A STUDY OF VARIOUS MODELING APPROACHES FOR ANALYSIS OF LAMINATED COMPOSITE PLATES

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In the design and certification process accurate computation of point-wise data like maximum transverse displacement, stress at a point play an important role. These point-wise data in-turn are used to determine the critical quantities, for example stresses are used in calculating the maximum failure index. Maximum transverse displacement may be used as an optimization constraint. Accurate evaluation of these quantities is important in order to make correct design decisions. The present paper emphasizes on the accuracy of different plate models in predicting these quantities. In this study three models (1) HSDT (Conventional), (2) Hierarchic and (3) Layerwise are compared for the transverse displacement and stresses. The in-plane stresses obtained by Conventional and Hierarchic models are in good agreement with the exact one while the transverse normal and shear stresses are significantly different both qualitatively and quantitatively. Whereas, all the stress components obtained by the present Layer-wise model are very close to the exact components. Further, for all the models the transverse normal and shear stresses were also computed using the equilibrium equations. It is seen that the stresses thus obtained for Conventional and Hierarchic models are qualitatively in agreement with the exact one but quantitatively they can be quite different. For the Layer-wise model these stresses are in good agreement both qualitatively and quantitatively. The effect of the accuracy of the computed stress quantities, on the first ply failure criterion is also demonstrated in this study.

Keywords: composite, transverse displacement, stresses, equilibrium stresses, first-ply-failure.

1 INTRODUCTION

Composites are increasingly used for fabrication of lightweight components. Thin plates and shells are often used in aircraft wings, fuselage and auxillary devices. In the analysis or design of laminated structures the critical quantities of interest like maximum transverse deflection, stress, buckling load, first ply failure load etc. have to be evaluated accurately.

Many modeling approaches are available in literature for the analysis and design of laminated plates. These are Higher Order Shear Deformable Theory (HSDT or Conventional) proposed by [1], Hierarchic model by [2] and Layerwise model by [3]. In the present study all these models are used to see their efficacy in evaluating these quantities of interest accurately.

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2 PLATE FORMULATION

2.1 HSDT and Hierarchic Plate Models

For these two models, we have

$$\mathrm{u}(x,y,z)=\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}=[\varphi(z)]\mathrm{U}(x,y) \tag{1}$$

where

$$[\phi(z)]=\begin{bmatrix} \phi_1(z) & 0 & \phi_3(z) & 0 & 0 & \phi_6(z) & 0 & 0 & \phi_9(z) & 0 & 0 & \dots \\ 0 & \phi_2(z) & 0 & \phi_4(z) & 0 & 0 & \phi_7(z) & 0 & 0 & \phi_{10}(z) & 0 & \dots \\ 0 & 0 & 0 & 0 & \phi_5(z) & 0 & 0 & \phi_8(z) & 0 & 0 & \phi_{11}(z) & \dots \end{bmatrix} \tag{2}$$

and

$$\mathbf{U}(\mathrm{x},\mathrm{y})=\{\mathbf{U}_1(\mathrm{x},\mathrm{y}) \quad \mathbf{U}_2(\mathrm{x},\mathrm{y}) \quad \mathbf{U}_3(\mathrm{x},\mathrm{y}) \quad \cdots \quad \mathbf{U}_{11}(\mathrm{x},\mathrm{y}) \quad \cdots\}^{\mathrm{T}} \tag{3}$$

Here $U_1(x,y)$, $U_3(x,y)$, $U_6(x,y)$, $U_9(x,y)$……are the in-plane components of $u(x,y,z)$ displacement. Similarly $U_2(x,y)$, $U_4(x,y)$, $U_7(x,y)$, $U_{10}(x,y)$…... are the components of $v(x,y,z)$ and $U_5(x,y)$, $U_8(x,y)$, $U_{11}(x,y)$….. are the components of $w(x,y,z)$. For Hierarchic model the 5-field model uses the first 5 terms, for 8-field first 8 terms and for 11-field first 11 terms. The transverse functions are given in terms of the normalized transverse coordinate $\hat{z}=(2/t)z$ (where t is the thickness of the laminate), as (see [2] for details)

