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Regionwise Modeling Approach for the Analysis of Layered Structures

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Abstract: In this paper families of equivalent single layer, intermediate and layerwise plate models are presented. The equivalent single layer models are accurate when used with equilibrium based postprocessing in the regions where three dimensional effects are less pronounced. Although, the layerwise model is very accurate pointwise with respect to all the stress components, this model is computationally expensive when number of layers are large. A novel regionwise modeling strategy is developed where any model can be put in any region of the domain. It is seen that by using layerwise model in the region where solution is predominantly three-dimensional in nature and a lower model elsewhere, same accuracy as layerwise model everywhere is achieved at significant lower computational cost.

Key words: equivalent single layer; intermediate; layerwise; regionwise model, pointwise stress, equilibrium based postprocessing, composite laminates

INTRODUCTION

The laminated structures are finding wide applications in manufacturing of critical components used in aerospace, marine and automobile industry. These structures are generally thin. Several dimensionally reduced models or 2*D* models have been proposed in the literature. These models (popularly known as plate models) are based on either the displacement based (see [1]-[2]) or mixed formulations. In this study, plate models based on the displacement formulation are dealt.

The attractive feature is that the computational cost does not depend on the number of layers. Some of the zig-zag models have also shown to be convergent to the three-dimensional elasticity solution with respect to the strain energy (energy norm). These models focus on representing the transverse shear effects more accurately, by enriching the representation field in the *z*-direction. Another important issue has been that of shear locking in the case of thin plates. Several "locking free" shear deformable models have been proposed in the literature [1].

A major drawback with these models has been that the transverse stresses, obtained using these models, are not accurate. An equilibrium based postprocessing approach can be (see [3]) used to extract the transverse stress components accurately. This approach is quite effective in most cases. However, for domains with unsymmetry in layup, existing delamination, ply level damage and re-entrant corners, the dimensionally reduced models are ineffective. In order to handle these situations a more refined analysis is desired. Layerwise models (for example see [3]) and intermediate models are often used to resolve this issue. In the layerwise models, the standard plate models are applied layerwise, and continuity of displacement (and transverse shear stresses in some cases) is imposed at the interlaminar interfaces. The intermediate model is based on using the dimensionally reduced models piecewise (eg. lump all laminae above and below a delamination separately). The use of these models leads to enhanced resolution of the local effects, but it also increases the size of the problem to be solved. In [3], a generalized layerwise model is given which is based on use of one-dimensional hierarchic basis functions in the transverse and the planar directions. These models allow for use of different approximation orders in the transverse and planar directions. Thus, the transverse order of approximation can be raised for each layer. These models are essentially three-dimensional models and have been shown to be very accurate with respect to any pointwise quantity of interest.

Generally, the strong three-dimensional effects are localized in the vicinity of boundaries (boundary-layer), vertices (vertex singularities), edges (edge and vertex edge singularities), free edges, parts of lamina/laminae (damaged lamina) and local interfaces (interlaminar delamination). An example of such a situation is shown in Figure 1. In these cases, beyond a local neighborhood of the regions of unsmooth behavior (eg. in the vicinity of damaged zone boundaries), the solution can be effectively represented using any of the families of dimensionally reduced models. Hence, the approximation is required to be enriched only in the regions where the solution is unsmooth. This is achieved in this paper by the proposed regionwise modeling approach. The novelty has been to give a generalized computationally implementable procedure to incorporate models of any complexity in any region of interest. As shown in this paper, this approach leads to tremendous savings in computational cost and gives accurate representation of the state of stress in the region of interest. This approach is a generalization of the planar constrained approximation approach of [4] and the $h-d$ approach of [5], given for homogeneous materials.

