ACCURATE COMPUTATION OF CRITICAL RESPONSE QUANTITES FOR LAMINATED COMPOSITE STRUCTURES

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ABSTRACT

Computations are often employed in the design and certification phase for aircraft components. With the advent of composite materials, conventional analysis tools need to be refined to accurately account for the complex response characteristics of components made of composite materials. For laminated composite plates, the first ply failure load and buckling load are taken as some of the critical quantities of interest. For these quantities, it will be shown that unless the model and the finite element discretisation are properly chosen, the predicted critical response quantities can have significant error. Further, the effect of randomness in the material and loading data on the critical quantities will also be shown. It will be demonstrated that a random analysis is essential to obtain the dispersion in the critical response quantities, due to inherent dispersion in the material and loading data.

INTRODUCTION

Composite materials are being used increasingly in the manufacture of structural components. Due to their high strength to weight ratio, composite materials are replacing conventional metals in the fabrication of aircraft components. Components with tailored response features are now possible. For example, the whole wing-box structure of an aircraft can now be co-cured, eliminating the need to produce the sub-components separately followed by assembly of the component. In the Indian aerospace industry, all-composite vehicles have become very popular. The light combat aircraft and the advanced light helicopter are examples of such developments. During detailed trials, these vehicles have proved to have better handling qualities.

Composite materials are heterogeneous media. Further, the composite structure is layered in nature, with several individual laminae stacked together . These features of composite structures make the analysis of the response more complicated. The response characteristics depend on the fibre and matrix material, the curing process, ply orientation, stacking sequence, inherent lamina and laminate level flaws introduced during the manufacturing process. Hence, the analysis of these structures becomes very challenging. Specialized analysis tools have to be developed for the analysis of composite structures. This study focuses on the issues of reliability of the computational tools. As examples, the static problem is discussed, with the goal being to accurately represent the macro-level response of thin, plate-like laminated structures. Linearized elasticity models will be employed throughout this study. The effect of the model and the finite element discretisation will be discussed. It will be shown that with proper model selection and mesh design, accurate computation of the desired response quantities is possible. Here, the first ply failure load and the buckling load have been taken as examples of desired macro-level quantities. Significant differences with the results reported in the literature will be demonstrated for these quantities.

Most often a deterministic analysis is reported. However, the material properties and loading data are random in reality. Dispersion in the data can severely alter the critical response characteristics of a structure. In order to get a realistic picture, dispersion in the critical quantities should also be reported along with the mean data. A perturbation based analysis model will be discussed in this study. This model will be employed to report the dispersion in the first ply failure load and the buckling load for a composite structure.

PLATE MODELS

The system of partial differential equations of three dimensional elasticity is generally intractable analytically, especially for a layered medium. The development of classical theories was motivated to alleviate these problems by reducing the dimension for analysis. For example, in case of plates and shells reduction from three to two dimension reduces the computational cost and enables the handling of a large class of problems. Traditionally, for the plate and shell like thin structures, several plate theories have been proposed. These can be broadly classified as:

1. *Shear deformable theories (HSDT);*
2. *Hierarchic plate theories and*
3. *Layerwise theories*

Shear Deformable Theory

Here, one such theory due to Reddy [1] is taken as representative theory from this group. It is a third order shear deformable theory. And imposes the condition of parabolic distribution of transverse shear strains through thickness of the plate to satisfy the zero transverse shear stress on the top and bottom face of the plate.

Hierarchic Plate Theory

In these, the displacement components have a zig-zag or hierarchic representation through the thickness. The hierarchic plate models are a sequence of mathematical models, the exact solutions of which constitute a converging sequence of functions in the norm or norms appropriate for the formulation and objectives of analysis. The construction of hierarchic models for laminated plates by Babuška et al. [2] and Actis et al. [3]. The solutions of the lower order models are embedded in the highest order model and these models can be adapted according to the requirement.

In these models the displacement field is given as product of functions that depend upon the variables associated with the plate, shell middle surface, and functions of the transverse variable. The transverse functions are derived on the basis of the degree to which the equilibrium equations of three-dimensional elasticity are satisfied.

Layerwise Theory

In these theories, the individual lamina has continuous through thickness representation of displacements. In the present study, the layer-by layer model proposed by Ahmed and Basu [4] is adapted. In this model, all the displacement components are represented as product of in-plane functions of same order and out-of-plane approximating functions of different order for $(u,v))$ and w. In the layerwise model proposed in [9] different transverse approximation order can be taken for $(u,v))$ and w.

MATHEMATICAL FORMULATION OF PLATE MODELS

The generic representation of the displacement field for the plate models is given as:

$$\mathbf{u}(x,y,z) = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = [\phi(z)]\mathbf{U}(x,y) \tag{1}$$

where

$$[\phi(z)] = \begin{bmatrix} \phi_1(z) & 0 & \phi_3(z) & 0 & 0 & \phi_6(z) & 0 & 0 & \cdots \\ 0 & \phi_2(z) & 0 & \phi_4(z) & 0 & 0 & \phi_7(z) & 0 & \cdots \\ 0 & 0 & 0 & 0 & \phi_5(z) & 0 & 0 & \phi_8(z) & \cdots \end{bmatrix} \tag{2}$$

and $\{U(x,y)\} = \{U_1(x,y) U_2(x,y) U_3(x,y) U_4(x,y) \cdots U_8(x,y)\}^T$ (3)

Note that $U_1(x,y), U_3(x,y), U_6(x,y) \cdots$, ……are the in-plane components of displacement terms $u(x,y,z)$. Similarly, $U_2(x,y), U_4(x,y), U_7(x,y) \cdots$ are the in-plane components of displacement terms $v(x,y,z)$. The in-plane components of transverse displacement $w(x,y,z)$ are given by

$U_5(x,y), U_8(x,y)\cdots$. The transverse functions are given in terms of the normalized transverse coordinate $\hat{z} = (2/t)z$ (where t is the thickness of the laminate).

