Stress analysis without Meshing
Iso-Geometric Analysis Finite Element Method (IGAFEM)

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Abstract—This research attempts to study and validate a substitute method for the meshing process used in finite element analysis. The method used here is Iso-Geometric analysis (IGA). IGA can used for solving various differential equations. In this paper it is used for solving Poisson’s equation with Dirichlet boundary condition. A tool for stress analysis of various geometries in 2-D and 3-D is developed. The 2-D Elasticity results have been compared using MSC Nastran Patran. An algorithm is developed for finding Non Uniform Rational B-spline (NURBs) curves for a given control points and knot vector. The applications of Non-Uniform B-splines (NURBs) are discussed. NURBs is developed as a tool for 3-D modelling. All the analysis is done using MATLAB®.

Keywords—IGAFEM, Mesh free method, Iso-Geometric analysis, B-splines, NURBs.

I. INTRODUCTION

The initial work in the field of Iso-geometric analysis was done by Tom Hughes and his group at University of Texas at Austin. Various Numerical methods such as finite difference method, finite-volume method, and finite element method were originally defined on meshes of data points. In a mesh, each point has a fixed number of predefined neighbors, and this connectivity between neighbors can be used to define mathematical operators like the derivative. These operators are then used to construct the equations to simulate such as the Euler equations or the Navier-Stokes equations. The technology used today by CAD to represent complex geometries is the Non Uniform B-splines. This allows certain geometries to be represented exactly that are only approximated by polynomial functions, including conic and circular sections. There is a vast array of literature focused on NURBS (e.g. [1], [2]) and as a result of several decades of research, many efficient computer algorithms exist for their fast evaluation and refinement. The key of Iso-Geometric analysis is to employ NURBS not only as a geometry discretization technology, but also as a discretization tool for analysis. The key is in attributing such methods to the field of ‘Iso-geometric Analysis’ (IGA). The current research in this field is to integrate the CAD tool to get the co-ordinates and the knot vector with a stress analysis tool to use b-splines as basis function. NURBS are also developed as a tool for 3-D modelling. The main advantage that NURBS provide is the correct boundary representation. Meshing approximates a curve with straight lines before analysing but with NURBS we can approximate a curve exactly. In simulations where the material being simulated can move around (as in computational fluid dynamics) or where large deformations of the material can occur (as in simulations of plastic materials), the connectivity of the mesh can be difficult to maintain without introducing error into the simulation. If the mesh becomes tangled or degenerate during simulation, the operators defined on it may no longer give correct values. The mesh may be recreated during simulation (a process called remeshing), but this can also introduce error, since all the existing data points must be mapped onto a new and different set of data points. Mesh free methods are intended to remedy these problems.

In contact formulations using conventional geometry discretization, the presence of faceted surfaces can lead to jumps and oscillations in traction responses unless very fine meshes are used. The benefits of using NURBS over such an approach are evident, since smooth contact surface are obtained, leading to more physically accurate contact stresses.

Iso-Geometric analysis can also be coupled with Boundary element method which only requires boundary specification. Through meshing we will not be able to get the exact results but with Iso-geometric analysis we can get exact results because we have exact boundary representation.

Meshing is difficult in complex problems such as bending simulation where the object may move out of alignment. Main advantage of B-splines used in Iso-Geometric analysis is its capability of correct boundary representation. In meshing, curves are approximated using straight lines which will not get us accurate results. Iso-Geometric analysis is used to solve 1-D differential equation.

II. LITERATURE REVIEW

Vinh Phu Nguyen studied about the implementation of Iso-geometric analysis (IGA) in computational analysis. In this paper he discusses about the advantages and disadvantages of IGA. He has applied this method in crack propagation.

Haojie Lian has discussed about the use of IGA instead of the conventional meshing in the stress analysis of various objects. He has discussed about B-splines curves which are used for the correct representation of the object. Finally he has applied IGA using Boundary element method and compared his solution which requires only boundary discretization to the conventional mesh method used in FEM.