$$\phi_1(\hat{z})=\phi_2(\hat{z})=\phi_5(\hat{z})=1; \quad \phi_3(\hat{z})=\phi_4(\hat{z})=\hat{z}\frac{t}{2};$$

$$\phi_6(\hat{z})=\frac{t}{2}\{\varphi_2(\hat{z})-\varphi_2(0)\}; \quad \phi_7(\hat{z})=\frac{t}{2}\{\psi_2(\hat{z})-\psi_2(0)\}; \quad \phi_8(z)=\frac{t}{2}\{\rho_1(\hat{z})-\rho_1(0)\};$$

$$\phi_9(\hat{z})=\frac{t^2}{4}\phi_3(\hat{z}); \qquad \phi_{10}(\hat{z})=\frac{t^2}{4}\psi_3(\hat{z}); \qquad \phi_{11}(\hat{z})=\frac{t^2}{4}\rho_2(\hat{z})$$

where

$$\varphi_2(\hat{z})=\int_{-1}^{\hat{z}}\frac{Q_{44}-Q_{45}}{Q_{44}Q_{55}-Q_{45}^2}d\bar{\hat{z}}; \quad \psi_2(\hat{z})=\int_{-1}^{\hat{z}}\frac{Q_{55}-Q_{45}}{Q_{44}Q_{55}-Q_{45}^2}d\bar{\hat{z}};$$

$$\rho_1(\hat{z})=\int_{-1}^{\hat{z}}\frac{1}{Q_{13}}d\bar{\hat{z}}$$

where Q_{ij} are the coefficients of constitutive relation, in the global *xyz*-coordinate system.

For HSDT model the transverse functions are given as

$$\phi_1(\hat{z})=\phi_2(\hat{z})=\phi_5(\hat{z})=1; \quad \phi_3(\hat{z})=\phi_4(\hat{z})=z; \quad \phi_9(\hat{z})=\phi_{10}(\hat{z})=z^3$$

And rest terms are zero.

For a given l^{th} lamina, the constitutive relationship in principal material directions is given as

$$\{\overline{\sigma}_{(l)}\} = \lfloor C_{(l)} \rfloor \{\overline{\varepsilon}_{(l)}\} \tag{4}$$

where $\overline{\sigma}_{(l)} = \{\sigma_{11}^{(l)}\ \sigma_{22}^{(l)}\ \sigma_{33}^{(l)}\ \sigma_{23}^{(l)}\ \sigma_{13}^{(l)}\ \sigma_{12}^{(l)}\}^T$ are the stress components for the layer, and $\{\varepsilon_{(l)}\} = \{\varepsilon_{11}^{(l)}\quad \varepsilon_{22}^{(l)}\quad \varepsilon_{33}^{(l)}\quad \gamma_{23}^{(l)}\quad \gamma_{13}^{(l)}\quad \gamma_{12}^{(l)}\}^T$ are the components of strain. The subscripts 1, 2 and 3 denotes the three principal material directions. The constitutive relationship in global *xyz* coordinates can be obtained by usual transformations.

2.2 Layerwise Plate Model

The displacement field for this model for the l^{th} layer is given by the product (see [3] for details)

$$u^l(x,y,z) = \sum_{j=1}^{(p_{xy}+1)(p_{xy}+2)} \sum_{k}^{p_z^u+1} u_{jk} N_j^l(x,y) M_k^l(z)$$

Where p_{xy} and p_z^u are the in-plane and transverse approximation order and $N_j^l(x,y)$ and $M_k^l(z)$ are the in-plane and transverse approximation functions, respectively. Similarly the other two components v^l and w^l can be expressed. The displacement *w(x,y,z)* can have different *z*-approximation than for $u^l(x,y,z)$ and $v^l(x,y,z)$ to take care of bending and membrane actions (see [4] for more details).

2.3 Finite Element Formulation

The potential energy, Π, for the laminate is given by

$$\Pi = \frac{1}{2}\int_V \{\sigma\}\{\varepsilon\} dv \quad - \int_{R^+ \cup R^-} q\, w\, ds \tag{5}$$

Where *V* is the volume enclosed by the plate domain, R^+ and R^- are the top and bottom faces of plate and *q(x,y)* is the transverse applied load. The solution to this problem $\mathbf{u}_{ex}$ is the minimizer of the potential energy Π.