For the higher order shear deformable model the functions $\phi(\hat{z})$ are given as:

$$\phi_1(z) = \phi_2(z) = \phi_5(z) = 1,\ \ \phi_3(z) = \phi_4(z) = z,$$
$$\phi_6(z) = \phi_7(z) = \phi_8(z) = \phi_{11}(z) = 0,\ \ \phi_9(z) = \phi_{10}(z) = z^3$$

For the hierarchic family of the plate models the transverse functions $\phi(\hat{z})$ are given as:

$$\phi_1(\hat{z}) = \phi_2(\hat{z}) = \phi_5(\hat{z}) = 1;\quad \phi_3(\hat{z}) = \phi_4(\hat{z}) = \hat{z}\frac{t}{2};$$
$$\phi_6(\hat{z}) = \frac{t}{2}\{\varphi_2(\hat{z}) - \varphi_2(0)\};\quad \phi_7(\hat{z}) = \frac{t}{2}\{\psi_2(\hat{z}) - \psi_2(0)\};\quad \phi_8(z) = \frac{t}{2}\{\rho_1(\hat{z}) - \rho_1(0)\};$$

where

$$\varphi_2(\hat{z}) = \int_{-1}^{\hat{z}} \frac{Q_{44} - Q_{45}}{Q_{44}Q_{55} - Q_{45}^2} d\bar{z};\quad \psi_2(\hat{z}) = \int_{-1}^{\hat{z}} \frac{Q_{55} - Q_{45}}{Q_{44}Q_{55} - Q_{45}^2} d\bar{z};\quad \rho_1(\hat{z}) = \int_{-1}^{\hat{z}} \frac{1}{Q_{13}} d\bar{z}$$

Here Q_{ij} are the coefficients of the global constitutive relation, in the global xyz-coordinate system. For other transverse functions see Actis et al.

The present layerwise plate model is an improvement over the model given In [4] as the original layerwise model had same order transverse representation for all three displacement components, whereas the present layerwise model can have different approximation in transverse direction for individual displacement components. The different approximation for displacement components is used as suggested by Schwab [5] for a single lamina, to separately account for bending and membrane actions. The displacement component u^l, for a prismatic element (i.e. triangular in-plane projection) in the l^{th} layer, is given as

$$u^l(x,y,z) = \sum_{j=1}^{(p_{xy}+1)(p_{xy}+2)} \sum_{k}^{p_z^u+1} u_{jk} N_j^l(x,y) M_k^l(z)$$

where p_{xy} and p_z^u are the in-plane and transverse approximation order (for component u^l) and $N_j(x,y)$ and $M_k(x,y)$ are in-plane and transverse approximation functions, respectively. Similarly the other components v^l and w^l can be expressed. The transverse approximation orders for u and v displacement components will be the same, while that for the component w can be different. In this study, $p_{xy} = 2$ or 3; p_z^u, $p_z^v = 1$, 2, 3; $p_z^w = 0$, 1, 2, 3 will be used.

FINITE ELEMENT FORMULATION

For a given l^{th} lamina, the constitutive relationship in principal material directions is given as:

$$\{\bar{\sigma}_{(l)}\} = [C_{(l)}]\{\bar{\varepsilon}_{(l)}\} \tag{4}$$

where $\bar{\sigma}_{(l)} = \{\sigma_{11}^{(l)}\ \sigma_{22}^{(l)}\ \sigma_{33}^{(l)}\ \sigma_{23}^{(l)}\ \sigma_{13}^{(l)}\ \sigma_{12}^{(l)}\}^T$ are the stress components for the layer, and $\{\varepsilon_{(l)}\} = \{\varepsilon_{11}^{(l)}\ \ \varepsilon_{22}^{(l)}\ \ \varepsilon_{33}^{(l)}\ \ \gamma_{23}^{(l)}\ \ \gamma_{13}^{(l)}\ \ \gamma_{12}^{(l)}\}^T$ are the components of strain. The subscripts 1, 2 and 3 denotes the three principal material directions. The constitutive relationship in global *xyz* coordinates can be obtained by usual transformations.

The potential energy, Π, for the laminate is given by

$$\Pi = \frac{1}{2}\int_V \{\sigma\}\{\varepsilon\} dv \quad - \int_{R^+ \cup R^-} q\ w\ ds \tag{5}$$

Where V is the volume enclosed by the plate domain, R^+ and R^- are the top and bottom faces of plate and $q(x,y)$ is the transverse applied load. The solution to this problem $\mathbf{u}_{ex}$ is the minimizer of the potential energy Π.