Carlos Pantaleón has discussed about the application of IGA with finite volume method. He has discussed various applications of Finite volume method with IGA.
II. Theory

A. B-splines and Non Uniform Rational B-splines (NURBs)

Iso-geometric analysis rely on the use of basis functions generated by CAD. The predominant functions are Non Uniform Rational B-splines (NURBs), but the algorithm used for their evaluation are extended from those used for B-splines. What is fundamental to all the discretization technology used in the present paper (and the majority of technology in the CAGD community), is the representation of geometry through parametric functions. These define a mapping from a given parameter to the desired geometry. It can imagine that as we move through the parameter domain, the parametric function ‘sweeps’ out the desired shape. Both B-splines and NURBS are based on parametric functions.

B. B-splines curves

B-splines can be considered a subset of NURBs. They are affine mappings from the parametric space to the physical space. The expression of a B-spline curve can therefore be written as

\[ C(\xi) = \sum_{i=1}^{n} N_{ap}(\xi)B_i \]

Where \( \xi \) denotes the parametric space coordinate, \( B_i \) is the control point co-ordinates, \( n \) the number of basis function, \( C \) the global coordinates interpolated by the curve and \( N_{ap} \) the B-spline basis function where \( a \) denotes the index of the basis function and \( p \) the order of the basis function.

To understand B-spline curves we need to understand the following three things

C. Control Points:

These do not necessarily lie on the boundary of the domain. The piecewise linear interpolation of the control points generates the control polygon. The control polygon is useful for interactive design because it provides intuitive geometrical information. In the figure below it can see that the lines in red are control polygons whereas the black curve is the corresponding B-spline curves. It can see that points on the control polygon may or may not lie on the B-spline curve.

D. Basis Functions:

Every basis function is associated with a control point. The basis function plays a key role in IGA.

E. Parametric Space:

The parametric space is always structured. It is a straight line, rectangle or cuboid in one-dimensional, 2D or 3D spaces respectively. In some cases, the physical space In this case, the problem is referred to as a multiple-element problem where each parametric space is called a control Polygon. Figure 1 is taken from [4].

F. Parametric Space:

Before the introduction of B-spline basis functions, it is necessary to start with the concept of a knot vector, which has a direct influence on the resulting basis functions. A knot vector is defined as a set of non-decreasing real numbers in the parametric space \( \{\xi_0, \xi_1, \ldots, \xi_{n+p+1}\} \) where \( \xi_i \in \mathbb{R} \)

Where \( i \) denotes the knot index, \( p \) is the curve order and \( n \) is the number of basis functions or control points. Each real number \( \xi_i \) is called a knot. The number of knots is given by \( m = n + p + 1 \). The half-open interval \( [\xi_i, \xi_{i+1}) \) is called a knot span.

Within the knot vector, knots can be repeated where, for example, \( [0, 0, 0, 1, 1, 2, 2, 3, 3, 3] \) is a valid knot vector. Knots with different values can be viewed as different break points that divide the parametric space into different elements. Hence, the physical interpretation of the knots can be explained as the parametric coordinates of the element edges, while the ‘knot span’ between two knots with different values can be viewed as the definition of elements in the parametric space. The insertion of a new knot will split an element, much like h-refinement in the FEM. The knot vector is open if its first and last knot values are repeated \( p + 1 \) times, such as \( [0, 0, 0, 1, 2, 3, 4, 4, 4] \). The open knot vector is the standard in CAD, so all the examples in this paper uses open knot vectors. The knot vector values can be normalized without affecting the resulting B-spline. Therefore, \( [0, 0, 0, 1, 2, 3, 4, 4, 4] \) is equivalent to \( [0, 0, 1/4, 2/4, 3/4, 1, 1, 1] \). It is called a uniform knot vector if the knots are uniformly spaced, for example \( [0, 0, 0, 1, 2, 3, 4, 5, 5, 5] \).

It is necessary to differentiate control points and knots in IGA with nodes in the standard FEM or BEM. In the standard FEM and BEM, nodes are placed on the domain or the boundary to discretize the geometry and the unknown fields. In IGA, the equivalent of a node is a control point, which may lie outside the domain. The knot values are used to divide the space into elements.

Knot vectors are not commonly used by CAD designers, and in most CAD software the ability to modify knot vectors is not provided. It is much more common to tailor the geometry through modification of the polynomial order, control points and weightings.

