Patch Test:

For D.E $\quad -\dfrac{d^2u}{dx^2} = f = \begin{cases} 0 & - \; u_{ex} \text{ is linear} \\ 1 & - \; u_{ex} \text{ is quadratic} \\ x & - \; u_{ex} \text{ is cubic} \end{cases}$

$0 \leq x \leq 1$

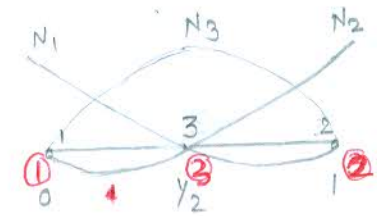
BC: $\quad u(0) = 0, \quad \left.\dfrac{du}{dx}\right|_{x=1} = 0$

u_h is also exact when $\quad f=0 \quad$ linear elements

$f=1 \quad$ quadratic "

$f=x \quad$ cubic " are used.

We find u_h for the case $f=1$ with one and two quadratic ~~linear~~ elements, respectively. $\quad \boxed{u_{ex} = x - \dfrac{x^2}{2}}$



Global nodes - ①, ② ③

local nodes - 1, 2, 3

$$N_1 = \frac{(x-1/2)(x-1)}{(0-1/2)(0-1)} = 2x^2 - 3x + 1$$

$$N_2 = \frac{(x-0)(x-1/2)}{(1-0)(1-1/2)} = 2x^2 - x$$

$$N_3 = \frac{(x-0)(x-1)}{(1/2-0)(1/2-1)} = 4x - 4x^2$$

$$u \approx u_h = u_1' N_1 + u_2' N_2 + u_3' N_3$$
$$= U_1 N_1 + U_2 N_2 + U_3 N_3$$

Weak form:

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}\,dx = \int_0^1 f \cdot w\,dx - \sum_{j=1}^{3} Q_j\, W(x_j)$$

putting $u = u_h$ and $w = N_i$, we get

$$k_{ij}' = \int_0^1 \frac{dN_j}{dx}\frac{dN_i}{dx}\,dx \qquad f_i = \int_0^1 f N_i\,dx, \quad Q_i =$$

putting N_i's we get

$$[K'] = \begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{8}{3} \\ \frac{1}{3} & \frac{7}{3} & -\frac{8}{3} \\ -\frac{8}{3} & -\frac{8}{3} & \frac{16}{3} \end{bmatrix}, \quad \{f'\} = \begin{Bmatrix} 1/6 \\ 1/6 \\ 4/6 \end{Bmatrix}, \quad \{Q'\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

Final system of equations:

$$\begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{8}{3} \\ \frac{1}{3} & \frac{7}{3} & -\frac{8}{3} \\ -\frac{8}{3} & -\frac{8}{3} & \frac{16}{3} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 1/6 \\ 1/6 \\ 4/6 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 = 0 \\ Q_3 = 0 \end{Bmatrix} \longrightarrow Q_3 = \frac{du}{dx}\Big|_{x=1} = 0$$

imposed

To impose $u(0) = 0 = U_1$

imposed

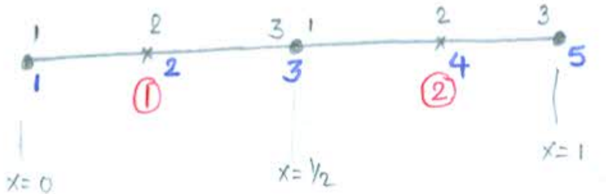
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7/3 & -8/3 \\ 0 & -8/3 & 16/3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1/6 \\ 4/6 \end{Bmatrix} + \begin{Bmatrix} Q_1 = 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{1}{3} u(0) \\ \frac{8}{3} u(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1/6 \\ 4/6 \end{Bmatrix}$$

Solving : $U_1 = 0$

$U_2 = \frac{1}{2}$ and $U_3 = \frac{3}{8}$

$$u_h = U_1 N_1 + U_2 N_2 + U_3 N_3 = x - \frac{x^2}{2}$$

F. E solution with two quadratic elements:



①, ② - element nos.

1, 2, 3, 4, 5 - global dof numbers

1, 2, 3 - local dof numbers

$$N_1^1 = \frac{(x-1/4)(x-1/2)}{(0-1/4)(0-1/2)} = 8x^2 - 6x + 1$$

$$N_2^1 = \frac{(x-0)(x-1/2)}{(1/4-0)(1/4-1/2)} = -16x^2 + 8x$$

$$N_3^1 = \frac{(x-0)(x-1/4)}{(1/2-0)(1/2-1/4)} = 8x^2 - 2x$$

$$N_1^2 = \frac{(x-3/4)(x-1)}{(1/2-3/4)(1/2-1)} = 8x^2 - 14x + 6$$

$$N_2^2 = \frac{(x-1/2)(x-1)}{(3/4-1/2)(3/4-1)} = -16x^2 + 24x - 8$$

$$N_3^2 = \frac{(x-1/2)(x-1)}{(1/4-1/2)(1/4-1)} = 8x^2 - 10x + 3$$

Element equations for the first element

$$k^1_{ij} = \int_0^{1/2} \frac{dN^1_j}{dx}\frac{dN^1_i}{dx}\,dx\,, \qquad f^1_i = \int_0^{1/2} f \cdot N^1_i\,dx$$