3. NUMERICAL EXAMPLES

The efficacy of all models in predicting the quantities of interest is elucidated with following numerical results.

3.1 Comparison of Transverse Deflection

The transverse deflection by all models is compared with [4]. The material properties are given in Table 2. Here, *S=a/t; b=3a;* where *a* and *b* are the plate dimension, h is total thickness of laminate and S is aspect ratio. The laminate sequence [0/90/0] with all laminae of equal thickness is considered. The top face of the plate is loaded with sinusoidal load, $q(x,y)=\sigma\sin(\pi x/a)\sin(\pi y/b)$. All edges are Soft Simple Supported (see Table 1). The transverse deflection is measured at (a/2,b/2, $\bar{z}$) and non-dimensionalised as $\bar{w}$=w$E_2$100/($\sigma h S^4$). The transverse deflection by all theories is given in Table 3.The bracketed terms shows the percentage error with respect to the exact one.

In the present example, the (3,3,0) Layerwise model is used, i.e. cubic transverse representation for *u* and *v* and constant for *w*. The 5 field model is used in case of Hierarchic model.

Boundary Condition	At y=0 and y=b	At x=0 and x=a
Soft Simple Support	v=w=0	u=w=0
Clamped	u=v=w=0	u=v=w=0

Table 1 Boundary conditions

Property	E_1	E_2	G_{12}	G_{23}	$\upsilon_{12}=\upsilon_{23}$
Value	25×10^6 psi	10^6 psi	0.5×10^6 psi	0.2×10^6 psi	0.25

Table 2 Material Properties for [4].

S=a/h	Exact [4]	Layer-wise	Conventional	Hierarchic
2	8.17	8.16 (0.12)	8.044 (1.54)	8.027 (1.75)
10	0.919	0.918 (0.11)	0.866 (5.76)	0.897 (2.39)
100	0.508	0.507 (0.19)	0.504 (0.78)	0.505 (0.59)

Table 3:Non-dimensional transverse deflection $(\bar{w})$ for [0/90/0] laminate

From Table 3 we observe that

1. The present layerwise model predicts the transverse deflection quite accurately. The percentage error varies from 0.11 to 0.19.
2. The Conventional and Hierarchic models predict the values quite close to exact. The percentage error ranges from 0.59 to 5.76.
3. As the aspect ratio, *S*, reduces all the models predict the deflection accurately.

3.2 Comparison of Stress Quantities For Symmetric Cross-ply Laminate

The problem description is same as given in previous section except the location at which stresses are plotted and the plate dimension. The thickness coordinate is non-dimensionalised as. $\bar{z}$=z/t.

3.2.1 Comparison of In-plane Stresses

Here square plate is considered. The in-plane normal and shear stresses are non-dimensionalised as $\left(\bar{\sigma}_{xx},\bar{\tau}_{xy}\right)=\left(\sigma_{xx},\tau_{xy}\right)/\left(\sigma S^2\right)$. The in-plane normal stress is plotted at (a/2,b/2, $\bar{z}$) while the in-lane shear stress is plotted at $(0,0,\bar{z})$

In all the following analysis 11 field Hierarchic model is used whereas the Layerwise model uses 332 model i.e. cubic in *u*, *v* and quadratic in *w*.

The stresses are shown in Fig. 1 and Fig. 2.

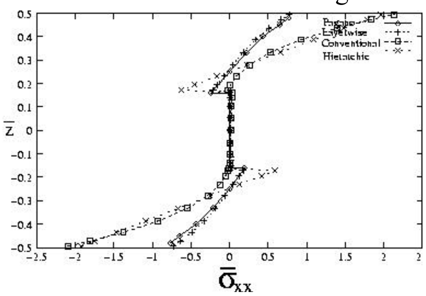

Fig. 1 In-plane Normal Stress

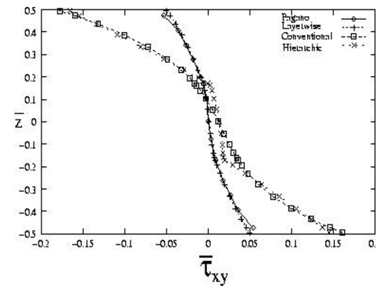

Fig. 2 In-plane Shear Stress

From the Fig. 1 and Fig. 2 we observe that

1. The present Layerwise model predicts the in-plane normal and shear stresses quite accurately.
2. The Conventional and Hierarchic models capture the trend but differ quantitatively from the exact ones.
3. The Conventional and Hierarchic models shows more jumps in the normal stresses at the interface than the Layerwise and exact ones.