With the concept of a knot vector, we can now define B-spline basis functions. There exist numerous definitions of B-spline basis functions but, for convenience in implementation, the Cox - de Boor recursive formula (Cox, 1971; de Boor, 1972) is used here

\[ N_{a0}(\xi) = 1 \quad \text{if} \quad \xi_a \leq \xi < \xi_{a+1} \]
\[ = 0 \quad \text{otherwise} \]

\[ N_{a,p}(\xi) = \frac{\xi - \xi_a}{\xi_{a+p} - \xi_a} N_{a,p-1}(\xi) + \frac{\xi_{a+p+1} - \xi}{\xi_{a+p+1} - \xi_{a+1}} N_{a+1,p-1}(\xi) \]

In which fractions of the form 0/0 are defined as zero.

During implementation, the derivatives of B-splines are required in both CAD and analysis to compute quantities such as tangent and normal vectors and field variable derivatives. The first derivative of a B-spline basis function is computed recursively from lower order basis functions as:

\[ B'_\xi = \sum_{i=0}^{n} N_{a,i}(\xi) B_i \]

\[ B''_\xi = \sum_{i=0}^{n} \frac{dN_{a,i}(\xi)}{d\xi} B_i \]

where the derivatives of \( N_{a,i}(\xi) \) are computed recursively.
Derivatives of higher order can be found in [1]. The following properties of B-spline basis functions can be observed,

Local support – the B-spline basis function $N_{a,p}$ is always non-negative in knot span $[\xi_{a}, \xi_{a+p-1}]$. This has an important significance for interactive design. The change of one control point only affects the local part of the curve, giving great convenience for curve modification.

Non-interpolatory – the B-spline basis functions do not interpolate the control points except at the start point, end point and any point whose knot value is repeated $p$ times.

G. Non Uniform Rational B-splines (NURBs)

NURBS are important parametric curves in CAD and are seen as the industry standard with implementation in several commercial software packages. NURBS are developed from B-spline curves but can offer significant advantages due to their ability to represent a wide variety of geometric entities. The expression defining NURBS interpolation is very similar to that of B-splines.

\[
C(\xi) = \sum_{a=1}^{n} w_a N_{a,p}(\xi) B_{a} 
\]

In which $B_a$ is the set of control point coordinates and $R_{a,p}$ are NURBS basis functions, defined as

\[
R_{a,p}(\xi) = \frac{N_{a,p}(\xi) \times w_a}{W(\xi)} \quad \text{where} \quad W(\xi) = \sum_{a=1}^{n} w_a N_{a,p}(\xi) w_a
\]

Weight = \{1,1,1,1,1\} \quad Weight = \{1,1,1,3,1,1\}

Figure 2

Where $N_{a,p}(\xi)$ is the standard B-spline basis function, $W(\xi)$ is the weighting function, and $w_a$ is the weight that is associated with $N_{a,p}$ which influences the distance between the curve and control points, with higher values drawing the curve closer to that point (see Figure 2). When all of the weights are equal to 1, the NURBS reduces to a B-spline curve. The NURBS basis function is a piecewise rational function. All the formulas given below can be found in [2]. Figure 2 is taken from [4].

H. Non Uniform Rational B-splines Surface

\[
S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} P_{i,j} R_{i,j}^{p,q}(\xi, \eta)
\]

Where $P_{i,j}$ is the control points and $w_{i,j}$ are the corresponding weights which are always positive. The basis function $R_{i,j}^{p,q}(\xi, \eta)$ is given by

\[
R_{i,j}^{p,q}(\xi, \eta) = \frac{N_{i}(\xi) M_{j}(\eta) w_{i,j}}{\sum_{k=1}^{n} \sum_{l=1}^{m} N_{i}(\xi) M_{l}(\eta) w_{i,l}}
\]

I. Non Uniform Rational B-splines volumes

\[
V(\xi, \eta, \zeta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} P_{i,j,k} R_{i,j,k}^{p,q,r}(\xi, \eta, \zeta)
\]

Where $P_{i,j,k}$ is the control point coordinates and $R_{i,j,k}^{p,q,r}(\xi, \eta, \zeta)$ is the basis function w.r.t the particular point $(\xi, \eta, \zeta)$, $w_{i,j,k}$ are the corresponding weights which are always positive. The basis function $R_{i,j,k}^{p,q,r}(\xi, \eta, \zeta)$ is given by

\[
R_{i,j,k}^{p,q,r}(\xi, \eta, \zeta) = \frac{N_{i}(\xi) M_{j}(\eta) P_{k}(\zeta) w_{i,j,k}}{\sum_{l=1}^{n} \sum_{m=1}^{m} \sum_{p=1}^{l} N_{l}(\xi) M_{m}(\eta) P_{p}(\zeta) w_{l,m,p}}
\]

