Proceedings of the National Symposium on
Recent Trends in Product Design(RTPD 2006),
DKTES's Textile and Engineering Institute, Ichalkarnaji (MS), India. Sept. 15 -1 6, 2006. pp.1 -6.

Accurate and Computationally Economical Analysis Tool for Layered Composites

Mohite PM[1] and **Upadhyay CS**[2]

[1]*Graduate Student, Department of Aerospace Engineering, Indian Institute of Technology Kanpur 208016, India*
[2]*Associate Professor, Department of Aerospace Engineering, Indian Institute of Technology Kanpur 208016, India*
E-mail: mohite @iitk.ac.in, shekhar@iitk.ac.in

Abstract

In this paper a novel region-by-region modeling approach for layered composite has been proposed. With this approach a Layerwise model (3D model) and equivalent single layer(2D Model) or intermediate (sublaminate) model can be put in any region of the domain. The three dimensional effects, in general, are localized at cutout boundaries, crack front, delaminated or damaged zones, re-entrant corners, free edges etc. It is shown that the Layerwise model is very accurate in such situations but very costly, computationally. Away from these structural details the equivalent or intermediate models are accurate when equilibrium based postprocessing is used for transverse stress components. It is shown that by judiciously using Layerwise model in the regions where solution is predominantly three dimensional in nature, and lower order models elsewhere, same accuracy (as that of Layerwise model) can be achieved at a significantly lower cost. A simple explicit modeling error indicator based on interelement jumps is used to adaptively select the models in the domain.

Keywords: layerwise; equivalent single layer; region-by-region model, modeling error, equilibrium based postprocessing

1.0 INTRODUCTION

The laminated structures are widely used for the fabrication of critical components used in aerospace, automobile and other applications. Several dimensionally reduced models have been proposed in the literature for the analysis of the layered medium (see [1] for a brief review). The displacement based models like shear deformable models and zig-zag models are very popular as their cost of computation is independent of numbers of layers in the laminate. The major drawback of these theories is that the transverse stresses are not accurately predicted by these models. The equilibrium based postprocessing approach works most of the cases for thaese models but fails in the case of laminate unsymmetry layup, cut-out boundaries. This problem is circumvented by using refined models like Layerwise model. In this, the standard models are used layer by layer where continuity of displacements (and is some cases the transverse stresses) is imposed. In intermediate models some of the laminae are lumped above and below a lamina of interest. The proposed models can be seen in [1].

The three dimensional effects are localized in the vicinity of boundaries (boundary layer), vertices, edges, damaged and delaminated zones (see Fig. 1). In the vicinity of these regions it is needed to enrich the approximation to capture these effects. This is achieved by proposed region-by-region model [2]. The proposed model is an adaptation of the planar constrained approximation of [3] and h-d approach of [4].

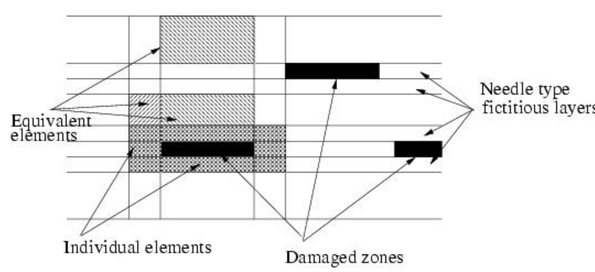


Fig. 1: General Scenario in Laminated Composites

2.0 PLATE MODELS

Here, the displacement based plate models are used for the analysis of laminated structures. The displacement field is given as a series in terms of products of the director functions in the z-direction and planar functions. The plate models used in this study are presented briefly.

2.1 Layerwise Models (LM)

This is the most general three-dimensional representation of the displacement field. The director functions are taken as one dimensional basis functions defined over each lamina. The displacement field is given as

$$\begin{aligned} u(x,y,z) &= \sum_{i=1}^{n_1} u_i(x,y)\overline{M}_i(z) \\ v(x,y,z) &= \sum_{i=1}^{n_2} v_i(x,y)\overline{M}_i(z) \\ w(x,y,z) &= \sum_{i=1}^{n_3} w_i(x,y)\overline{M}_i(z) \end{aligned} \quad (1)$$

Where $n_1 = n_2$ and n_3 depends upon the order of transverse approximation $p_z^u = p_z^v$ and p_z^w for *u*, *v* and *w*, respectively and the number of layers *nl* in the laminate. Hence, the number of unknowns grows with number of layers. Here, $\overline{M}_i(z)$ are director functions in transverse direction defined as the one dimensional hierarchical basis functions. These are shown in Fig. 2(a). $u_i(x,y)$, $v_i(x,y)$ and $w_i(x,y)$ are the planar functions for three displacement components.