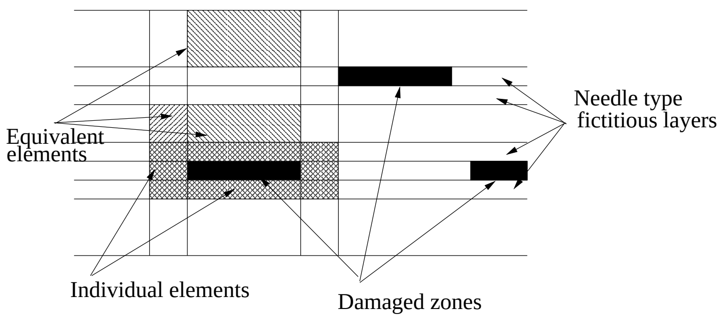


Figure 1: General scenario in laminated composites

PLATE MODELS

Analysis of thin laminated structures is based on using predefined director functions in the z-direction, with the displacement field given as a series in terms of products of the director functions and planar functions. Various families of plate models can be defined based on the specific definitions of the director functions. The plate models employed in this study belong to the families of plate models given below. These models are developed by authors in [3].

1. LAYERWISE MODELS (LM) This is the most general three-dimensional representation of the displacement field. Each lamina is taken as a separate group and the director functions are defined as the one dimensional basis functions defined over the lamina. From Figure 2(a), it can be seen that the representation of the displacement field is given by:

$$\begin{aligned} u_1(x,y,z) &= \textstyle\sum_{i=1}^{n_1} u_{1i}(x,y)\,\overline{M}_i(z) \\ u_2(x,y,z) &= \textstyle\sum_{i=1}^{n_2} u_{2i}(x,y)\,\underline{M}_i(z) \\ u_3(x,y,z) &= \textstyle\sum_{i=1}^{n_3} u_{3i}(x,y)\,\overline{M}_i(z) \end{aligned} \tag{1}$$

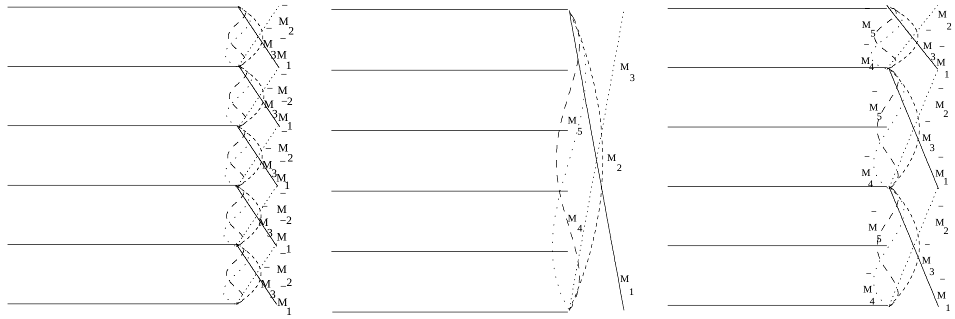
where $n_1 = n_2$ and n_3 depend on the order of approximation $p_z^1 = p_z^2$, p_z^3 and the number of laminae (or layers) nl in the laminate. Hence, here the number of unknowns increases with the number of laminae. Members of this family of models will be represented by $LMp_{xy}p_z^1p_z^2p_z^3$.

2. EQUIVALENT MODELS (EQ) These are conventionally the most popular plate models, with CLPT and HSDT models as special cases. The director functions are polynomials defined over the full thickness as shown in Figure 2(b). Members of this family of models will be represented by $EQp_{xy}p_z^1p_z^2p_z^3$.

3. INTERMEDIATE MODELS (IM) Generally, the critical local quantities of interest are desired in particular lamina or at the interface of two laminae. In this case, the equivalent models cannot be used. The intermediate models are based on defining the director functions for a group of laminae and not the full laminate. The director functions for an intermediate model is shown in Figure 2(c). Members of this family of models will be represented by $IMp_{xy}p_z^1p_z^2p_z^3$.