J. NURBS as a basis for analysis: Iso-geometric Finite Element formulation

Our attention now focusses on the use of B-splines and NURBS as a discretization tool for analysis, outlining the core concepts of Iso-geometric analysis. In this section the important spaces and mappings are defined, followed by the Iso-geometric FEM formulation in which we use NURBS as a basis for analysis.

Relevant spaces:

Familiarity must be gained with the spaces that are commonplace in Iso-geometric analysis and the relationships that exist between each. Those that are considered presently in the context of B-splines and NURBS include: index, parameter, physical and parent space.

Index space is formed through the specified knot vectors by giving each knot value a distinct coordinate, regardless of whether the knot is repeated or not. As an example, consider a NURBS patch defined through bivariate NURBS basis functions with knot vectors $\xi_1 = \{0,0,0,1,2,3,3\}$, $\xi_2 = \{0,0,1,1\}$ in each of the parametric directions $\xi$, $\eta$ respectively. This will form the index space as illustrated in Figure 3 where the presence of repeated knots leads to several regions of zero parametric area.

Index space is often used during implementation, discarding elements of non-zero parametric area, but in the present work we choose to only consider those elements that have a non-zero parametric area, obviating the need for index space.

Parametric space

Parametric space (sometimes referred to as the 'pre-image' of the NURBS mapping) is formed by considering only the non-
zero intervals between knot values. For the knot vectors considered previously, the parametric space is illustrated in Fig. 6 which can subsequently be reduced to a unit square through appropriate normalization. All parametric spaces can be reduced to a unit interval ($d_p = 1$), square ($d_p = 2$) or cube ($d_p = 3$) in this manner. We define the parametric space as $\Omega \subset \mathbb{R}^{d_p}$ with an associated set of parametric coordinates $\xi = (\xi, \eta, \zeta) = (\xi^1, \xi^2, \xi^3) \subset \Omega$ ($d_p = 3$). If normalization is performed, $\Omega \subset \{0; 1\}^6$.

Figure 4 also reveals that regions bounded by knot lines with non-zero parametric area lead to a natural definition of element domains. More formally, a set $S_{\xi}$ of unique knot values can be defined as

$$S_{\xi} = \{\xi_1, \xi_2, \cdots, \xi_{ns-1}, \xi_{ns}\}$$

where $\xi_i = \xi_{i+1} - 1 \leq i \leq ns - 1$

Where ns is the number of unique knot values. This is generalized to $S_{\xi} \subset \Theta$ which represents the unique knot values for each parametric direction $i = 1, 2, \ldots, d_p$.

Elements can now be defined in the general multivariate case as

$$\omega^f = (\xi_p, \xi_{p+1}) \times (\eta_p, \eta_{p+1}) \times (\zeta_k, \zeta_{k+1})$$

$1 \leq i \leq n^1 - 1$

$1 \leq k \leq n^3 - 1$

$\xi_i \in S^1$, $\eta_j \in S^2$ and $\zeta_k \in S^3$

Where $n^1$, $n^2$ and $n^3$ represent the number unique knots in the $\xi$, $\eta$ and $\zeta$ parametric directions respectively. This leads to a natural numbering scheme for elements over a patch as

$$\phi = k(n^3 - 1)(n^2 - 1) + j(n^2 - 1) + i$$

Before outlining the details of Iso-geometric analysis in a FE context, it is instructive to consider the similarities and differences of IGA over conventional discretization technology. Lagrangian basis functions are most commonly used to discretize both the geometry and unknown fields in an isoparametric fashion. In this way, exactly the same basis functions are used for both. For the majority of cases, the geometry is always approximated incurring no geometrical error. But the key compelling feature of IGA is the unified nature of design and analysis.

We can summarize these points as:

- Conventional finite element analysis: the basis which is chosen to approximate the unknown field is also used to approximate the known geometry. This most commonly takes the form of (low order) Lagrangian basis functions. In most cases the geometry is only approximated. CAD and analysis are disparate.