ERROR ESTIMATOR FOR LOCAL QUANTITY OF INTEREST

In the analysis of laminates for first-ply failure, accurate computation of state of stress at a point is essential. When the finite element analysis is employed, the issue of control of modeling error (error due to model employed in the analysis of laminate, as compared to three dimensional elasticity) and discretization error becomes important. Various smoothening based a-posteriori error estimation techniques for laminated composites have been proposed by the authors for the local quantity of interest [6]. Further, estimation and control of the error in the quantity of interest and "one shot" adaptive approach for the control of discretization error was proposed in [7],[8]. Further, this approach is used for accurate computation of critical local quantities in [9]. The quantity of interest is the stress component, which contributes maximum to the Tsai-Wu [10] first ply failure index. Similar procedures can be used for the Hoffman and other failure theories.

TSAI-WU FAILURE CRITERION

The Tsai-Wu criterion is given by

$$\mathrm{FI}_{\mathrm{TW}} = \mathrm{F}_{\mathrm{i}}\sigma_{\mathrm{i}} + \mathrm{F}_{\mathrm{ij}}\sigma_{\mathrm{i}}\sigma_{\mathrm{j}} \geq 1 \qquad (6)$$

where $F_{i,}\ F_{ij}$ are the strength tensor terms and σ_i are the stress components.

NUMERICAL RESULTS

One of the major goals of this paper is to do a critical analysis of various families of plate models, with respect to the quality of the point-wise stresses obtained using the models. The effect of in-plane approximation order, model order and type will also be investigated here. All the models are subjected to rigorous numerical studies to compare the transverse deflection and stress profiles for numerous ply orientations, stacking sequences and boundary conditions under transverse loadings. In the present study uniform pressure, sinusoidal transverse and constant in-plane loadings are considered.

Effect of Model on Accuracy of Point-wise Data

Comparison of Transverse Deflections

The goal of this numerical experiment is to compare the value of transverse displacement components obtained using various models, and in-plane discretization, with the exact three-dimensional elasticity results reported in [11], for cross-ply laminate sequence with material properties given in table 1. The plate has dimension a along x-axis and b along y-axis, and is subjected to sinusoidal loading of the form

$$\mathrm{q}(\mathrm{x},\mathrm{y}) = \mathrm{q}_0(\mathrm{x},\mathrm{y})\,\mathbf{sin}(\pi\mathrm{x}/\mathrm{a})\,\mathbf{sin}(\pi\mathrm{y}/\mathrm{b}) \qquad (7)$$

All edges of the plate are simply supported (see table 2 for all BC's used). The transverse deflection at $(\mathrm{a}/2, \mathrm{b}/2, 0)$ is reported in tables 3 and 4. Note that in all the computations the layerwise model uses (3,3,2) model (unless specified), that is, transverse approximation for u and v is cubic and quadratic for w. For the hierarchic family 11 field model is used, while for the HSDT model (3,3,0) approximation is used.

Table 1 Material Properties for [11, 12].

Property	E_1	E_2	G_{12}	G_{23}	$\upsilon_{12}=\upsilon_{23}$
Value	25×10^6 psi	10^6 psi	0.5×10^6 psi	0.2×10^6 psi	0.25

Table 2 Boundary conditions

Boundary Condition	At y=0 and y=b	At x=0 and x=a
Soft Simple Support	v=w=0	u=w=0
Clamped	u=v=w=0	u=v=w=0
Free	u, v, w≠0	u, v, w≠0

In this study following case has been studied:

Square plate with cross ply laminae, such that outer laminae with orientation 0° and total thickness of 0° laminae is equal to total thickness of 90° laminae. Also laminae with same orientation have equal thickness. In this study, 7 layered laminate is used. Transverse deflection is nondimensionalised as $w^* = \dfrac{\pi^4 Qw}{12 q_0 S^4 t}$, $Q = 4G_{12} + \left[E_{11} + E_{22}\left(1 + 2\nu_{23}\right)\right]/\left(1 - \nu_{12}\nu_{21}\right)$. Here, $p_{xy} = 3$ is used for all models. Numbers in parenthesis show the % error with respect to exact solution given in [11].

Table 3: Non-dimensional transverse deflection $\left(w^*\right)$ for 7 layered cross-ply laminate.

S	Exact [11]	Layer-wise	HSDT	Hierarchic
10	1.529	1.529 (0.00)	1.417 (7.33)	1.444 (5.56)
50	1.021	1.021 (0.00)	1.005 (1.56)	1.017 (0.39)
100	1.005	1.005 (0.00)	0.993 (1.19)	1.004 (0.09)

From these tables we observe that:

1. The layerwise model predicts the transverse deflection accurately for all the aspect ratios.
2. For the HSDT and hierarchic model with aspect ratios $S > 10$ the displacement is close to exact. The error is 0.1-3 %.

