rotation of the whole system by an arbitrary angle.

Let us rotate the system by angle $\delta \phi$ about an axis. The vector $\delta \phi$ is a vector aligned along the axis, and its magnitude is $\delta \phi$. Under this operation, each particle is shifted by

$$\delta \mathbf{r}_a = \delta \phi \times \mathbf{r}_a \text{ and } \delta \mathbf{v}_a = \delta \phi \times \mathbf{v}_a.$$ 

Therefore, the change in the Lagrangian under the above operation is

$$\delta L = \sum_a \frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} \cdot \delta \mathbf{v}_a = 0 \text{ or }$$

$$\delta L = \sum_a \dot{p}_a \cdot \delta \phi \times \mathbf{r}_a + p_a \cdot \delta \phi \times \mathbf{v}_a = 0$$

Therefore,

$$\delta \phi \cdot \frac{d}{dt} \sum_a \mathbf{r}_a \times \mathbf{p}_a = 0.$$ 

Since $\delta \phi$ is arbitrary, the total angular momentum of the system about the origin

$$L = \frac{d}{dt} \sum_a \mathbf{r}_a \times \mathbf{p}_a$$

is a constant. This is related to the isotropy of space.

If the system is invariant under rotation about an axis, the angular momentum about the axis is conserved.

The above three symmetries (homogeneity and isotropy of space, and homogeneity in time) have never been broken. So far, we have not observed any violation of conservation laws of energy, linear momentum, and angular momentum.

Robust conservation

Example:

**Galilean invariance:**

$V_r$ is the relative velocity between the two inertial frames. For a set of particles,

$$L = \sum_a \frac{1}{2} m_a v_a^2 \text{ and } L' = \sum_a \frac{1}{2} m_a (v_a + V_r)^2$$

Hence

$$\delta L = L' - L = \sum_a \delta \left[ \frac{1}{2} m_a (v_a + V_r)^2 \right] = \left[ \sum m_a v_a \right] \cdot V_r = p_{CM} \cdot V_r$$

for small $V_r$. 

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We do not set \(\delta L\) zero like in earlier cases because it will lead to \(P_{CM} = 0\), a special case. We set

\[
\delta L = \frac{dF}{dt} = \frac{d}{dt}[R_{CM} \cdot V_r]
\]

Comparing the above we obtain

\[
P_{CM} = \frac{dR_{CM}}{dt},
\]

which is an identity.

**Self similarity**

**Exercise:**

1. Construct conserved quantities for two masses coupled by a spring.

2. A particle moves in the field for the following system. Construct the conserved quantities of the charged particle:
   - Infinite charged plate, infinite line charge, two charges of equal magnitude, infinite half-plane charge, charged helix, charged torus.

3. Construct conserved quantities for the system of Exercise 1 of Section 3.2 as well as a system of two masses \(m_1\) and \(m_2\) coupled by a spring of a spring constant \(k\).

4. What are the conserved quantities for a particle moving on the inner side of the cone?
Nonrelativistic particle dynamics

**Time Reversal Symmetry:**

\[ U(-t) = U(t) , \]

hence the Lagrangian is symmetric under time reversal.

We cannot distinguish between the forward and time-reversed dynamics.

**Parity or Mirror Symmetry**

\[ r \rightarrow -r = [(x, y, z) \rightarrow (x, -y, z)] + [(x, -y, z) \rightarrow (-x, -y, -z)] \]

Parity = mirror (about xz plane) + rotation about the y axis by \( \pi \).

symmetry of mirror reflection.

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Can’t figure out who is the real gun man!

Can’t figure out the real projectile motion
Proper and pseudo vectors

When a mirror is placed as an $xz$ plane, the symmetry of mirror reflection is mathematically expressed as

$z' = z, \ y' = -y, \ x' = x; \ t' = t$

Hence,

$v'_x = v_x, \ v'_y = -v_y, \ v'_z = v_z$

$a'_x = a_x, \ a'_y = -a_y, \ a'_z = a_z$

A vector that transforms as above is called \textit{proper vector} or \textit{real vector}.