REGIONWISE MODEL

1. MOTIVATION In a structural component, the “hot-spot” are generally localized in the vicinity of structural details, boundaries of the domain (faces and edges), re-entrant corners, cut-outs, existing delaminations and ply-failure zones. The solution is unsmooth in the vicinity of these details, while it is very smooth in the remaining part of the domain (see Figures 1 and 3). In order to get an accurate representation of the solution everywhere, it is desirable to use an



(a) Layerwise (b) Equivalent (c) Intermediate

Figure 2: Director approximations over layerwise, equivalent and intermediate models

enriched approximation model (LM or IM with sublaminae if desired) only in the vicinity of the "hot-spot", while in the rest of the domain, a lower order model will suffice. In this study, p_{xy} is uniform over the whole domain, while the approximation enrichment is done by using either a higher value of p_z^i and/or a more refined model, e.g. IM or LM. Thus it is important to build the capability to put any desired model in a specified region, rather than doing an overkill by using a higher model everywhere in the domain (which will be computationally very expensive). This concept has been introduced through the regionwise modeling approach described in this section.

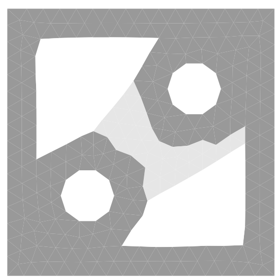


Figure 3: A typical plate domain with cut-outs

2. CONCEPT OF REGIONS Let us consider the domain given in Figure 3, with the two circular cut-outs shown. In the vicinity of the cut-outs and the outer boundaries of the domain (shown grey shaded in Figure 3), the solution is expected to be unsmooth, have severe boundary layer effect, and possibly be three-dimensional in nature locally. Hence along with a refinement of the mesh, enrichment of the model will also be desired in the shaded regions. Thus, the domain is divided into multiple regions (three regions shown by different shades of grey). The plate model is then fixed for each region. For example, for the domain of Figure 3, $EQp_{xy}p_z^1p_z^2p_z^3$ may be used in the unshaded region; $IMp_{xy}p_z^1p_z^2p_z^3$ may be used in the lighter grey region and $LMp_{xy}p_z^1p_z^2p_z^3$ may be used in the region shaded dark grey.

3. CONCEPT OF GROUPS Let the laminate have nl layers, or laminae (this also includes the sublaminae). Since all the models given above have the same representation in terms of the one-dimensional hierarchic basis functions defined over groups of laminae, it is advantageous to define a generic representation of the group structure. The base two-dimensional mesh T_{2D} (with nel_{2D} number of elements) is made first over the projected two-dimensional surface. In this study, meshes of triangles are used. Using the base two-dimensional mesh, the three-dimensional mesh T_{3D} of prismatic elements is made over the whole domain, layer by layer. Hence the number of elements in T_{3D} is $nel_{2D} \times nl$. Each two-dimensional element $\tau_{2D} \subseteq T_{2D}$ is assigned the set of all the nl three-dimensional elements $\tau_{3D} \subseteq T_{3D}$, whose projection on the plane is τ_{2D}. This set is denoted as $P_{\tau_{2D}}$. For each element τ_{2D}, the type of model to be used through the thickness is then specified. The model is fixed by the regionwise allocation described above. Note that two contiguous two-dimensional elements may have the same or different models by this strategy. For the element τ_{2D} we specify the number of groups $ng_{\tau_{2D}}$. For each group $g_{i,\tau_{2D}}, i = 1,2,...,ng_{\tau_{2D}}$, the three-dimensional elements $\tau_{3D} \subseteq P_{\tau_{2D}}$ contained, are specified. Note that the group will contain one or more three-dimensional elements that are stacked on top of each other. Illustrations of some possible groups is given in Figure 4, through a frontal view. The figure also demonstrates the possible interfaces between neighboring groups. Note that in this study, case (g) is not considered.