- Iso-geometric analysis: the basis is generated by CAD which captures the geometry exactly. This basis is also used to approximate the unknown field. Refinement may be required for the unknown fields, but the exact geometry is maintained at all stages of analysis. CAD and analysis are combined to form a unified process.

Isogeometric discretization

The B-spline and NURBS discretization are written in terms of parametric coordinates, but to use such discretization for analysis, we must provide a mapping that allows us to operate at the parent element level. For now let us assume that B-spline and NURBS basis functions can be written in terms of parent coordinates. This allows us to state the Iso-parametric discretization used to approximate both the geometry and fields in IGA.

IV. ALGORITHMS USED FOR 1-D AND 2-D ELASTICITY PROBLEMS

The algorithm for 1-D IGA formulation used in this paper is in [2]

Loop over all the elements

BOX 1 Element stiffness matrix evaluation for element $\Omega$ = ($\xi_p, \xi_{p+1}$)

1. $P = \{P1; P2; P3\}$
2. Store the connectivity of the element in $sctr$.
3. Set $k_e = 0$
4. Loop over Gauss points (GPs) ($\xi_j, \omega_j$) $j = 1, 2, \ldots, n_{gp}$ where $n_{gp}$ is number of gauss points.

a) Compute parametric coordinate $\xi = \phi^a(\xi_j)$

b) Compute derivatives $R_{\xi}^a(\xi)$ ($a = 1, 2, 3$) at point $\xi$

c) Define vector $R_{\xi} = [R_{\xi1}^a \, R_{\xi2}^a \, R_{\xi3}^a]$

d) Compute $|J_{\xi}| = |R_{\xi}P|$

e) Compute $|J_{\xi}| = 0.5(\xi_{i+1} - \xi_i)$

f) Compute shape function derivatives $R_{\xi} = |J_{\xi}|O_j$

g) $k_e = k_e + R_{\xi}R_{\xi}^TJ_{\xi}|J_{\xi}|\omega_j$

5. End loop over GPs

6. Assemble $k_e$ into the global matrix as $K(sctr, sctr) = K(sctr, sctr) + k_e$

End loop over all the elements

The algorithm for 2-D IGA elasticity problem used in this paper is in [2].

Loop over all the elements
1. Determine NURBS coordinates
\((\xi_j^e, \eta_j^e)^T = (\eta_{j+1}^e, \eta_{j+1}^e)^T)\) using elRangeU and elRangeV.
2. Store the connectivity of the element in an array named sctrB (of size nn).
3. Define strain displacement matrix B of size \((1, 2*nn)\).
4. Set \(k_e = 0\).
5. Loop over Gauss points (GPs) \((\xi_j, \omega_j)\) \(j = 1, 2, ..., n_{gp}\) where \(n_{gp}\) is number of gauss points
   a) Compute parametric coordinate \(\xi\) corresponding to \(\xi_j\).
   b) Compute \(|J|\) corresponding to equations.
   c) Compute derivatives of shape function \(R_{\xi}^{e}\) and \(R_{\eta}^{e}\) at point \(\xi, \eta\).
   d) Compute \(J\) using controlPts(sctr(:,e)) \(R_{\xi}^{e}\) and \(R_{\eta}^{e}\).
   e) Find \(J^{-1}\) and determinant \(|J|\).
   f) Compute shape function derivatives \(R_{\xi}^{e}\) and \(R_{\eta}^{e}\).
   g) Use \(R_{\xi}^{e}\) to build the strain displacement matrix B.
   h) \(k_e = k_e + B^TDB|J| J\xi| \omega_j^r\).
6. End loop on gauss points.
7. Assemble \(k_e\) into the global matrix as \(K(sctr, sctr) = K(sctr, sctr) + k_e\).
8. End loop over all the elements.