Comparison of Stresses

In this case [0/90/0], square laminate with all edges simple supported is considered. All the laminae are of equal thickness. The sinusoidal loading is of the form as in above subsection. The in-plane stresses are nondimensionalised as $\left(\bar{\sigma}_{xx}, \bar{\tau}_{xy}\right) = 1/\left(q_0 S^2\right)\left(\sigma_{xx}\left(a/2, b/2, \bar{z}\right), \tau_{xy}\left(0, 0, \bar{z}\right)\right)$ and the transverse stresses as $\left(\bar{\tau}_{xz}\right) = 1/\left(q_0 S\right)\left(\tau_{xz}\left(0, b/2, \bar{z}\right)\right)$. The in-plane stress components are shown in fig. 1 and transverse stress component is shown in fig. 2. These are compared with exact ones given in [12].

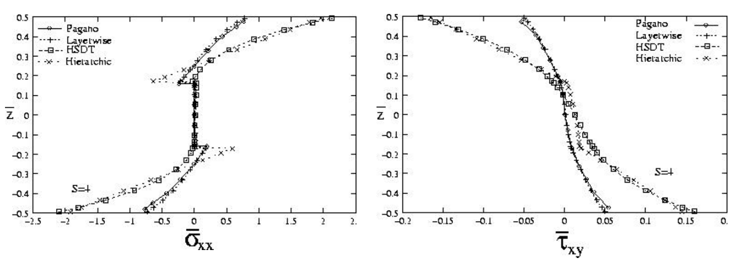
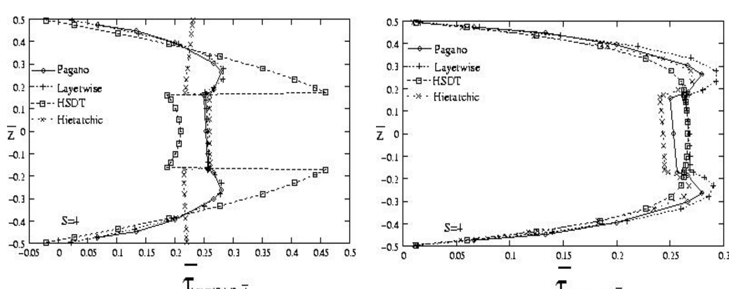


Fig. 1 [0/90/0] laminate; SSSS b.c.



Direct stresses   Equilibrium stresses

Fig. 2 [0/90/0] laminate; SSSS, transverse stress

Effect of Models on Accuracy of Predicted Failure Load

The laminates considered are $[0/90]_S$ and [-45/45/-45/45]. The plate is clamped on all edges. The top face of the plate is subjected to uniform transverse load $q(x, y) = q_0$. The plate dimensions are $a = 228.9\ mm\ (9\,in)$ and $b = 127\ mm\ (5\ in)$. The material properties are given in table 4. The first-ply failure load is nondimensionalised as $\mathrm{FLD} = \mathrm{q_0 S^4 / E_{22}}$. The results obtained from the present analysis are compared with those reported in [13].

Table 4: Material properties for T300/5208 Graphite/Epoxy (Pre-preg) [13].

Property	Value	Property	Value
E_{11}	132.5 GPa	X_T	1515 MPa
$E_{22} = E_{33}$	10.8 GPa	X_C	1697 MPa
$G_{12} = G_{13}$	5.7 GPa	$Y_T = Y_C = Z_T = Z_C$	43.8 MPa
G_{23}	3.4 GPa	R	67.6 MPa
$\nu_{12} = \nu_{13}$	0.24	$S = T$	86.9 MPa
ν_{23}	0.49	Ply thickness, t_i	0.127 mm

The computed failure load depends on the accuracy of the lamina level stress. Hence, an adaptive approach with the capability to estimate error in the local stresses and refine mesh accordingly to bring the error down to acceptable tolerance, is required. For the fixed model, the focussed adaptive approach is employed to recompute the failure load. Here, the stress component contributing maximum to the Tsai-Wu first-ply failure criterion is used as the quantity of interest. In table 5 the first-ply failure loads are given. In these tables, the superscript a shows all the values of failure loads and corresponding failure index obtained using uniform mesh, the superscript b shows the value of the failure index obtained with the same load as in a (see fig. 3) and the adapted mesh and the superscript c shows the first-ply failure load for the adapted mesh.

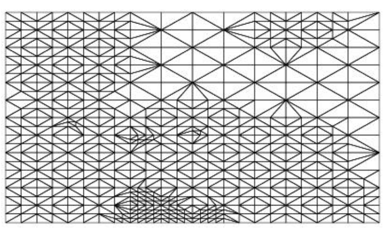


Fig. 3 Adapted mesh with 11-field hierarchic for [-45/45/-45/45] clamped laminate.

Table 5: First-ply failure loads; all edges clamped, [-45/45/-45/45] laminate under uniform transverse loading, (equilibrium stresses) $p_{xy} = 2$.

