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AE 681 Composite Materials**

Generalized Hooke's Law

The generalized Hooke's law for a material is given as

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad i, j, k, l = 1, 2, 3 \quad (1)$$

where, σ_{ij} is a second order tensor known as stress tensor and its individual elements are the stress components. ε_{ij} is another second order tensor known as strain tensor and its individual elements are the strain components. C_{ijkl} is a fourth order tensor known as stiffness tensor. In the remaining section we will call it as stiffness matrix, as popularly known. The individual elements of this tensor are the stiffness coefficients for this linear stress-strain relationship. Thus, stress and strain tensor has $(3 \times 3 =)$ 9 components each and the stiffness tensor has $((3)^4 =)$ 81 independent elements. The individual elements are referred by various names as elastic constants; moduli and stiffness coefficients. The reduction in the number of these elastic constants can be sought with the following symmetries.

Stress Symmetry:

The stress components are symmetric under this symmetry condition, that is, $\sigma_{ij} = \sigma_{ji}$. Thus, there are six independent stress components. Hence, from Eq. (1) we write

$$\sigma_{ji} = C_{jikl} \varepsilon_{kl} \quad (2)$$

Subtracting Eq. (2) from Eq. (1) leads to the following equation

$$0 = (C_{ijkl} - C_{jikl}) \varepsilon_{kl} \Rightarrow C_{ijkl} = C_{jikl} \quad (3)$$

There are six independent ways to express i and j taken together and still nine independent ways to express k and l taken together. Thus, with this symmetry the number of independent elastic constants reduces to $(6 \times 9 =)$ 54 from 81.

Strain Symmetry:

The strain components are symmetric under this symmetry condition, that is, $\varepsilon_{ij} = \varepsilon_{ji}$. Hence, from Eq. (1) we write

$$\sigma_{ij} = C_{ijkl} \varepsilon_{lk}$$

Subtracting Eq. (3) from Eq. (2) we get the following equation

$$0 = (C_{ijkl} - C_{ijlk}) \varepsilon_{kl} \Rightarrow C_{ijkl} = C_{ijlk} \quad (4)$$

It can be seen from Eq. (3) that there are six independent ways of expressing i and j taken together when k and l are fixed. Similarly, there are six independent ways of expressing k and l taken together when i and j are fixed in Eq. (4). Thus, there are $6 \times 6 = 36$ independent constants for this linear elastic material with stress and strain symmetry.

With this reduced stress and strain components and reduced number of stiffness coefficients, we can write Hooke's law in a contracted form as

$$\sigma_i = C_{ij} \varepsilon_j \quad (i, j = 1, 2, \dots, 6) \quad (5)$$

where,

$$\begin{aligned} \sigma_1 &= \sigma_{11} & \varepsilon_1 &= \varepsilon_{11} \\ \sigma_2 &= \sigma_{22} & \varepsilon_2 &= \varepsilon_{22} \\ \sigma_3 &= \sigma_{33} & \varepsilon_3 &= \varepsilon_{33} \\ \sigma_4 &= \sigma_{23} & \varepsilon_4 &= 2\varepsilon_{23} \\ \sigma_5 &= \sigma_{13} & \varepsilon_5 &= 2\varepsilon_{13} \\ \sigma_6 &= \sigma_{12} & \varepsilon_6 &= 2\varepsilon_{12} \end{aligned}$$

Here, it should be noted that the shear strains are the engineering shear strains.

In order Eq. (5) to be solvable for strains in terms of stresses, the determinant of the stiffness matrix must be nonzero, that is, $|C_{ij}| \neq 0$.

The number of independent elastic constants can be reduced further, if there exists a *strain energy density function* W , given as below.

Strain Energy Density Function (W):

The strain energy density function W is given as

$$W = \frac{1}{2} C_{ij} \varepsilon_i \varepsilon_j \quad (7)$$

with the property that

$$\sigma_i = \frac{\partial W}{\partial \varepsilon_i} \quad (8)$$

It is seen that W is a quadratic function of strain. A material with existence of W with property in Eq. (8) is called as *Hyperelastic Material*.

The W can also be written as

$$W = \frac{1}{2} C_{ji} \varepsilon_j \varepsilon_i \quad (9)$$

Subtracting Eq. (9) from Eq. (7) we get

$$0 = (C_{ij} - C_{ji}) \varepsilon_i \varepsilon_j \quad (10)$$

which leads to the identity $C_{ij} = C_{ji}$. Thus, the stiffness matrix is symmetric. This symmetric matrix has 21 independent elastic constants. The stiffness matrix is given as follows:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ \text{Symmetric} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \quad (11)$$

The existence of the function W is based upon the first and second law of thermodynamics. Further, it should be noted that this function is positive definite. Also, the function W is an invariant (An invariant is a quantity which is independent of change of reference).