$$[k^1] = \begin{bmatrix} 14/3 & -16/3 & 2/3 \\ -16/3 & 32/3 & -16/3 \\ 2/3 & -16/3 & 14/3 \end{bmatrix}, \quad \{f^1\} = \begin{Bmatrix} 1/12 \\ 1/3 \\ 1/12 \end{Bmatrix}, \quad \{Q^1\} = \begin{Bmatrix} Q^1_1 \\ Q^1_2 \\ Q^1_3 \end{Bmatrix}$$

for 2nd element

$$k^2_{ij} = \int_{1/2}^{1} \frac{dN^2_j}{dx}\frac{dN^2_i}{dx}\,dx\,, \qquad f^2_i = \int_{1/2}^{1} f\,N^2_i\,dx$$

$$[k^2] = \begin{bmatrix} 14/3 & -16/3 & 2/3 \\ -16/3 & 32/3 & -16/3 \\ 2/3 & -16/3 & 14/3 \end{bmatrix}, \quad \{f^2\} = \begin{Bmatrix} 1/12 \\ 1/3 \\ 1/12 \end{Bmatrix}, \quad \{Q^2\} = \begin{Bmatrix} Q^2_1 \\ Q^2_2 \\ Q^2_3 \end{Bmatrix}$$

Assembling element stiffness matrices

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 14/3 & -16/3 & 2/3 & 0 & 0 \\ -16/3 & 32/3 & -16/3 & 0 & 0 \\ 2/3 & -16/3 & 14/3 + 14/3 & -16/3 & 2/3 \\ 0 & 0 & -16/3 & 32/3 & -16/3 \\ 0 & 0 & 2/3 & -16/3 & 14/3 \end{bmatrix} \end{array} = [K]_{5\times5}$$

Assembling element f vectors

$$\{f\} = \begin{Bmatrix} 1/12 \\ 1/3 \\ 1/12 + 1/12 \\ 1/3 \\ 1/12 \end{Bmatrix}$$

Assembling element $\{\dot{Q}\}$ vectors

$$\{Q\} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_2^2 \\ Q_3^2 \end{Bmatrix}$$

Imposition of boundary conditions:

$$\left.\frac{du}{dx}\right|_{x=1} = F = D = Q_3^2 = 0$$

There are not externally applied point forces. Hence, all other entries of $\{Q\}$ are zero.

$$u(0) = 0 = U_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 32/3 & -16/3 & 0 & 0 \\ 0 & -16/3 & 28/3 & -16/3 & 2/3 \\ 0 & 0 & -16/3 & 32/3 & -16/3 \\ 0 & 0 & 2/3 & -16/3 & 14/3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1/3 \\ 1/6 \\ 1/3 \\ 1/12 \end{Bmatrix}$$

solving, we get $U_1 = 0, \quad U_2 = \frac{7}{32}, \quad U_3 = \frac{3}{8}, \quad U_4 = \frac{15}{32}, \quad U_5 = \frac{1}{2}$

Solution interpolation:

$$u_h^1(x) = \sum_{i=1}^{3} u_i^1 N_i^1(x) = U_1 N_1^1 + U_2 N_2^1 + U_3 N_3^1 = x - \frac{x^2}{2}$$

$$u_h^2(x) = \sum_{i=1}^{3} u_i^2 N_i^2(x) = U_3 N_1^2 + U_4 N_2^2 + U_5 N_3^2 = x - \frac{x^2}{2}$$

New global dof numbering

Assembling element stiffness matrices

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
14/3 & 2/3 & 0 & -16/3 & 0 \\
2/3 & 14/3+14/3 & \frac{2}{3} & -16/3 & -16/3 \\
0 & 2/3 & 0 & 14/3 & -16/3 \\
-16/3 & -16/3 & 0 & 32/3 & 0 \\
0 & -16/3 & -16/3 & & 32/3
\end{bmatrix}
\end{array}
= [K]
$$

Assembling element $\{f\}$ and $\{Q\}$ vectors

$$
\{Q\} = \begin{Bmatrix} Q_1^1 \\ Q_3^1 + Q_1^2 \\ Q_3^2 \\ Q_2^1 \\ Q_2^2 \end{Bmatrix}, \quad
\{f\} = \begin{Bmatrix} f_1^1 \\ f_3^1 + f_1^2 \\ f_3^2 \\ f_2^1 \\ f_2^2 \end{Bmatrix} = \begin{Bmatrix} 1/12 \\ 1/12 + 1/12 \\ 1/12 \\ 1/3 \\ 1/3 \end{Bmatrix}
$$

Applying the BC's; the resulting final system is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
2/3 & 28/3 & 2/3 & -16/3 & -16/3 \\
0 & 2/3 & 14/3 & \cancel{14/3}\ 0 & -16/3 \\
0 & -16/3 & 0 & 32/3 & 0 \\
0 & -16/3 & -16/3 & 0 & 32/3
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 1/6 \\ 1/12 \\ 1/3 \\ 1/3 \end{Bmatrix}
$$

solving: $U_1 = 0$, $U_2 = 3/8$, $U_3 = 1/2$, $U_4 = 7/32$, $U_5 = 15/32$

$$u_h^1(x) = U_1 N_1^1 + U_4 N_2^1 + U_2 N_3^1 = x - x^2/2$$

$$u_h^2(x) = U_2 N_1^2 + U_5 N_2^2 + U_3 N_3^2 = x - x^2/2$$