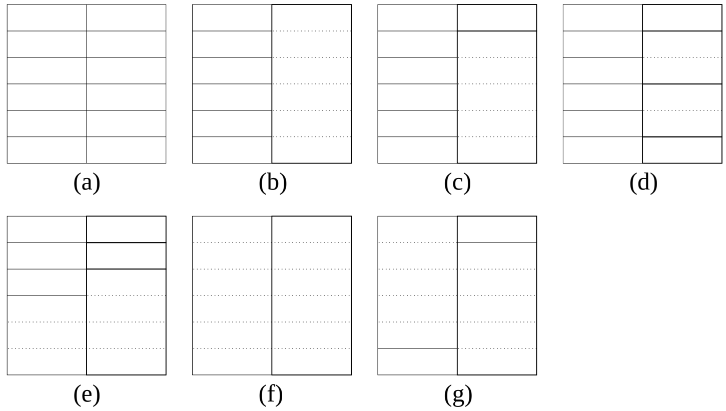


Figure 4: Grouping strategies for regionwise model

4. IMPOSITION OF CONSTRAINTS In this section the concept of constrained approximation will be discussed. The ideas are generalization of the concept introduced in Demkowicz et al [4]. In order to fix ideas let us consider a one-dimensional example. Let us take an interval $(0,L)$ with one element, as shown in Figure 5(a). Let us also assume that piecewise linear basis functions (i.e. $p=1$) are defined over this mesh. Let

$$v(z) = \sum_{i=1}^{p+1} a_i M_i(z) \tag{2}$$

be the representation of a function over this domain. Here, $M_i(z)$ are the linear basis functions defined as shown in Figure 5(a). Let us now subdivide this element into two equal sub-elements, with $\frac{L}{2}$. Over this new mesh of two elements, let the function $v(z)$, given above, be represented in terms of the piecewise linear basis functions (as shown in Figure 5(b)) as

$$v(z) = \sum_{i=1}^{2p+1} \bar{a}_i \bar{M}_i(z) \tag{3}$$

where, $\bar{M}_i(z)$ are the piecewise linear basis functions defined over the new mesh. Since both equations (2) and (3) represent the same function, the coefficients $\bar{a}_i$ can be expressed in terms of the coefficients a_j. It is obvious that

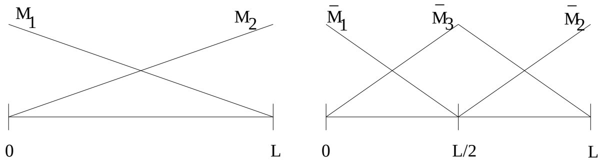
$$\bar{a}_1 = a_1; \bar{a}_3 = \frac{a_1 + a_2}{2}; \bar{a}_2 = a_2 \tag{4}$$

Similarly, the representation of $v(z)$ over any finer mesh can be obtained in terms of the representation over the coarser mesh, with the new fine mesh coefficients $\bar{a}_i$ constrained by the values of the coefficients a_j for the coarser mesh. This can be easily extended to any p-order approximation defined over the coarse and fine meshes. The transverse representation of the finite element solution is defined over a group. However, the basic building block in the analysis is the individual three-dimensional element τ_{3D}. Hence, the approach given above will be employed to represent the element degrees of freedom in terms of the group degrees of freedom.

NUMERICAL RESULTS

The major goal of this paper is to present an approach through which the local three-dimensional state of stress can be obtained in a laminated plate structure, in a designed region of interest, with optimal computational effort. The efficacy of the proposed regionwise model is demonstrated by following two numerical examples. This modeling approach is compared with EQ, LM and IM models.

1. ANTISYMMETRIC LAMINATE UNDER CYLINDRICAL BENDING In this case $[165/-165]$ laminate is considered. The cylindrical bending load is of the form $T_3(x,y) = q_0 \sin\left(\frac{\pi x}{a}\right)$. All the laminae are of equal thickness.