The material with 21 independent elastic constants is called as *Anisotropic* or *Aelotropic Material*.

Further reduction in number of independent elastic constants can be obtained with the use of planes of material symmetry as follows.

Material Symmetry:

It should be recalled that both the stress and strain tensor follow transformation rule and so is the stiffness tensor. The transformation rule for these quantities (as given in Eq. (1)) is known as follows:

$$\begin{aligned} \sigma'_{ij} &= a_{ki} a_{lj} \sigma_{ij} \\ \varepsilon'_{ij} &= a_{ki} a_{lj} \varepsilon_{ij} \\ C'_{ijkl} &= a_{mi} a_{nj} a_{rk} a_{sl} C_{mnr} \end{aligned} \quad (12)$$

where a_{ij} are the direction cosines from i to j coordinate system. The prime indicates the quantity in new coordinate system.

When the function W given in Eq. (9) is expanded using the contracted notations for strains and elastic constants given in Eq. (11) W has the following form:

$$W = \frac{1}{2} \begin{bmatrix} C_{11}\varepsilon_1^2 + 2C_{12}\varepsilon_1\varepsilon_2 + 2C_{13}\varepsilon_1\varepsilon_3 + 2C_{14}\varepsilon_1\varepsilon_4 + 2C_{15}\varepsilon_1\varepsilon_5 + 2C_{16}\varepsilon_1\varepsilon_6 + \\ C_{22}\varepsilon_2^2 + 2C_{23}\varepsilon_2\varepsilon_3 + 2C_{24}\varepsilon_2\varepsilon_4 + 2C_{25}\varepsilon_2\varepsilon_5 + 2C_{26}\varepsilon_2\varepsilon_6 + \\ C_{33}\varepsilon_3^2 + 2C_{34}\varepsilon_3\varepsilon_4 + 2C_{35}\varepsilon_3\varepsilon_5 + 2C_{36}\varepsilon_3\varepsilon_6 + \\ C_{44}\varepsilon_4^2 + 2C_{45}\varepsilon_4\varepsilon_5 + 2C_{46}\varepsilon_4\varepsilon_6 + \\ C_{55}\varepsilon_5^2 + 2C_{56}\varepsilon_5\varepsilon_6 + \\ C_{66}\varepsilon_6^2 \end{bmatrix} \quad (13)$$

Thus, from Eq. (13) it can be said that the function W has the following form in terms of strain components:

$$W = W \begin{bmatrix} \varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \varepsilon_6^2, \\ \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_1\varepsilon_4, \varepsilon_1\varepsilon_5, \varepsilon_1\varepsilon_6, \\ \varepsilon_2\varepsilon_3, \varepsilon_2\varepsilon_4, \varepsilon_2\varepsilon_5, \varepsilon_2\varepsilon_6, \\ \varepsilon_3\varepsilon_4, \varepsilon_3\varepsilon_5, \varepsilon_3\varepsilon_6, \\ \varepsilon_4\varepsilon_5, \varepsilon_4\varepsilon_6, \\ \varepsilon_5\varepsilon_6 \end{bmatrix} \quad (14)$$

With these concepts we proceed to consider the planes of material symmetry. The planes of the material, also called as *elastic*, symmetry are due to the symmetry of the structure of anisotropic body. In the following, we consider some special cases of material symmetry.

(A) Symmetry with respect to a Plane:

Let us assume that the anisotropic material has only one plane of material symmetry. A material with one plane of material symmetry is called as *Monoclinic Material*.

Let us consider the x_1 - x_2 ($x_3=0$) plane as the plane of material symmetry. This is shown in Figure 1. This symmetry can be formulated with the change of axes as follows:

$$x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = -x_3 \quad (15)$$

With this change of axes,

$$a_{ij} = \frac{\partial x'_i}{\partial x_j} \quad \text{and} \quad \frac{\partial x'_i}{\partial x_j} = \delta_{ij} \quad \text{for } j = 1, 2 \quad \text{and} \quad \frac{\partial x'_i}{\partial x_3} = -\delta_{i3} \quad (16)$$

This gives us along with the use of the second of Eq. (12)

$$\varepsilon'_{11} = \varepsilon_{11}, \varepsilon'_{22} = \varepsilon_{22}, \varepsilon'_{33} = \varepsilon_{33}, \varepsilon'_{23} = -\varepsilon_{23}, \varepsilon'_{13} = -\varepsilon_{13}, \varepsilon'_{12} = \varepsilon_{12} \quad (17)$$