3.2.2 Comparison of Transverse Shear Stresses

The problem description is same as given in section 3.2.1. The stress $\bar{\tau}_{xz}$ is plotted at the location0 (0,b/2, $\bar{z}$). The non-dimensionalization is done as $\left(\bar{\tau}_{xz}\right)=\left(\tau_{xz}\right)/\left(\sigma S\right)$. The stress profile is shown in Fig. 3. The stress profile obtained using equilibrium approach of post-processing is shown in Fig. 4. From these figures we observe that

1. The Layerwise stresses are close to the exact values. The stresses at top and bottom faces of laminate are zero.
2. The Conventional and Hierarchic models are quite different both quantitatively and qualitatively.
3. The stresses from Conventional and Hierarchic are not continuous at interface.

For the same problem these stresses are plotted using the equilibrium equations for laminate. It can be observed that the transverse stresses obtained by equilibrium approach are in good agreement with exact one.

3.3 Comparison of Stress Quantities For Anti-symmetric Angle-ply Laminate

The material properties are same as given in Table. 2. The [165/-165] laminate is considered with edges $x = 0, 1$ Soft Simple Supported, plate is infinite in *y*-direction. The top face of the plate is loaded with sinusoidal load $q(x,y)=\sigma\sin(\pi x/a)$. All the examples are done for *S*=4. The stresses are non-dimensionalised as given in previous section. The stresses are compared with [4,7]

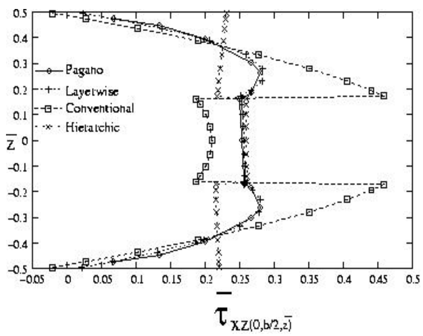

Fig. 3 Transverse Shear Stress

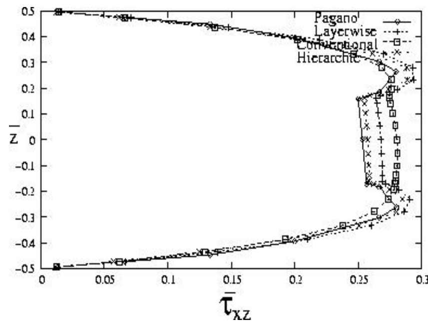

Fig. 4 Transverse Shear Stress Distribution by equilibrium equations

3.3.1 Comparison of Transverse Shear Stresses

The stresses are plotted at $(0,\bar{z})$ and shown in Fig. 5. From this figure it can be observed that

1. The present Layerwise model captures the exact accurately.
2. The Conventional model is quite different both qualitatively and quantitatively from exact one.
3. The Hierarchic models is quite different both qualitatively and quantitatively from exact one.

This stress is also obtained using equilibrium approach of post-processing and is shown in fig. 6. The transverse shear stresses are also obtained using equilibrium equations. From this figure we observe that

1. The present Layerwise model is in good agreement with exact one both qualitatively and quantitatively.
2. The Conventional and Hierarchic models are different quantitatively.