V. MAPPINGS:

The main aim of this mapping is to simplify the calculations and convert it into the form of \(Ax = B\). The use of NURBS basis functions for discretization introduces the concept of parametric space which is absent in conventional FE implementations. The consequence of this additional space is that an additional mapping must be performed to operate in parent element coordinates. Taking the case \(dp = ds = 2\), an element defined by \(\Omega = (\xi, \eta, \xi_{1+e}, \eta_{1+e})\) is mapped from parent space to parametric space through
\[
\mathbf{\phi}^e(\xi') = \begin{pmatrix}
\frac{1}{2}(\xi_{1+e} - \xi)\
\frac{1}{2}(\eta_{1+e} - \eta)
\end{pmatrix}
\]
The Jacobian determinant is given by
\[|J| = \frac{1}{2}(\xi_{1+e} - \xi)(\eta_{1+e} - \eta).
\]
The Jacobian transformation of 2-d elasticity problem is defined as
\[
J = \begin{pmatrix}
\frac{dx}{d\xi} & \frac{dx}{d\eta} \\
\frac{dy}{d\xi} & \frac{dy}{d\eta}
\end{pmatrix}
\]
Where the derivatives are calculated as
\[
\frac{dx}{d\xi} = \sum_{a=1}^{nn} \frac{\partial R_a^e(\xi)}{\partial \xi},
\]
The determinant is determined by \(|J|\).

The Jacobian Determinant of this mapping is given by,
\[|J| = |J| \cdot |J|.
\]
With this final mapping and Jacobian determinant, it is possible to integrate a function \(f: \Omega \rightarrow \mathbb{R}\) over the physical domain as
\[
\int f(\xi) \, d\Omega = \sum_{\alpha=1}^{n_{el}} \int f(\xi) \, d\Omega
\]
\[= \sum_{\alpha=1}^{n_{el}} f(\xi) \, |J| \, d\xi \, d\eta\]
\[= \sum_{\alpha=1}^{n_{el}} f(\xi(\xi^e, \eta^e)) \, |J| \, d\xi \, d\eta\]

Where \(n_{el}\) is the number of elements. All this can be found in [2]. The numerical integral method used will be Gauss Legendre Quadrature rule.

VI. FINITE ELEMENT METHOD ALGORITHM

The algorithm is divided into 3 steps:

Pre-Processing

Analysis

Post-Processing

In the Pre-Processing step we define all the connectivity as well as do the refinement required. In addition to that we initialize the values of the stiffness and the displacements. In the Analysis step we calculate the stiffness for a particular element and assemble it form a stiffness matrix. The numerical integration used here is gauss quadrature rule. In the Post-processing step we will first apply the boundary conditions on force as well as stiffness and then use the assembled values of stiffness obtained from the analysis step to find the displacement and the stress. The algorithm is same as used in conventional FEM techniques the only change is instead of using the conventional basis function we will be using the NURBS basis function.

VII. PROPOSED ALGORITHM
A. NURBS as a tool for modelling

Various tools such as CATIA has an option of using NURBS to draw a surface or a volume. We can integrate this feature with our stress analysis tool to get better results than the current meshing techniques. As discussed before NURBS can be used to draw curves and this can be used to correctly approximate the curved boundaries of a complex 3-D object. In this figure we can see that between [0, 1] there are 4 basis functions and since the co-ordinate matrix is a linear combination of the basis function therefore we can better approximate a surface using NURBS than using current techniques. Here a tool is developed to generate the corresponding B-spline curves or volumes. The results from MSC Nastran Patran are also analyzed. Basic shapes such as cylinder, truncated cone with circular and parabolic cross-sections have been generated from the algorithm that has been developed. All the analysis are done using MATLAB®. The results are also validated using Open Source code by Vin Phu Nyugen

B. 2-D modelling

The basis necessity of any modelling technique is its ability to generate curves such as circle, parabola etc. First developed is an algorithm for 2-D modelling followed by its extension in 3-D. In NURBS we usually play with weights and the control points but not with knot vector. And we have control Points which are connected by straight line and by assigning weights to these control Points we are able to get the curve we want. Even more basic curve is a circular or parabolic arc. Figure 6 is a circle obtained from a triangle. Here the control Points in order are [3, 0; 0, 0; 1.5, 3*√3/2; 3,3 √3; 4.5, 3 √3/2; 6, 0; 3, 0]. The knot vector is [0; 0; 0; 1/3; 1/3; 2/3; 2/3; 1; 1; 1] and the corresponding weights are [1; 1/2; 1; 1/2; 1; 1/2; 1]. The order of B-splines used is 2. Here we can observe that just with 6-control points we can create a curve. This circle is also the incircle of these polygons. In general we require 3 points to create a circular arc. Thus figure 6 is divided in 3 circular arcs. Figure 7 shows the basis function variation of in the intervals of the knot vector in parabolic using a triangle.