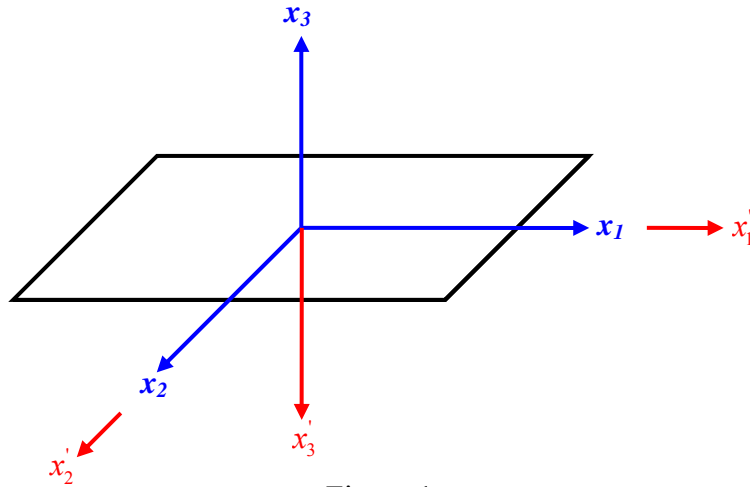


Figure 1

First Approach: Invariance Approach

Now, the function W can be expressed in terms of the strain components ε'_{ij} . If W is to be invariant, then it must be of the form

$$W = W \left[\varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \varepsilon_6^2, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_1\varepsilon_6, \varepsilon_2\varepsilon_3, \varepsilon_2\varepsilon_6, \varepsilon_3\varepsilon_6, \varepsilon_4\varepsilon_5 \right] \quad (18)$$

Comparing this with Eq. (13) it is easy to conclude that

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0 \quad (19)$$

Thus, for the monoclinic materials the number of independent constants are 13. With this reduction of number of independent elastic constants the stiffness matrix is given as

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ \text{Symmetric} & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad (20)$$

Second Approach: Stress Strain Equivalence Approach

The same reduction of number of elastic constants can be derived from the stress strain equivalence approach. From Eq. (12) and Eq. (16) we have

$$\sigma'_{11} = \sigma_{11}, \sigma'_{22} = \sigma_{22}, \sigma'_{33} = \sigma_{33}, \sigma'_{23} = -\sigma_{23}, \sigma'_{13} = -\sigma_{13}, \sigma'_{12} = \sigma_{12} \quad (21)$$

The same can be seen from the stresses on a cube inside such a body with the coordinate systems shown in Figure 1. Figure 2(a) shows the stresses on a cube with the coordinate system x_1, x_2, x_3 and Figure 2(b) shows stresses on the same cube with the coordinate system x'_1, x'_2, x'_3 . Comparing the stresses we get the relation as in Eq. (21).