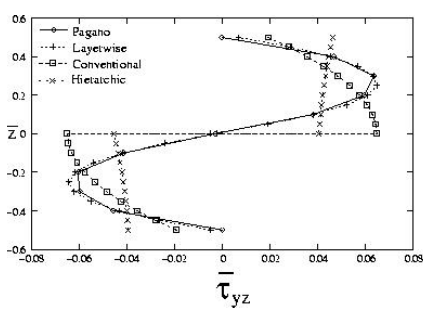

Fig. 5 Transverse Shear Stress

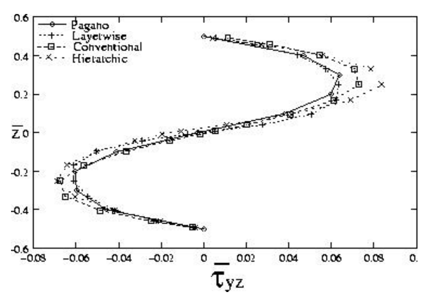

Fig. 6 Transverse Shear Stress distribution by equilibrium

3.4 Effect of Stresses on Prediction of First-ply Failure Loads

In the present study the Tsai-Wu first-ply failure criterion is used (see [6] for details).The first-ply failure loads are compared with Reddy[6]. The material properties and strengths are given in Table 4. The plate dimensions are 228.9×127 mm. All edges are clamped and laminate is subjected to uniform transverse load on top face. In this example, quadratic finite element approximation (i.e.p_{xy}=2) is used.

Properties	Values	Properties	Values
E_1	132.5 GPa	X_T	1515 MPa
E_2=E_3	10.8 GPa	X_C	1697 MPa
G_{12}=G_{13}	5.7 GPa	Y_T=Y_C=Z_T=Z_C	43.8 MPa
G_{23}	3.4 GPa	R	67.6 MPa
υ_{12}=υ_{13}	0.24	S=T	86.9 MPa
υ_{23}	0.49	Ply Thickness h_i	0.127 mm

Table 4 Material properties of T300/5208 graphite/epoxy pre-prg [6]

The failure load ($\overline{q}$)obtained is non-dimensionalized as $\overline{q} = \sigma S^4 / E_2$ .In Tables 5 through 7 the superscript *D* denotes the direct stress for that model (e.g. $HSDT^D$ denotes HSDT model uses direct stresses) and *E* denotes the equilibrium stresses (e.g. $HSDT^E$). From these Tables we observe that:

1. The failure loads obtained with direct stresses using Conventional and Hierarchic models are close to the one obtained in [6].
2. The failure load with equilibrium stresses is lower than that obtained with the direct stresses. The reduction is about 20%
3. The failure locations obtained by Conventional and Hierarchic models are in accordance with [6]. Due to symmetry of problem in all respect, symmetric failure locations are also predicted by these models.
4. All the models predict the failure either on the top face or on the bottom face of the laminate.

Model	Load	x-coord	y-coord	Layer	z-location
Reddy [6]	39354.80	$\approx$115	$\approx$125	1	bottom
$HSDT^D$	39419.20	107.51	0.56	1	bottom
$HSDT^E$	31870.65	121.83	126.43	4	top
Hier-5f^D	38917.90	116.85	126.33	1	bottom
Hier-5f^E	31488.97	121.38	126.43	4	top
Hier-8f^D	38728.10	112.04	0.66	1	bottom
Hier-8f^E	31298.13	121.38	126.43	4	top
Hier-11f^D	39297.60	121.38	126.43	1	bottom
Hier-11f^E	31756.15	121.38	126.43	4	top
$Layerwise^D$	39581.40	107.52	0.56	1	Bottom
$Layerwise^E$	32546.20	107.52	0.56	1	bottom

Table 5 [-45/45-45/45] laminate under uniform transverse loading, CCCC, p_{xy}=2.

4. CONCLUSION

1. All the models predict the maximum transverse deflection accurately.
2. The Conventional and Hierarchic models predicts the in-plane stresses quite accurately but transverse stresses are quite different both qualitatively and quantitatively.
3. As Hierarchic model is designed for symmetric laminate, it fails to predict the stresses in anti-symmetric laminates.
4. The transverse stresses obtained by Conventional and Hierarchic models using equilibrium equations are in good agreement with the exact one but differ quantitatively.
5. The present Layerwise model predicts the stress quantities quite accurately both qualitatively and quantitatively.
6. The transverse stresses obtained by present Layerwise model using equilibrium stresses are accurate.
7. Conventional and Hierarchic models predict the failure loads close to [6] using direct stresses.
8. The failure loads obtained by Conventional and Hierarchic models using equilibrium stresses are quite lower (20%) than those given in [6].

5. REFERENCES

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