(a) (b)

Figure 5: Constraint imposition

Here, we have taken $t = 2nl$ ($t = 2\,mm$ here) and $a = St$. Further, in the y-direction the plate is taken to be sufficiently long with $b = 20a$. At $x = 0, a$ the edge is point supported while at $y = 0, b$ the edge is free. The normalised stress $\overline{\tau}_{yz}$ is plotted at $\left(0, \frac{b}{2}, \overline{z}\right)$. The regionwise scheme (RR) for this case is shown in Figure 6. In the darker region $LM3332$ model is used. In the region shown by grey shading $LM3112$ model is used followed by the $EQ3112$ model in the remaining region. The stress components for $S = 4$ are compared with exact solution of [6] and are shown in Figure 7. The number of unknowns $LM3332$, $EQ3332$ and regionwise scheme are given in Table 1.

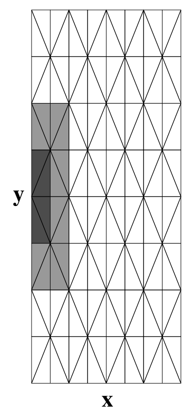
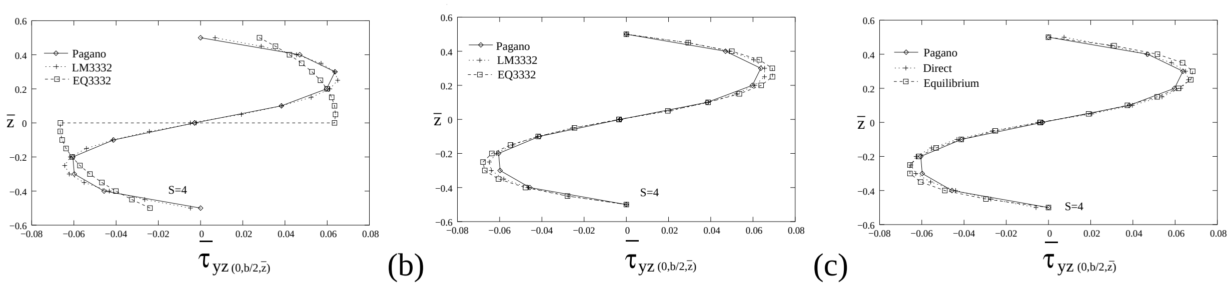


Figure 6: Regionwise scheme for transverse shear stress for $[165/-165]$ laminate



(a) (b) (c)

(a) Direct stresses (b) Equilibrium based postprocessed stresses and (c) Regionwise scheme

Figure 7: Comparison of transverse shear stress for $[165/-165]$ laminate under cylindrical bending

Table 1: Number of unknowns

Model	$LM3332$	$EQ3332$	RR
Unknowns	11875	6875	4863

From the results it is observed that:

1. The stress components obtained directly by the $LM3332$ model are close to the exact values.

2. When the equilibrium based postprocessing is employed the $LM3332$ and $EQ3332$ models give very good values of transverse stress.

3. The stresses obtained both directly and with equilibrium based postprocessing using regionwise scheme are close to the exact values.

2. DOMAIN WITH MULTI-MATERIAL Let us take the rectangular domain of Figure 8(c). The dimensions of the plate are $a = 100\,mm$, $b = 10\,mm$ and $t = 0.254\,mm$. The plate is clamped along all the edges and is subjected to a uniform transverse load of intensity $q_0 = 1\,N/mm^2$. The plate has two lamina with $[0/90]$ in the region $0 \le x \le \frac{a}{2}$, $0 \le y \le b$. The material properties for each lamina are as given in Table 2. In the region $\frac{a}{2} \le x \le a$, $0 \le y \le \frac{b}{2}$, the plate has a bottom layer of epoxy (with $E_{11} = E_{22} = E_{33} = 4600\,MPa$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.36$) for $-\frac{t}{2} \le z \le 0$. For $z \ge 0$, lamina with 90° orientation with material properties given in Table 2 is present (see Figure 8(a)). The plate essentially mimics a L-shaped domain in two-dimensions. For this domain the exact solution will have an edge singularity along the line given by $x = \frac{a}{2}$, $z = 0$. To solve this problem we will use $LM3333$, $EQ3333$ models and the regionwise schemes given by:

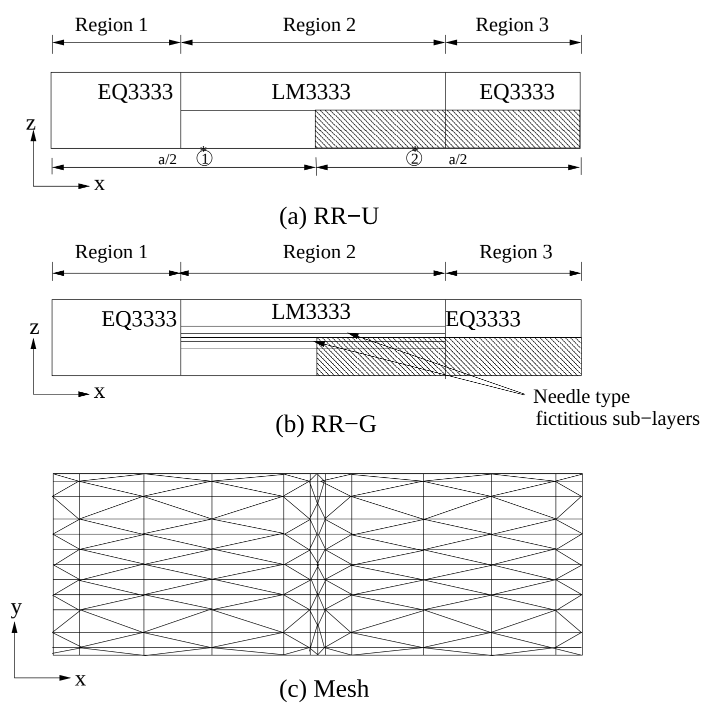


Figure 8: Beam geometry and mesh. (a) $RR-U$ model with point 1 and 2, where through thickness variation is given, (b) $RR-G$ model and (c) the two-dimensional mesh

Table 2: Material properties for $T300/5208$ Graphite/Epoxy (Pre-preg)

Property	Value	Property	Value
E_{11}	132.5 GPa	X_T	1515 MPa
$E_{22} = E_{33}$	10.8 GPa	X_C	1697 MPa
$G_{12} = G_{13}$	5.7 GPa	$Y_T = Z_T = Y_C = Z_C$	43.8 MPa
G_{23}	3.4GPa	R	67.6 MPa
$\nu_{12} = \nu_{13}$	0.24	$S = T$	86.9 MPa
ν_{23}	0.49	Ply thickness, t_i	0.127 mm

1. $RR-U$ (Figure 8 (a)): $EQ3333$ in region 1 and 3, $LM3333$ model in region 2.

2. $RR-G$ (Figure 8 (b)): $EQ3333$ in region 1 and 3, $LM3333$ model in region 2, with geometrically graded sublaminae (with factor $q = 0.15$) near $z = 0$.

Note that in the RR strategies the $3D$ model is used in the vicinity of the singular edge only. Elsewhere, lower models are used. The two-dimensional mesh is as shown in Figure 8(c), with geometrically graded elements (with grading factor q=0.15) in the vicinity of the line $x = \frac{a}{2}$. The energy norm of the discretization error for this mesh was 2.84% (when the $EQ3333$ model is used everywhere).