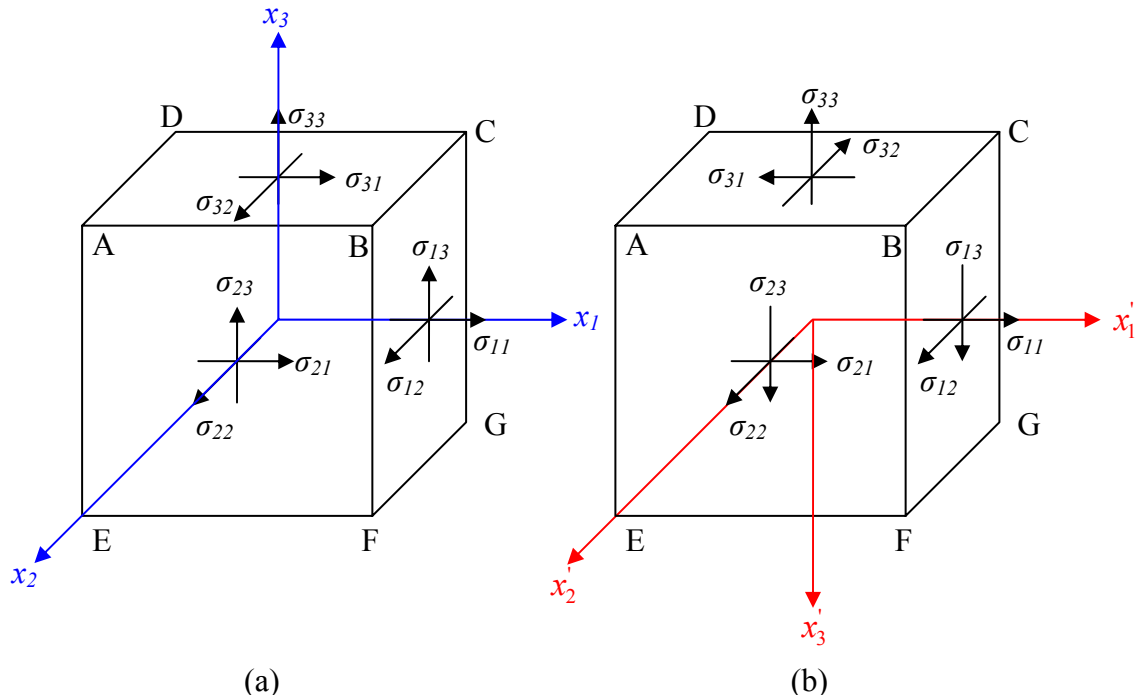


Figure 2

Now using the stiffness matrix as given in Eq. (11), strain term relations as given in Eq. (17) and comparing the stress terms in Eq. (21) as follows:

$$\sigma_{11} = \sigma'_{11}$$

$$C_{11}\epsilon_1 + C_{12}\epsilon_2 + C_{13}\epsilon_3 + C_{14}\epsilon_4 + C_{15}\epsilon_5 + C_{16}\epsilon_6 = C'_{11}\epsilon'_1 + C'_{12}\epsilon'_2 + C'_{13}\epsilon'_3 + C'_{14}\epsilon'_4 + C'_{15}\epsilon'_5 + C'_{16}\epsilon'_6$$

Using the relations from Eq. (17), the above equations reduces to

$$C_{14}\varepsilon_4 + C_{15}\varepsilon_5 = C'_{14}\varepsilon'_4 + C'_{15}\varepsilon'_5$$

Noting that $C_{ij} = C'_{ij}$, this holds true only when $C_{14} = C_{15} = 0$

Similarly,

$$\sigma_{22} = \sigma'_{22} \text{ gives } C_{24} = C_{25} = 0$$

$$\sigma_{33} = \sigma'_{33} \text{ gives } C_{34} = C_{35} = 0$$

$$\sigma_{23} = \sigma'_{23} \text{ gives } C_{46} = 0$$

$$\sigma_{13} = \sigma'_{13} \text{ gives } C_{56} = 0$$

This gives us the C_{ij} matrix as in Eq. (20).

(B) Symmetry with respect to two Orthogonal Planes:

Let us assume that the material under consideration has one more plane, say x_2 - x_3 is plane of material symmetry along with x_1 - x_2 as in (A). These two planes are orthogonal to each other. This transformation is shown in Figure 3.

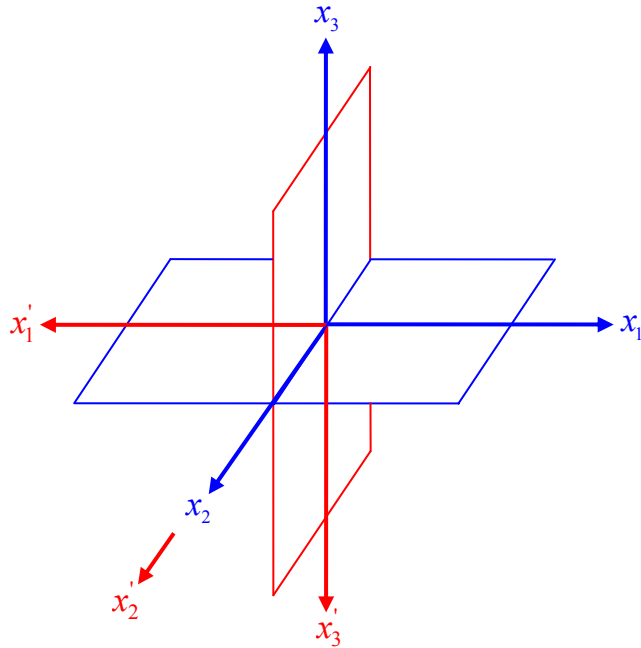


Figure 3

This can be mathematically formulated by the change of axes as

$$x'_1 = -x_1, x'_2 = x_2, x'_3 = -x_3 \quad (22)$$

And

$$a_{ij} = \frac{\partial x'_i}{\partial x_j} \quad \text{and} \quad \frac{\partial x'_i}{\partial x_j} = -\delta_{ij} \quad \text{for } j = 1, 3 \quad \text{and} \quad \frac{\partial x'_i}{\partial x_2} = \delta_{i2} \quad (23)$$

This gives us the required strain relations as (from Eq. (12))

$$\varepsilon'_{11} = \varepsilon_{11}, \varepsilon'_{22} = \varepsilon_{22}, \varepsilon'_{33} = \varepsilon_{33}, \varepsilon'_{23} = -\varepsilon_{23}, \varepsilon'_{13} = \varepsilon_{13}, \varepsilon'_{12} = -\varepsilon_{12} \quad (24)$$