Model	FLD	Xco	Yco	Layer	Location	FI_{TW}	Max. σ
Ref. 13	39354.8	≈115.00	≈125.00	1	bottom	-	
HSDT[a]	31463.7	107.51	0.56	4	top	1.00	σ_{yy}
HSDT[b]	31463.7	112.71	0.14	4	top	1.82	
HSDT[c]	23377.6	112.71	0.14	4	top	1.00	
11-field[a]	31672.2	121.38	126.43	4	top	1.00	σ_{yy}
11-field[b]	31672.2	116.18	126.85	4	top	1.75	
11-field[c]	23955.1	116.18	126.85	4	top	1.00	
Layer	32549.2	107.51	0.56	1	bottom	1.00	σ_{yy}

With equilibrium stresses we observe that:

1. For the initial mesh, the failure loads predicted by all the models are lower than those obtained by [13] (shown with superscript a) and those obtained by using direct stresses.
2. The locations predicted by all the models are either close to one obtained by [13] or are corresponding symmetric points. The locations for both direct stresses and equilibrium stresses are same (or corresponding symmetry points).
3. Failure loads predicted by the HSDT and hierarchic models are close while those predicted by layerwise are slightly higher than these.
4. When the discretization error control is used the failure index, for the failure load obtained using adapted mesh, increases upto 82%. This is due to the increased flexibility of the numerical solution for the adapted mesh.
5. With the adapted mesh the error in the failure load computations can be close to 25%.
6. The failure locations for the HSDT and hierarchic models are in the same region before and after the use of discretization error control.

It is obvious that a suitably refined mesh, along with proper post-processed values of the transverse stresses, is necessary to obtain reliable values of the first-ply failure load.

Here, it can be seen that a suitably refined analysis is necessary for the standard analysis. In real life problems, however, even the material and the load data is not deterministic. A proper random analysis is required to get the true picture of the influence of the dispersion of the input data. In the following, properly controlled finite element computations are employed to carry out a random analysis. The effect of randomness on critical design data, like the first-ply failure load and the buckling load, are studied.

STOCHASTIC FINITE ELEMENT APPROACH

Consider a plate with stochastically varying material properties, subjected to random transverse loading. The plate is assumed to be linearly elastic with a stochastic elasticity tensor field C_{ijkl} . The goal is to find the statistics of failure index under transverse loading and the statistics of buckling strength in the case of in-plane loading. The following Green's strain-displacement relationship is used:

$$\{\varepsilon\} = \{\varepsilon^{L}\} + \{\varepsilon^{NL}\} \tag{8}$$

Here $\{\varepsilon^{L}\}$ is the linear strain and $\{\varepsilon^{NL}\}$ is the geometric nonlinear strain with components:

$$\varepsilon_{ij}^{L} = \frac{1}{2}\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\}; \qquad \varepsilon_{ij}^{NL} = \frac{1}{2}\{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\} \qquad (\, i, j, k = 1, 2, 3\,) \tag{9}$$

The total potential energy corresponding to the linear state of the system for uncertain stiffness can be written as [15]:

$$\Pi(u_i) = \int_{\Omega} C_{ijkl}\varepsilon_{ij}^{L}\varepsilon_{kl}^{L} d\Omega - \int_{\Gamma_1} q_i\, u_i\, d\Gamma_1 \qquad (\, i, j, k, l = 1, 2, 3\,) \tag{10}$$

The exact solution minimizes Π on the set of all kinematically admissible functions denoted by V, i.e. $u \in V$ such that $V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$. This yields:

$$\delta\Pi(u_i) = \int_{\Omega} C_{ijkl}\varepsilon_{ij}^{L}\delta\varepsilon_{kl}^{L} d\Omega - \int_{\Gamma_1} q_i\, \delta u_i\, d\Gamma_1 = 0 \qquad (\, i, j, k, l = 1, 2, 3\,) \tag{11}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$ represents the surface of the body Ω . Γ_0 denotes the Dirichlet part and Γ_1 denotes the Neumann part of the boundary of the body.

According to the standard stochastic variational formulation in conjunction with Taylor series expansion, the zeroth- and first-order variational statements are written as:

Zeroth-order: $$\int_{\Omega} C^{0}_{ijkl}\varepsilon^{0(L)}_{ij}\delta\varepsilon^{0(L)}_{kl}d\Omega = \int_{\Gamma_1} q^{0}_{i}\delta u^{0}_{i}d\Gamma_1 \qquad (i,j,k,l=1,2,3) \tag{12}$$

First-order: $$\int_{\Omega} C^{0}_{ijkl}\varepsilon^{,r(L)}_{ij}\delta\varepsilon^{0(L)}_{kl}d\Omega + \int_{\Omega} C^{,r}_{ijkl}\varepsilon^{0(L)}_{ij}\delta\varepsilon^{0(L)}_{kl}d\Omega = \int_{\Gamma_1} q^{,r}_{i}\delta u^{0}_{i}d\Gamma_1$$
$$(i,j,k,l=1,2,3;\ r=1,2,...,R) \tag{13}$$

In the case of buckling analysis a linear elastostatic problem is first solved for the reference loading q^{ref} with the given constraints for the plate. The linear solution is used for computing the initial stress tensor σ^{ref}_{ij} and then these stresses are used to compute the geometric matrix. The zeroth- and first-order variational statements can be written as:

Zeroth-order: $$\int_{\Omega} C^{0}_{ijkl}\varepsilon^{0(L)}_{ij}\delta\varepsilon^{0(L)}_{kl}d\Omega + \lambda^{0}_{cr}\int_{\Omega}\sigma^{0(ref)}_{ij}\delta\varepsilon^{0(NL)}_{ij}d\Omega = 0 \quad (i,j,k,l=1,2,3) \tag{14}$$

First-order: $$\int_{\Omega}(C^{,r}_{ijkl}\varepsilon^{0(L)}_{ij} + C^{0}_{ijkl}\varepsilon^{,r(L)}_{ij})\delta\varepsilon^{0(L)}_{kl}d\Omega + \lambda^{0}_{cr}\int_{\Omega}\sigma^{,r(ref)}_{ij}\delta\varepsilon^{0(NL)}_{ij}d\Omega + \lambda^{,r}_{cr}\int_{\Omega}\sigma^{0(ref)}_{ij}\delta\varepsilon^{0(NL)}_{ij}d\Omega = 0$$
$$(i,j,k,l=1,2,3;\ r=1,2,...,R) \tag{15}$$

where R is the number of basic random variables chosen for the analysis.