A cross product of two proper vectors transforms very differently under mirror transformation. Let us consider two proper vectors $A$ and $B$. A cross product of these two vectors is

$C = A \times B$

$= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$ \hspace{1cm} (8.8.4)

Since $A$ and $B$ are proper vectors, the rules of mirror reflections of Eq. (8.8.1) yield

$C'_x = -C_x, \ C'_y = C_y, \ C'_z = -C_z$ \hspace{1cm} (8.8.5)

A vector that transforms like $C$ is called \textit{pseudo vector} or \textit{axial vector}. Thus,

\textit{Cross product of two proper vectors is a pseudo vector.}

Two corollaries of the above statements are

\textit{A cross product of a proper vector and a pseudo vector is a proper vector.}

\textit{A cross product of two pseudo vectors is a proper vectors.}

Examples of pseudo vectors

(1) Angular momentum $L = r \times p$ since the linear momentum $p = mv$ is a proper vector. Also, spin angular momentum of a body.

(2) Angular velocity $\Omega$, which is defined using $v = \Omega \times r$.

(3) Magnetic field $B$ since

$B = \frac{\mu_0}{4\pi} \int \frac{Idl \times r}{r^3}$
Wu and coworkers discovered that the electrons (beta) are emitted preferentially opposite to the direction of the spin of the Cobalt-60 nucleus (see Fig. (a)). Note that the spin vector perpendicular to the mirror does not change sign under mirror reflection, but the velocity vector does. Hence in the mirror image, the electrons would move preferentially in the direction of the spin, as shown in Fig. (b). Thus the law that the beta particles have a preferential velocity opposite to their spin is violated in the mirror reflection. This is how mirror symmetry is violated in a beta decay experiment.

Mathematically, the violation of the mirror symmetry is due to the appearance of nonzero value of the pseudo scalar $Q = S \cdot v$ in the Lagrangian. Here $S$ is the spin of the Cobalt-60 nucleus, and $v$ is the velocity of the electron. Such quantities vanish in experiments respecting mirror symmetry (e.g., gravitational and electromagnetic interactions). The potential of weak nuclear force however is a combination of a proper scalar and a pseudo scalar. This kind of potential was first proposed by Marshak and Sudarshan, and Feynman and Gell-mann in 1957.
**Charge Conjugation**

\[ q \rightarrow -q \]

In nuclear physics, C, P, T all are violated, but CPT together is not violated.

**Exercises**

1. Which of the following forces would violate mirror symmetry?

   (a) \[ F = q \mathbf{v} \times \mathbf{B} \]
   (b) \[ F = q \mathbf{v} \times \mathbf{E} \]
   (c) \[ F = q \mathbf{v} \times \mathbf{B} + mg \]
   (d) \[ F = \mathbf{E} + \mathbf{B} \]
   (e) \[ F = \mathbf{E} \times \mathbf{B} \]

   In the above, \( \mathbf{E} \) is the electric field, \( \mathbf{B} \) is the magnetic fields, and \( g \) is the acceleration due to gravity.
We can derive conservation laws for continuous transformation. A general transformation is

\[ t' = t + \epsilon \tau \]

\[ q'^\mu = q^\mu + \epsilon \zeta^\mu \]

The new action is

\[ \int L' dt' - \int L dt = \int \left[ L' \frac{dt'}{dt} - L \right] dt \]

For minimum,

\[ \frac{d}{de} \left[ L' \frac{dt'}{dt} - L \right]_{\epsilon=0} = 0 \]

or

\[ L \frac{d}{de} \left[ \frac{dt'}{dt} \right]_{\epsilon=0} + \left[ \frac{dL'}{de} \right]_{\epsilon=0} = 0 \]

Note:

\[ \frac{\partial L'}{\partial \epsilon} = \frac{\partial L'}{\partial t'} \frac{dt'}{de} + \frac{\partial L'}{\partial q^\mu} \frac{dq^\mu}{de} + \frac{\partial L'}{\partial \dot{q}^\mu} \frac{d\dot{q}^\mu}{de} \]

\[ = \frac{\partial L'}{\partial t'} \tau + \frac{\partial L'}{\partial q^\mu} \zeta^\mu + \frac{\partial L'}{\partial \dot{q}^\mu} \dot{q}^\mu \]

Using \[ \frac{dq^\mu}{dt'} = \frac{dq^\mu + e d\zeta^\mu}{dt + e d\tau} \]

we obtain \[ \frac{d}{de} \left( \frac{dt'}{dt'} \right) = \frac{d}{de} (1 + \epsilon \dot{\tau}) = \dot{\tau} \]

Also, \[ \frac{d}{de} \left( \frac{dt'}{dt'} \right) = \frac{d}{de} (1 + \epsilon \dot{\tau}) = \dot{\tau} \]

Therefore

\[ L \dot{\tau} + \frac{\partial L}{\partial \tau} \tau + \dot{p}_\mu \zeta^\mu + p_\mu (\dot{\zeta}^\mu - \dot{q}^\mu \dot{\tau}) = \frac{dF}{dt} \] or

\[ -H \dot{\tau} - \frac{\partial H}{\partial \tau} \tau + \dot{p}_\mu \zeta^\mu + p_\mu \dot{\zeta}^\mu = \frac{dF}{dt} \] or

\[ p_\mu \zeta^\mu - H \tau - F = \text{const.} \]
This is the Noether’s theorem.