First Approach: Invariance Approach

We can get the function W simply by substituting ε'_{ij} in place of ε_{ij} and using contracted notations for the strains in Eq. (18). Noting that W is invariant, its form in Eq. (18) must now be restricted to functional form

$$W = W[\varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \varepsilon_6^2, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3] \quad (25)$$

From this it is easy to see that

$$C_{16} = C_{26} = C_{36} = C_{45} = 0$$

Thus, the number of independent constants reduces to 9. The resulting stiffness matrix is given as

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ \text{Symmetric} & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad (26)$$

When a material has (any) two orthogonal planes as planes of material symmetry then that material is known as *Orthotropic Material*. It is easy to see that when two orthogonal planes are planes of material symmetry, the third mutually orthogonal plane is also plane of material symmetry and Eq. (26) holds true for this case also.

Unidirectional fibrous composites are an example of orthotropic materials.

Second Approach: Stress Strain equivalence Approach

The same reduction of number of elastic constants can be derived from the stress strain equivalence approach. From the first of Eq. (12) and Eq. (23) we have

$$\sigma'_{11} = \sigma_{11}, \sigma'_{22} = \sigma_{22}, \sigma'_{33} = \sigma_{33}, \sigma'_{23} = -\sigma_{23}, \sigma'_{13} = \sigma_{13}, \sigma'_{12} = -\sigma_{12} \quad (27)$$

The same can be seen from the stresses on a cube inside such a body with the coordinate systems shown in Figure 3. Figure 4(a) shows the stresses on a cube with the coordinate system x_1, x_2, x_3 and Figure 4(b) shows stresses on the same cube with the coordinate system x'_1, x'_2, x'_3 . Comparing the stresses we get the relation as in Eq. (27).

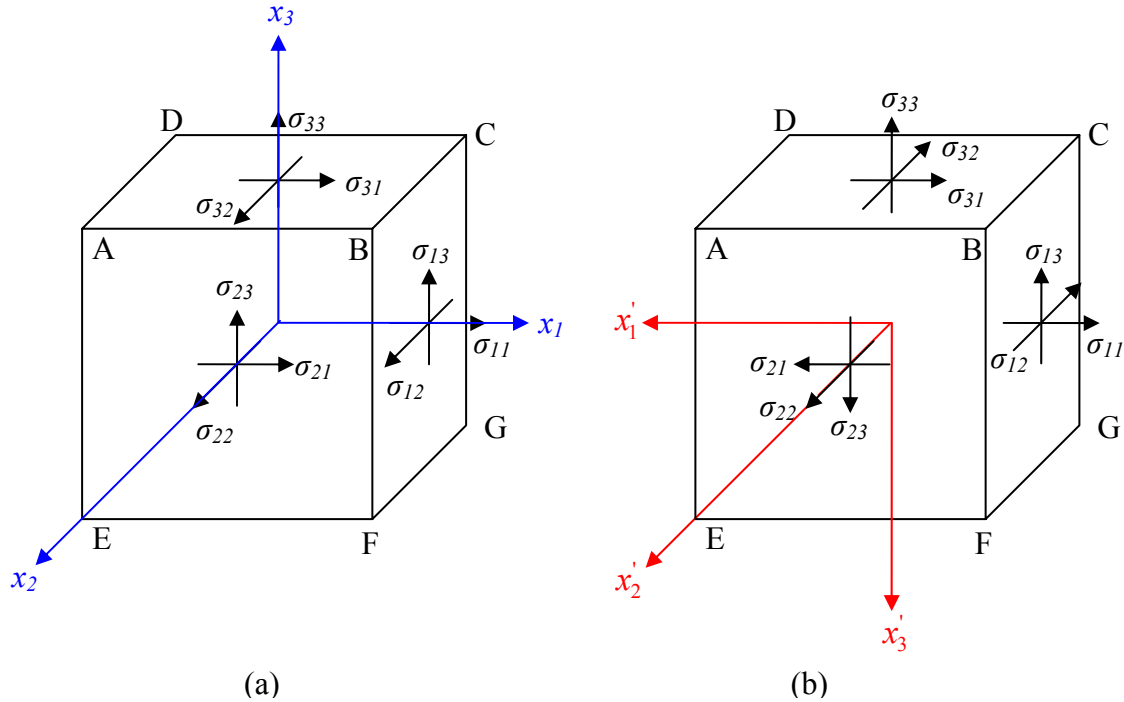


Figure 4

Now using the stiffness matrix given in Eq. (20) and comparing the stress equivalence of Eq. (27) we get the following:

$$\sigma_{11} = \sigma'_{11}$$

$$C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + C_{16}\varepsilon_6 = C'_{11}\varepsilon'_1 + C'_{12}\varepsilon'_2 + C'_{13}\varepsilon'_3 + C'_{16}\varepsilon'_6$$

