APPENDIX A

A.1. NORMED LINEAR SPACES, LINEAR FUNCTIONALS, AND BILINEAR FORMS

In the following the definitive properties of normed linear spaces, linear functionals, and bilinear forms are listed. α and β denote real numbers.

A.1.1. Normed Linear Spaces

A normed linear space X is a family of elements $u, v, \ldots$ which have the following properties:

1. If $u \in X$ and $v \in X$ then $(u + v) \in X$.
2. If $u \in X$ then $\alpha u \in X$.
3. $u + v = v + u$.
4. $u + (v + w) = (u + v) + w$.
5. There is a unique element in X, denoted by 0, such that $u + 0 = u$ for any $u \in X$.
6. Associated with every element $u \in X$ is a unique element $-u \in X$ such that $u + (-u) = 0$.
7. $\alpha(u + v) = \alpha u + \alpha v$.
8. $(\alpha + \beta)u = \alpha u + \beta u$.
9. $\alpha(\beta u) = (\alpha\beta)u$.
10. $1 \cdot u = u$.
11. $0 \cdot u = 0$.

12. With every $u \in X$ we associate a real number $\|u\|_X$, called *the norm*. The norm has the following properties:

    (a) $\|u + v\|_X \leq \|u\|_X + \|v\|_X$. This is called the *triangle inequality*.
    (b) $\|\alpha u\|_X = |\alpha| \, \|u\|_X$.
    (c) $\|u\|_X \geq 0$.
    (d) $\|u\|_X \neq 0$ if $u \neq 0$.

A.1.2. Linear Functionals

Let X be a normed linear space and $\mathscr{F}(v)$ a process which associates with every $v \in X$ a real number $\mathscr{F}(v)$. $\mathscr{F}(v)$ is called a *linear functional* or *linear form* on X if it has the following properties:

1. $\mathscr{F}(v_1 + v_2) = \mathscr{F}(v_1) + \mathscr{F}(v_2)$.
2. $\mathscr{F}(\alpha v) = \alpha \mathscr{F}(v)$.
3. $|\mathscr{F}(v)| \leq C\|v\|_X$ with C independent of v. The smallest possible value of C is called the *norm* of $\mathscr{F}$.

A.1.3. Bilinear Forms

Let X and Y be normed linear spaces and $\mathscr{B}(u, v)$ a process which associates with every $u \in X$ and $v \in Y$ a real number $\mathscr{B}(u, v)$. $\mathscr{B}(u, v)$ is a *bilinear form* on $X \times Y$ if it has the following properties:

1. $\mathscr{B}(u_1 + u_2, v) = \mathscr{B}(u_1, v) + \mathscr{B}(u_2, v)$.
2. $\mathscr{B}(u, v_1 + v_2) = \mathscr{B}(u, v_1) + \mathscr{B}(u, v_2)$.
3. $\mathscr{B}(\alpha u, v) = \alpha \mathscr{B}(u, v)$.
4. $\mathscr{B}(u, \alpha v) = \alpha \mathscr{B}(u, v)$.
5. $|\mathscr{B}(u, v)| \leq C\|u\|_X \|v\|_Y$ with C independent of u and v. The smallest possible value of C is called the *norm* of $\mathscr{B}$.

The space X is called *trial space* and functions $u \in X$ are called *trial functions*. The space Y is called *test space* and functions $v \in Y$ are called *test functions*. $\mathscr{B}(u, v)$ is not necessarily symmetric.

A.2. CONVERGENCE IN THE SPACE *X*

A sequence of functions $u_n \in X$ $(n = 1, 2, \dots)$ converges in the space X to the function $u \in X$ if for every $\varepsilon > 0$ there is a number n_ε such that for any $n > n_\varepsilon$ the following relationship holds:

$$\|u - u_n\|_X < \varepsilon.$$

A.3. THE SCHWARZ INEQUALITY FOR INTEGRALS

Definition: the function $f(x)$ defined on the interval $a < x < b$ is *square integrable* if:

$$\int_a^b f^2\, dx < \infty.$$

Let $f(x)$ and $g(x)$ be square integrable functions defined on the interval $a < x < b$. Then:

$$\left| \int_a^b fg\, dx \right| \leq \left(\int_a^b f^2\, dx \right)^{1/2} \left(\int_b^b g^2\, dx \right)^{1/2}. \tag{A.3.1}$$

This is the Schwarz inequality for integrals. To prove this inequality we observe that:

$$\int_a^b (f + \lambda g)^2\, dx \geq 0 \quad \text{for any } \lambda \tag{A.3.2a}$$

and therefore:

$$\int_a^b f^2\, dx + 2\lambda \int_a^b fg\, dx + \lambda^2 \int_a^b g^2\, dx \geq 0 \quad \text{for any } \lambda. \tag{A.3.2b}$$

On the left of this inequality is a quadratic expression for λ. To find the roots of this expression we need only to compute:

$$\lambda = \frac{-\int_a^b fg\, dx \pm \sqrt{(\int_a^b fg\, dx)^2 - \int_a^b g^2\, dx \int_a^b f^2\, dx}}{\int_a^b g^2\, dx}. \tag{A.3.3}$$

Denoting the roots by λ_1 and λ_2 (A.3.2b) can be written as: $(\lambda - \lambda_1)(\lambda - \lambda_2) \geq 0$. We now observe that the roots cannot be real and simple because then we could select any λ so that $\lambda_1 < \lambda < \lambda_2$ and we would have $(\lambda - \lambda_1)(\lambda - \lambda_2) < 0$. Therefore the radicand must be less than or equal to zero. This completes the proof.

A.4. LEGENDRE POLYNOMIALS

The Legendre polynomials $P_n(x)$ are solutions of the *Legendre differential equation* for $n = 0, 1, 2, \ldots$:

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0, \qquad -1 \leq x \leq 1. \tag{A.4.1}$$

The first eight Legendre polynomials are:

$$P_0(x) = 1 \tag{A.4.2a}$$

$$P_1(x) = x \tag{A.4.2b}$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1) \tag{A.4.2c}$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \tag{A.4.2d}$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \tag{A.4.2e}$$

$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \tag{A.4.2f}$$

$$P_6(x) = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \tag{A.4.2g}$$

$$P_7(x) = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x). \tag{A.4.2h}$$

Legendre polynomials can be generated from *Bonnet's recursion formula*:

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x), \qquad n = 1, 2, \ldots \tag{A.4.3}$$

and Legendre polynomials satisfy the following relationship:

$$(2n + 1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \qquad n = 1, 2, \ldots \tag{A.4.4}$$

where the primes represent differentiation with respect to x. Legendre polynomials satisfy the following orthogonality property:

$$\int_{-1}^{+1} P_i(x)P_j(x)\,dx = \begin{cases} \dfrac{2}{2i+1} & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{A.4.5}$$

All roots of Legendre polynomials occur in the interval $-1 < x < +1$. The n roots of $P_n(x)$ are the abscissas x_i for the n-point *Gaussian integration*:

$$\int_{-1}^{+1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i). \tag{A.4.6}$$

The abscissas x_i and weight factors w_i for Gaussian integration are listed in Table A.4.1.