Statistics of first-ply failure load under transverse loading

Tsai-Wu and Hoffman failure criteria are adopted for the present analysis to predict the failure of a laminate based on first-ply failure analysis.

$$f(\sigma) = F_i\sigma_i + F_{ij}\sigma_i\sigma_j \qquad (i,j=1,2,...,6) \tag{16}$$

The mean and variance of the failure index are expressed as:

$$E[f(\sigma)] = F_i\sigma^0_i + F_{ij}\sigma^0_i\sigma^0_j \tag{17}$$

$$\begin{aligned} Var[f(\sigma)] = [&F_iF_l\sigma^{,r}_i\sigma^{,s}_l + F_iF_{lm}\sigma^{,r}_i\sigma^0_l\sigma^{,s}_m + F_iF_{lm}\sigma^{,r}_i\sigma^{,s}_l\sigma^0_m + F_lF_{ij}\sigma^{,s}_l\sigma^0_i\sigma^{,r}_j \\ &+ F_{ij}F_{lm}\sigma^0_i\sigma^{,r}_j\sigma^0_l\sigma^{,s}_m + F_{ij}F_{lm}\sigma^0_i\sigma^{,r}_j\sigma^{,s}_l\sigma^0_m + F_{ij}F_l\sigma^{,r}_i\sigma^0_j\sigma^{,s}_l \\ &+ F_{ij}F_{lm}\sigma^{,r}_i\sigma^0_j\sigma^0_l\sigma^{,s}_m + F_{ij}F_{lm}\sigma^{,r}_i\sigma^0_j\sigma^0_m\sigma^{,s}_l]\,E[(b_r - b^0_r)(b_s - b^0_s)] \end{aligned} \tag{18}$$

Statistics of buckling strength

The total minimum eigenvalue or load parameter takes the following form:

$$\lambda_{cr} = \lambda^0_{cr}(b^0_l) + \lambda^{'l}_{cr}(b_l - b^0_l) \tag{19}$$

The mean and variance of the minimum eigenvalue can be expressed as:

$$E[\lambda_{cr}] = \lambda^0_{cr}; \qquad Var[\lambda_{cr}] = \lambda^{'r}_{cr}\lambda^{'s}_{cr}E[(b_r - b^0_r)(b_s - b^0_s)] \tag{20}$$

Results and discussions

Statistics of failure load under transverse loading

The laminates considered for generating the results are made of T300/5208 graphite/epoxy material with properties as listed below:

$E_{11} = 132.5\,GPa$; $E_{22} = E_{33} = 10.8\,GPa$; $G_{23} = 3.4\,GPa$; $G_{12} = G_{13} = 5.7\,GPa$; $\nu_{12} = \nu_{13} = 0.24$; $\nu_{23} = 0.49$

The ultimate strengths for the above material which are used to calculate the strength parameters are defined as:

$$X_T = 1515\,MPa\,;\; X_C = 1697\,MPa\,;\; Y_T = Z_T = 43.8\,MPa\,;$$

$$Y_C = Z_C = 43.8\,MPa\,;\; R = 67.6\,MPa\,;\; S = T = 86.9\,MPa$$

In the present analysis the elastic moduli (E_{11}, E_{22}, G_{12}, G_{23}, ν_{12}, ν_{23}) of the material are treated as independent random variables. The laminated plate is subjected to a uniform distributed random load.

Validation - Simply supported (SSSS) symmetric cross-ply laminated plate

A thin square laminated plate consisting of four layers $[0^0/90^0/90^0/0^0]$ of equal thickness with $b/h=100$ having all edges simply supported is considered for the present validation. A uniformly distributed load is applied on the top surface. Both Tsai-Wu and Hoffman failure criteria are used to compare the mean failure load and the statistics of the failure index. In order to validate the layer-wise model implementation, the failure loads obtained using the layerwise model are compared with that obtained using a closed form solution and Kirchoff-Love (KL) model. From the results given in Table 6, it can be observed that the mean failure load is close to that obtained using a closed-form solution. The layer-wise model gives lower failure loads because this model is less stiffer than the KL model. Further for thin plates the behavior is accurately predicted by the KL model, i.e. shear effect are negligible. This is the reason why the layer-wise model and KL model give very close values of the failure load.

Table 6: Comparison of the mean failure load for $[0^0/90^0/90^0/0^0]$ square laminate with $b/h=100$ having SSSS boundary condition.