This hold true when $C_{16} = 0$. Similarly,

$$\sigma_{22} = \sigma'_{22} \text{ gives } C_{26} = 0$$

$$\sigma_{33} = \sigma'_{33} \text{ gives } C_{36} = 0$$

$$\sigma_{23} = -\sigma'_{23} \text{ (or } \sigma_{13} = \sigma'_{13}) \text{ gives } C_{45} = 0$$

This gives us the C_{ij} matrix as in Eq. (26).

Alternately, if we consider x_1 - x_3 as the second plane of material symmetry along with x_1 - x_2 as shown in Figure 5, then

$$x'_1 = x_1, x'_2 = -x_2, x'_3 = -x_3 \quad (28)$$

And

$$a_{ij} = \frac{\partial x'_i}{\partial x_j} \quad \text{and} \quad \frac{\partial x'_i}{\partial x_j} = -\delta_{ij} \quad \text{for } j = 2, 3 \quad \text{and} \quad \frac{\partial x'_i}{\partial x_1} = \delta_{i1} \quad (29)$$

This gives us the required strain relations as (from Eq. (12))

$$\varepsilon'_{11} = \varepsilon_{11}, \varepsilon'_{22} = \varepsilon_{22}, \varepsilon'_{33} = \varepsilon_{33}, \varepsilon'_{23} = \varepsilon_{23}, \varepsilon'_{13} = -\varepsilon_{13}, \varepsilon'_{12} = -\varepsilon_{12}$$

Substituting these in Eq. (18) the function W reduces again to the form given in Eq. (25) for W to be invariant. Finally, we get the reduced stiffness matrix as given in Eq. (26).

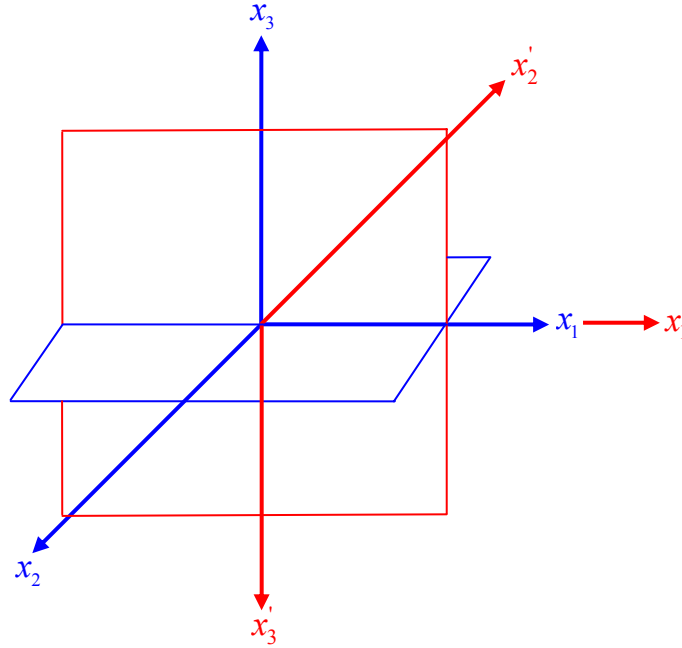


Figure 5

The stress transformations for these coordinates transformations are (from the first of Eq. (12) and Eq. (29))

$$\sigma'_{11} = \sigma_{11}, \sigma'_{22} = \sigma_{22}, \sigma'_{33} = \sigma_{33}, \sigma'_{23} = \sigma_{23}, \sigma'_{13} = -\sigma_{13}, \sigma'_{12} = -\sigma_{12}$$

Same can be seen from the stresses shown on the same cube in x_1, x_2, x_3 and x'_1, x'_2, x'_3 coordinate systems in Figure 6(a) and (b), respectively. The comparison of the stress terms leads to the stiffness matrix as given in Eq. (26).

Note: It is clear that if any two orthogonal planes are planes of material symmetry the third mutually orthogonal plane has to be plane to material symmetry. We have got the same stiffness matrix when we considered two sets of orthogonal planes. Further, if we proceed in this way considering three mutually orthogonal planes of symmetry then it is not difficult to see that the stiffness matrix remains the same as in Eq. (26).