The through thickness variation of the stress components is given for two points P_1 and P_2. The equlibrium based postprocessed stress σ_{zz} obtained with $EQ3333$, $LM3333$ and $RR-U$ model is shown in Figure 9. From the figures we observe that:

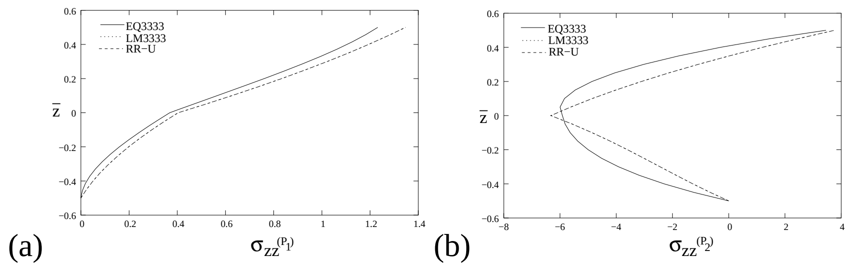


Figure 9: Transverse normal stress (equilibrium based post-processing) for beam without needle elements. (a) At point 1 and (b) at point 2

1. All the stress components, obtained by the $EQ3333$ models, are quantitatively and qualitatively different from those obtained by the $LM3333$ model.

2. The $RR-U$ strategy gives stress values that are very close to those obtained by the $LM3333$ model.

Here, the $LM3333$ model was our benchmark. The pointwise stresses due to the $RR-U$ scheme are very close those obtained by the $LM3333$ model.

In order to resolve the unsmoothness in the solution better, graded sublaminae were used in the $RR-G$ model in the vicinity of the singular edge. The $RR-G$ strategy is compared with the $RR-U$ strategy in Figure 10. From the figures we observe that the $RR-U$ and $RR-G$ strategy give postprocessed stresses very close (overlapping lines). The example clearly demonstrates the ineffectiveness of the EQ models. It further demonstrates the effectiveness of the $RR-U$ and $RR-G$ strategies. Note that with the $RR-G$ strategy, the value of σ_{zz} at the top reaches closer to the exact value of 1. Further graded sublaminae and mesh has to be used in the vicinity of singular edge to get the pointwise value of the transverse stress components at the given point more accurately. This can be achieved by using a suitable a-posteriori error estimator for modeling error, along with the $RR-G$ strategy.

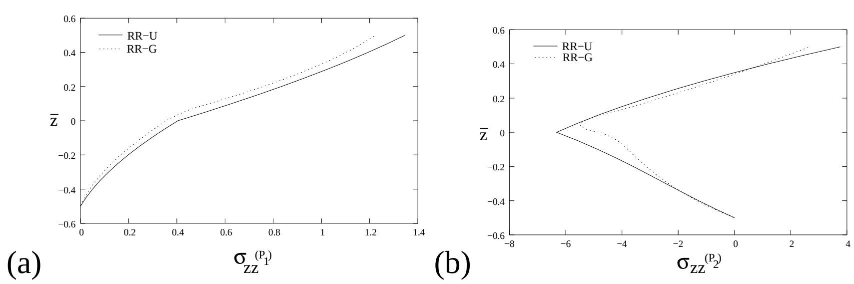


Figure 10: Transverse normal stress (equilibrium based post-processing) for beam with needle elements. (a) At point 1 and (b) at point 2

In Table 3, the number of unknowns for each of the solution strategies, is given. From the table it is clear that the $RR-U$ or $RR-G$ strategy leads to significant savings in the computational cost as compared to the LM model.

Table 3: Number of unknows for domain with multi-material region

Model	Throughout $LM3333$	Throughout $EQ3333$	$RR-U$	$RR-G$
Unknowns	50421	28812	38955	92757

CONCLUSIONS

In this study a new regionwise modeling approach was introduced. The major conclusions resulting from this study are given below.

1. The layerwise model accurately captures the local state of stress.
2. The transverse stress components computed by direct use of finite element data for equivalent model are significantly different both qualitatively and quantitatively. However, when these stresses are computed using equilibrium based postprocessing approach they are in good agreement with exact one.
3. The concept of regionwise modeling, with different models in different regions of the domain has been proposed and implemented successfuly.
4. The regionwise modeling approach is as accurate as layerwise model and computationally very economical.

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