Failure criteria	Mean failure load (MPa)	
	SFEM	Closed form
Tsai-Wu	0.07292	0.07306
Hoffman	0.06246	0.06250

In order to validate the SFEM implementation the effect of randomness of the material data and loading, on the calculated failure load, is calculated. In Table 7 the coefficient of variation (COV) of the failure load is reported, with respect to change in the random input variables, for the thin symmetric cross ply laminate considered above. From the result it can be noted that:

(1) The COV of failure load obtained using SFEM is close to that obtained using the closed form solution.

(2) The failure load is very sensitive to change in the input data. For example, for a COV of 4% in input material data, the COV of failure load is ≈13%.

(3) The Tsai-Wu and Hoffman measures give similar COV for the failure load

This example validates the SFEM implementation of the present study. It should be noted that the two failure models, i.e. Tsai-Wu and Hoffman, give different values of the mean failure load (Table 6). It is also note worthy that effect of material defects on the failure load is significant.

Table7: Comparison of COV of failure index for $[0^0/90^0/90^0/0^0]$ square laminate with $b/h=100$ having SSSS boundary condition.

sd/mean of all random variables	sd/mean of failure index $f(\sigma)$			
	Tsai-Wu		Hoffman	
	SFEM	Closed form	SFEM	Closed form
0.00	0.0000	0.0000	0.0000	0.0000
0.08	0.2599	0.2597	0.2343	0.2343
0.16	0.5197	0.5195	0.4685	0.4685
0.20	0.6497	0.6494	0.5856	0.5856

Second order failure statistics of anti-symmetric laminated plates

The probabilistic failure of the angle-ply laminated plates with three different boundary conditions SSSS, SCSC and SFSF are used in this study. The plate thickness ratio $b/h = 100$ has been used in this study. Two different ply schemes for the plate used in the study are:

Ply scheme 1: [0/90]; Ply scheme 2: [-45/45]

The mean failure load predicted by Tsai-Wu and Hoffman criteria by keeping failure index equal to 0.8 are listed in Table 8 for square plate with different lay-ups and boundary conditions. These failure loads ensure that if a factor of safety of 1.2 were chosen for the design, the plates would still be assumed to be safe in the deterministic environment.

Table 8: Mean failure load for square plate with different lay-ups and boundary conditions.

Ply schemes	Failure criteria	Mean failure load for boundary conditions		
		SSSS	SCSC	SFSF
[0/90]	Tsai-Wu	0.02469	0.01869	0.01113
	Hoffman	0.02827	0.02397	0.01274
[-45/45]	Tsai-Wu	0.04323	0.03059	0.01412
	Hoffman	0.05526	0.04098	0.01589

Simultaneous Variation of All BRVs

The effects of material properties on failure index of composite laminated plates under transverse random loading are now presented. The variations of non-dimensionalised failure index (FI) with dispersion in all the basic random variables (BRV) changing simultaneously for ply schemes 1 and 2 having SSSS and SCSC boundary conditions are presented in Figures 4 and 5 respectively. It is found that:

(1) Angle ply is more affected by dispersion in the input variables compared to cross-ply laminate.
(2) Boundary conditions also play an important role in the stochastic analysis.
(3) The variation in FI is most sensitive for SCSC boundary condition.

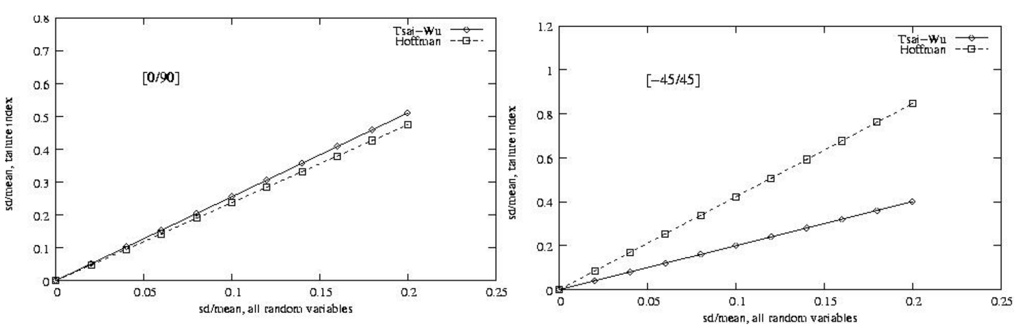


Figure 4 - Influence of SD/mean of all basic random inputs changing simultaneously on COV of failure index for different laminates with SSSS boundary condition and *b/h*=100.

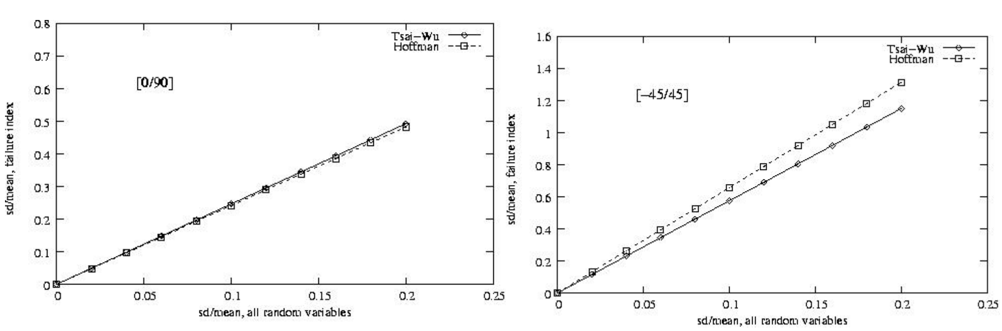


Figure 5 - Influence of SD/mean of all basic random inputs changing simultaneously on COV of failure index for different laminates with SCSC boundary condition and *b/h*=100.