Figure 6

Figure 7

Figure 8

Figure 9
Now we can also create a circle using another polygon, a rectangle. Figure 8 shows the modelling of circle. The control Points used are [1,0,0,0;0,1,0,2;2,2,1;2,0,1,0] Knot vector used is [0,0,0,1/4,1/4,1/2,1/2,3/4,3/4,1,1,1]. Weights used are [1,1/√2,1,1/√2,1,1/√(2 ),1,1/√(2 ),1] [7]. Figure 9 shows the basis function variation of in the intervals of the knot vector in circle using a rectangle.

Just by modifying the weights we can create different shapes that we want. One such example is a parabolic curve. Figure 10 gives three parabolic curve by using the weight as [1, 1, 1, 1, 1] keeping the control points and the knot vector same. This has roughly created an egg shaped structure. Now this can be used to generate 3-D model of a cylinder a truncated cone with different cross-section. Figure 11 shows the basis function variation in the intervals of the knot vector in parabolic using a triangle.

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C. 3-D modelling

The application on 2-D modelling can be extended on 3-D modelling. We can create complex shapes by using B-spline surface or volume. Here we have created simple geometries such as cylinder and truncated cone which can be divided into series of 2-d problems and we can simulate the model cross-section by cross-section. Figure 12 shows a cylinder of radius 5m and height 5m. We can specify the number of

D. Solving Differential equation using Iso-geometric analysis:

The main purpose of this exercise is to see the accuracy of IGA method. The Poisson equation under consideration is

\[ \nabla^2 u = -x \]

with Dirichlet Boundary Condition \( u(0) = 0, u(1) = 0 \)

The exact solution found using simple integration is \( u = x^2/6 + x/6 \)
Figure 15 shows the solution of Poisson’s equation. The red curve is the actual solution and the square box is the solution obtained from IGA.

\[ \text{Solution of poisson equation using IGA method} \]

\[ \begin{array}{c|c|c|c|c|c|c} 
 X-coordinate & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
 Y-coordinate (Displacement solution) & 0.02 & 0.04 & 0.06 & 0.08 & & \\
\end{array} \]

\[ \text{Figure 15} \]

\[ \text{Figure 15 shows the solution of Poisson’s equation. The red curve is the actual solution and the square box is the solution obtained from IGA.} \]

\[ \text{E. 2-D Elasticity problems} \]

\[ \text{Stress analysis of plate:} \]

\[ \text{The length of the plate is 1m. Properties of the material are, Young’s modulus is 70 GPa and poisons ratio is 0.33. The detailed procedure and the algorithm is given in the previous sections. The left boundary is fixed and we have a force of 1000 N on the right boundary equally distributed on each control point there. Here we are going to discuss the results obtained from stress analysis of a plate and compare it with results from MSC Nastran Patran.} \]

\[ \text{Figure 16} \]

\[ \text{Figure 16 are the control Polygons used in Iso-geometric analysis and Figure 17 is the mesh of the plate using MSC Nastran Patran they look similar but the main difference between conventional FEM and mesh free FEM can be clearly seen in the next section Plate with a hole. Figures 18 and 19 shows the x-displacement plot for the plate under uniaxial tension using IGA and Nastran Patran Respectively. We will compare the values from Nastran Patran and my code on iso-geometric analysis in the table as shown. The pattern of displacement obtained is similar and therefore it can be said that the tool developed is working properly.} \]

\[ \text{Figure 17} \]

\[ \text{Figure 17 shows the mesh of the plate using MSC Nastran Patran.} \]

\[ \text{Figure 18} \]

\[ \text{Figure 18 shows the x-displacement plot for the plate under uniaxial tension using IGA and Nastran Patran Respectively.} \]

\[ \text{Comparison of maximum values from Nastran Patran and from Iso-Geometric analysis are as follows,} \]

<table>
<thead>
<tr>
<th>Displacement</th>
<th>From IGA</th>
<th>From Nastran Patran</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>$1.6319 \times 10^{-8}$ m</td>
<td>$1.62 \times 10^{-8}$ m</td>
<td>0.73 %</td>
</tr>
</tbody>
</table>