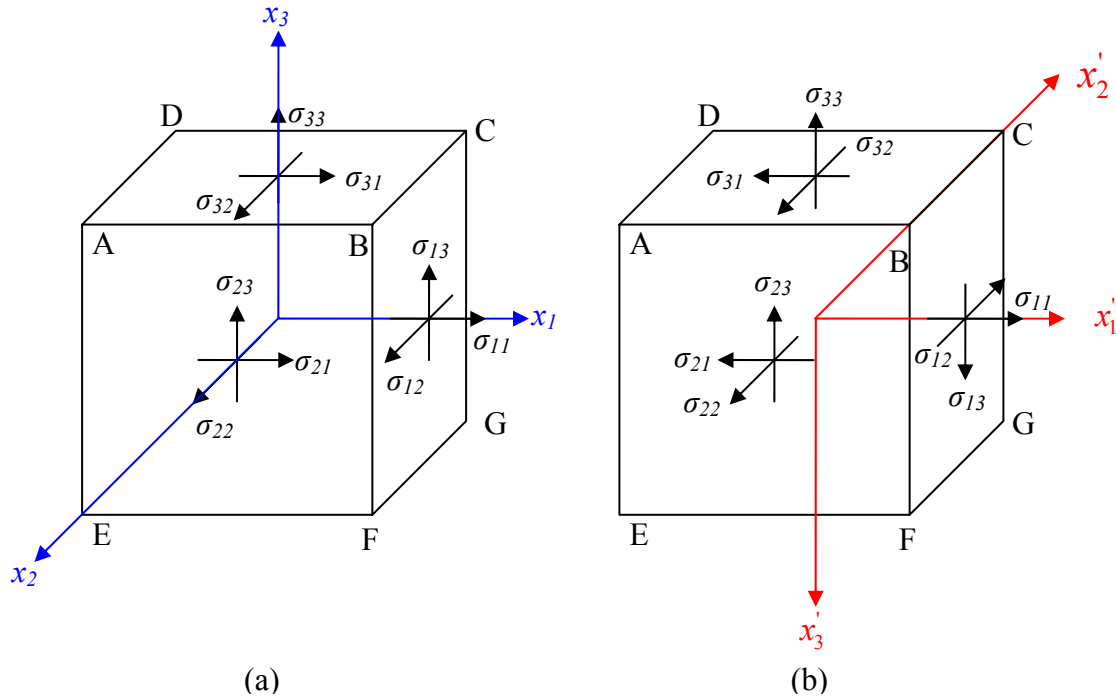


Figure 6

Transverse Isotropy:

First Approach: Invariance Approach

This is obtained from an orthotropic material. Here, we develop the constitutive relation for a material with transverse isotropy in x_2 - x_3 plane (this is used in lamina/laminae/laminate modeling). This is obtained with the following form of the change of axes

$$\begin{aligned}
 x'_1 &= x_1 \\
 x'_2 &= x_2 \cos \alpha + x_3 \sin \alpha \\
 x'_3 &= -x_2 \sin \alpha + x_3 \cos \alpha
 \end{aligned}
 \tag{30}$$

Now, we have

$$\frac{\partial x'_1}{\partial x_1} = 1, \quad \frac{\partial x'_2}{\partial x_2} = \frac{\partial x'_3}{\partial x_3} = \cos \alpha, \quad \frac{\partial x'_2}{\partial x_3} = -\frac{\partial x'_3}{\partial x_2} = \sin \alpha,$$

$$\frac{\partial x'_1}{\partial x_2} = \frac{\partial x'_1}{\partial x_3} = \frac{\partial x'_2}{\partial x_1} = \frac{\partial x'_3}{\partial x_1} = 0$$

From this, the strains in transformed coordinate system are given as:

$$\begin{aligned} \varepsilon'_{11} &= \varepsilon_{11} \\ \varepsilon'_{22} &= \varepsilon_{22} \cos^2 \alpha + 2\varepsilon_{23} \cos \alpha \sin \alpha + \varepsilon_{33} \sin^2 \alpha \\ \varepsilon'_{33} &= \varepsilon_{22} \sin^2 \alpha - 2\varepsilon_{23} \cos \alpha \sin \alpha + \varepsilon_{33} \cos^2 \alpha \\ \varepsilon'_{23} &= (\varepsilon_{33} - \varepsilon_{22}) \cos \alpha \sin \alpha + \varepsilon_{23} (\cos^2 \alpha - \sin^2 \alpha) \\ \varepsilon'_{13} &= -\varepsilon_{12} \sin \alpha + \varepsilon_{13} \cos \alpha \\ \varepsilon'_{12} &= \varepsilon_{12} \cos \alpha + \varepsilon_{13} \sin \alpha \end{aligned} \quad (31)$$