Statistics of buckling strength under in-plane compressive loading

Validation

A graphite/epoxy antisymmetric cross-ply laminated plates with E_{11}/E_{22}=40, G_{12}/E_{22}=0.6, G_{23}/E_{22}=0.5, $\nu_{12}=\nu_{23}$=0.25 and b/h=50 are considered under uniaxial compressive loading.

Table 4 shows mean normalized buckling load of two-layer antisymmetric cross-ply square plate with b/h=5, having different boundary conditions. The effect of both pre-buckled stress and uniform stress assumption on the buckling load is compared with those obtained using Reddy's plate theories [14]. Further the effect of equivalent layer, by assuming both material layers as an equivalent solution layer, on the buckling load using uniform stress assumption is also presented in the table. It is observed that conventional 2D plate models overpredict buckling loads compared to layer-wise plate model for the plate considered. Also the reference uses uniform stress assumption which is only true for specific laminates with specific boundary condition and is not true in general. Buckling load of a plate using pre-buckled stress is found to be smaller compared to analytical solution using uniform stress assumption with equivalent single layer consideration.

Table 4: Comparison of mean buckling load for [0/90] laminates with different supports and b/h=5

Various Boundary Conditions	Mean Buckling load $\hat{N} = \lambda^{cr} b^2 /(E_{22} h^3)$		
	Present result (layer-wise plate model)		Reddy's result [14] (various 2D plate model)
	with pre-buckled stress (layer-by-layer)	with uniform stress (equivalent layer)	
SSSS	7.874	8.473	8.769 (HSDT) 8.277 (FSDT) 12.957 (CPT)
SCSC	8.945	10.652	11.490 (HSDT) 9.757 (FSDT) 31.280 (CPT)

Variance of buckling strength

Second order statistics of buckling load is evaluated using stochastic finite element method. In the present analysis the elastic moduli (E_{11}, E_{22}, G_{12}, G_{23}, ν_{12}, ν_{23}) of the material are treated as independent random variables. The variations of buckling load with 10% dispersion in all the basic inputs for antisymmetric cross-ply and angle ply square plate with SSSS boundary conditions is presented in Figures 6 (a) and (b). 1σ limit is used to study the effect of material properties on critical buckling load. It is observed that the effect of dispersion in material properties on critical buckling load is significant for

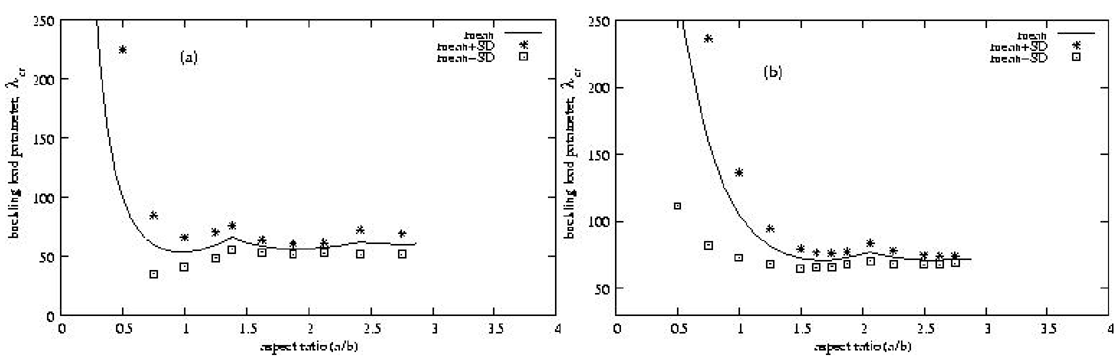


Figure 6: Influence of all basic random inputs changing simultaneously on critical buckling load parameter for different laminates with SSSS boundary condition (a) [0/90] (b) [-15/15].

plates with lower aspect ratios and it decreases with increase in aspect ratio of the plate.

CONCLUSIONS

In this study, a systematic procedure is outlined for the reliable analysis of composite laminated structures. It has been shown that the discretisation error has to be controlled in order to get accurate values of the response quantities of interest. A-priori the right choice of the mesh, model and approximation are not obvious. Heuristic reasoning can be disastrous. As has been demonstrated a feedback-based analysis, where the error in the desired quantity of interest is controlled, is mandatory. A more fundamental issue is the choice of the right model, to represent the physics (at the desired scale) accurately. This is the subject of ongoing research initiatives.

Most often a deterministic engineering analysis is done. This can be very far from the reality, especially for composite materials, where dispersion in the lamina and laminate level material properties can be significant. Further, the loading can never be deterministic. Hence, a meaningful analysis should give an indication of the dispersion in the computed response quantities, along with the mean values. Here, we have shown that the effect of the dispersion in the material and load data can be significant for the failure load, which is a local quantity. While for the buckling load, which is a global quantity, the effect of the dispersion is small.

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