Examples:

(1): By choosing $\tau = 1, \zeta^\mu = 0, F = 0$, we obtain $H = \text{constant}$.

(2): By choosing $\tau = 0, \zeta^\mu = 1, F = 0$, we obtain $p_\mu = \text{constant}$.

(3) Choose $\tau = 0, F = 0, x' = x - \epsilon y, y' = \epsilon x + y$ that yields $M_\zeta = \text{constant}$.

(4) Galilean invariance: $\tau = 0, F = R_{CM} \cdot V_r, \zeta^\mu = V_r^\mu t$. It yields a conserved quantity as

$$P_{CM} \cdot V_r^t - R_{CM} \cdot V_r = \text{const}$$

which is identically zero since $R_{CM} = P_{CM} t$.

(5) Damped linear oscillator:

$$L = \left( \frac{1}{2} m x^2 - \frac{1}{2} k x^2 \right) \exp(bt/m)$$

The above Lagrangian is invariant under the transform $t' = t + \epsilon \tau$ and $x' = x + \epsilon \zeta$ with $\tau = 1$ and $\zeta = -bx/2m$. Note that

$$\dot{x}' = \frac{\dot{x} + \epsilon \zeta}{1 + \epsilon \dot{t}} = \dot{x} \left( 1 - \epsilon \frac{b}{2m} \right)$$

Substitution of the above implies the invariance of $L$ under the above transformation.

Therefore, the conserved quantity with $F = 0$ is

$$p_\zeta - H = \text{constant}$$

where $p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \exp(bt/m)$.

Hence

$$p_\zeta = m \dot{x} \left( -\frac{b x}{2m} \right) \exp(bt/m) = -b x \dot{x} \exp(bt/m).$$

Also

$$H = p_\zeta - L = \left( \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \right) \exp(bt/m)$$

Therefore the conserved quantity is

$$\left( \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 + \frac{1}{2} b x \dot{x} \right) \exp(bt/m)$$
Relativistic Dynamics

A particle follows a trajectory for which the elapsed time is minimum. This action is Lorentz invariant.

\[ S = -\alpha \int d\tau \]

\[ = -\alpha \int \sqrt{1 - v^2/c^2} dt \]

In the nonrelativistic limit,

\[ S = -\alpha \int \left[ 1 - \frac{v^2}{2c^2} \right] dt. \]

Hence, when we choose \( \alpha = m c^2 \), we obtain the usual Lagrangian as \( L = mv^2/2 \). Hence, the relativistic Lagrangian is

\[ L_0 = -mc^2\sqrt{1 - v^2/c^2} \]

The linear momentum is

\[ p_0 = \frac{\partial L}{\partial v} = \frac{mv}{\sqrt{1 - v^2/c^2}} \]

Therefore the energy or the Hamiltonian of the free particle is

\[ H_0 = p \cdot v - L = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \]

For a charged particle in an electromagnetic field

\[ L = L_0 - qA^\mu v_\mu \]

\[ = L_0 - q(\phi - A \cdot v/c) \]

The generalized momentum is

\[ p = \frac{\partial L}{\partial v} = p_0 + qA/c = \gamma mv + qA/c \]

Hence \( \gamma^2 = 1 + (p - qA/c)^2/m^2c^2 \).

Hence the energy of the system is

\[ H = p \cdot v - L = \gamma mc^2 + q\phi = \sqrt{m^2c^4 + (pc - qA)^2} + q\phi \]
In the nonrelativistic limit,

\[ H = \frac{(p - qA/c)^2}{2m} + q\phi \]

Similarly the Lagrangian is

\[ L = \frac{1}{2} mv^2 - q(\phi - A \cdot v/c) \]

The equation of motion is

\[ \frac{d}{dt}(mv_i + qA_i/c) = -\partial_i\phi + (\partial_j A_j)v_j \]

Using \( \frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_j}v_j \), we obtain

\[ \frac{d}{dt}(mv_i) = -q \left[ \partial_i\phi + \frac{\partial A_j}{\partial t} \right] + (v \times B)_i = q(E + v \times B)_i \]

which is the Lorentz equation.