This is the stress analysis of plate subjected to uniaxial tension. We will compare the values from Nastran Patran and my code on iso-geometric analysis in the table as shown. Figures 20 and 21 shows the stress plot for the plate under uniaxial tension from MATLAB code and MSC Nastran Patran respectively. The pattern and the stress concentration obtained are similar. We can see that stress is concentrated at the corners and at the fixed corner points which matches with the stress concentration obtained from Nastran Patran. The maximum values are analyzed here.
Stress analysis of plate with a hole:
The length of the plate is 4m. Properties of the material are,
Young’s modulus is 70 GPa and poisons ratio is 0.33. The
detailed procedure and the algorithm is given in the previous
sections. The left boundary is fixed and we have a force of
1100 N on the right boundary equally distributed on each
control Point there. Here we are going to discuss the results
obtained from stress analysis of a plate and compare it with
results from MSC Nastran Patran. Figure 22 is the control
Polygons used in Iso-geometric analysis and Figure 23 is the
mesh of the plate using MSC Nastran Patran.
Here we can actually see the difference. Nastran Patran uses
a non-uniform meshing without curves whereas the control
Polygons that we choose in iso-geometric analysis can be
curved. Also we can see that with iso-geometric analysis we
can correctly represent a boundary with a circle rather than
with line as in case of meshing.

<table>
<thead>
<tr>
<th></th>
<th>From IGA</th>
<th>From Nastran Patran</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>XX-Stress</td>
<td>2202.9 N/m²</td>
<td>2269.5 N/m²</td>
<td>2.93 %</td>
</tr>
</tbody>
</table>

Figure 24 and 25 shows the X-displacement plot for the plate
with a hole under uniaxial tension from MATLAB code and
MSC Nastran Patran respectively.
Figure 26 and 27 shows the stress plot for the plate with a hole under uniaxial tension from code in MATLAB and MSC Nastran Patran respectively. The stress concentration on the left corner is due to the aspect ratio.

The pattern and the stress concentration obtained are similar. We can see that stress is concentrated at the corners and at the curved position which matches with the stress concentration obtained from Nastran Patran. The values are ought to be different since the number of elements in both the cases are different.

F. 3-D Elasticity problems

Stress analysis of a quarter cylinder:
The radius of the cylinder is 300 m and its height is 300m. The points with Z = 0 are fixed and the one with Z=300 are forced. Thickness is taken to be 3m. Properties of the material are, Young’s modulus is 70 GPa and Poisson’s ratio is 0.33. Here we are going to discuss the results obtained from stress analysis of a cylinder. Since the force is applied only in Z-direction we will only see the ZZ component of the stress and compare the maximum values of each component. Figure 28 shows the Control Polygons used in Iso-geometric analysis.
Figure 31 shows the stress plot (ZZ) for the cylinder uniaxial tension from back view.

<table>
<thead>
<tr>
<th>Stress Component</th>
<th>Maximum Value (in N/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>XX</td>
<td>93.35</td>
</tr>
<tr>
<td>YY</td>
<td>93.35</td>
</tr>
<tr>
<td>ZZ</td>
<td>397.53</td>
</tr>
<tr>
<td>XY</td>
<td>37.14</td>
</tr>
<tr>
<td>XZ</td>
<td>74.53</td>
</tr>
<tr>
<td>YZ</td>
<td>77,2073</td>
</tr>
</tbody>
</table>

IX. CONCLUSIONS

A substitute method for conventional meshing called Iso-Geometric analysis is studied. An algorithm is designed for generating B-spline curves for a given control points. Iso-Geometric analysis coupled with finite Element method is implemented to solve various problems like stress analysis of plate, plate with a hole and cylinder. The method is also used to solve 1-D differential equation like Poisson’s equation. NURBs as a tool for 3-D modelling is developed. Future work on this would be to make a graphic user interface which will help us to get a common platform for analysis. Next would be developing IGABEM (Iso-Geometric Analysis Boundary Element Method) in which only boundary Discretization is required. We can also try updating the code for stress analysis of any complex object.

X. REFERENCES

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[9] Carlo Lovadina, Alessandro Reali, Giancarlo Sangalli, What is Isogeometric Analysis?

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