Here, it is to be noted that the shear strains are the tensorial shear strain terms. For any angle α ,

$$\begin{aligned} \varepsilon'_{22} + \varepsilon'_{33} &= \varepsilon_{22} + \varepsilon_{33}, & \varepsilon'_{22}\varepsilon'_{33} - (\varepsilon'_{23})^2 &= \varepsilon_{22}\varepsilon_{33} - (\varepsilon_{23})^2, \\ (\varepsilon'_{12})^2 + (\varepsilon'_{13})^2 &= (\varepsilon_{12})^2 + (\varepsilon_{13})^2 & |\varepsilon'_{ij}| &= |\varepsilon_{ij}| \end{aligned} \quad (32)$$

and therefore, W must reduce to the form

$$W = W(\varepsilon_{22} + \varepsilon_{33}, \varepsilon_{22}\varepsilon_{33} - \varepsilon_{23}^2, \varepsilon_{33}, \varepsilon_{12}^2 + \varepsilon_{13}^2, |\varepsilon_{ij}|) \quad (33)$$

Now, we put Eq. (31) in the function W using the C_{ij} matrix as given in Eq. (26) and express it terms of ε'_{ij} . Then, for W to be invariant follow the following:

1. If we observe the terms containing $(\varepsilon_{11})^2$ and $(\varepsilon'_{11})^2$ then we conclude that C_{11} is unchanged.
2. Now compare the terms containing $\varepsilon_{12}, \varepsilon_{13}$ and $\varepsilon'_{12}, \varepsilon'_{13}$. From this comparison we see that $C_{55} = C_{66}$.
3. Now compare the terms containing $\varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}$ and $\varepsilon'_{22}, \varepsilon'_{33}, \varepsilon'_{23}$. This comparison leads to $C_{12} = C_{13}, C_{22} = C_{33}$ and $C_{44} = \frac{1}{2}(C_{22} - C_{23})$ and C_{23} is unchanged.

Thus, for transversely isotropic material (in plane x_2 - x_3) the stiffness matrix becomes as

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{22} - C_{23}) & 0 & 0 \\ \text{Symmetric} & & & & C_{66} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad (34)$$

Thus, there are only 5 independent elastic constants for a transversely isotropic material.

Second Approach: Comparison of constants

This can also be verified from the elastic constants expressed in terms of engineering constants like E , ν and G . Recall the constitutive equation for orthotropic material expressed in terms of engineering constants. For the transversely isotropic materials the following relations hold.

$$E_2 = E_3 \quad \nu_{12} = \nu_{13}$$

$$G_{12} = G_{13} \quad G_{23} = \frac{E_2}{2(1 + \nu_{23})}$$

When these relations are used in the constitutive equation for orthotropic material expressed in terms of engineering constants, the stiffness matrix relations in Eq. (34) are verified.

Isotropic Bodies:

If the function W remains unaltered in form under all possible changes to other rectangular Cartesian systems of axes, the body is said to be *Isotropic*. In this case, W is a function of the strain invariants. Alternatively, from the previous section, W must be unaltered in form under the transformations

$$\begin{aligned} x_1' &= x_1 \cos \alpha + x_2 \sin \alpha \\ x_2' &= -x_1 \sin \alpha + x_2 \cos \alpha \\ x_3' &= x_3 \end{aligned} \quad (35)$$

And

$$\begin{aligned}
 x_3' &= x_3 \cos \alpha + x_1 \sin \alpha \\
 x_1' &= -x_3 \sin \alpha + x_1 \cos \alpha \\
 x_2' &= x_2
 \end{aligned} \tag{36}$$

In other words, W when expressed in terms of ε_{ij}' must be obtained from Eq. (33) simply by replacing ε_{ij} by ε_{ij}' . By analogy with the previous section it is seen that for this to be true under the transformation Eq. (35)

$$\begin{aligned}
 C_{13} &= C_{23} \\
 C_{11} &= C_{22}
 \end{aligned} \tag{37}$$

$$C_{66} = C_{44} = \frac{1}{2}(C_{11} - C_{12})$$

It follows automatically that W is unaltered in form under the transformation in Eq. (36). Thus, the stiffness matrix for isotropic material becomes as

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ \text{Symmetric} & & & & \frac{1}{2}(C_{11} - C_{12}) & 0 \\ & & & & & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \tag{38}$$

Thus, for an isotropic material there are only two independent elastic constants. It can be verified that W is unaltered in form under all possible changes to other rectangular coordinate systems, that is, it is the same function of ε_{ij}' as it is of ε_{ij} when x_i is changed to x_i' .