The members of this family are denoted by the $LMp_{xy}p_z^u p_z^v p_z^w$ where p_{xy} is the in-plane approximaion order.

2.1 Equivalent Models (EQ)

These are most popular models with, HSDT as a special case. The displacement fields corresponding to these models is represented as

$$\begin{aligned} u(x,y,z) &= \sum_{i=1}^{n_1} u_i(x,y)\overline{M}_i(z) \\ v(x,y,z) &= \sum_{i=1}^{n_1} v_i(x,y)\overline{M}_i(z) \\ w(x,y,z) &= \sum_{i=1}^{n_1} w_i(x,y)\overline{M}_i(z) \end{aligned} \quad (2)$$

Here, the number of unknowns is independent of number of layers. The director functions for this model are shown in Fig. 2(b).

The members of this family are denoted by the $EQp_{xy}p_z^u p_z^v p_z^w$.

2.2 Intermediate Models (IM)

Generally, the critical local quantities of interest like stress of state or damage are desired in a particular lamina or at the interface of two laminae. Analysis of initially damaged laminates with embedded lamina level damage or delamination requires accurate representation of the local displacement, strain and stress components. In such a case, the equivalent models cannot be used. The intermediate models are based on defining the director functions for a group of laminae and not the full laminate. The director functions are given as

$$\begin{aligned} u(x,y,z) &= \sum_{i=1}^{n_1} u_i(x,y)\overline{M}_i(z) \\ v(x,y,z) &= \sum_{i=1}^{n_2} v_i(x,y)\overline{M}_i(z) \\ w(x,y,z) &= \sum_{i=1}^{n_3} w_i(x,y)\overline{M}_i(z) \end{aligned} \quad (3)$$

Where $n_1 = n_2$ and n_3 depends upon the order of transverse approximation and number of lamina groups in thickness direction. The director functions for this model are shown in Fig. 2(c).

The members of this family are denoted by the $IMp_{xy}p_z^u p_z^v p_z^w$.

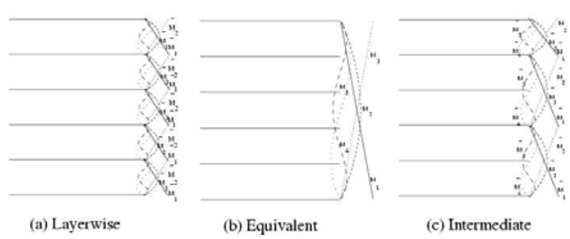
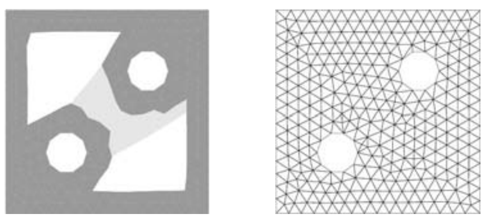


Fig. 2: Director Functions Over Laminate Thickness for (a) Layerwise (b) Equivalent and (c) Intermediate Models

3.0 REGION-BY-REGION MODEL (RR)

3.1 Motivation

In a structural component, the “hot-spots” are generally localized in the vicinity of structural details, boundaries of the domain (faces and edges), re-entrant corners, Cut-outs, existing delaminations and ply failure zones. The solution is unsmooth in the vicinity of these details, while it is very smooth in the remaining part of the domain (see Fig. 1 and 3(a)). In order to get an accurate representation of the solution everywhere, it is desirable to use an enriched approximation model (*LM* or *IM* with sublaminae (i.e. to divide a lamina into two or more laminae) if desired) only in the vicinity of the "hot-spot", while in the rest of the domain, a lower order model will suffice. In this study, p_{xy} is uniform over the whole domain, while the approximation enrichment is done by using either a higher value of p_z^i and/or a more refined model, e.g. *IM* or *LM*. Thus it is important to build the capability to put any desired model in a specified region, rather than doing an overkill by using a higher model everywhere in the domain (which will be computationally very expensive). This concept has been introduced through the region-by-region modeling approach described in this section.



(a) (b)

Fig. 3: (a) Typical Plate Domain with Cut-outs (b) and Meshing

3.2 Concept of Regions

Let us consider the domain given in Fig. 3(a), with the two circular cut-outs shown. In the vicinity of the cut-outs and the outer boundaries of the domain (shown grey shaded in Fig. 3), the solution is expected to be unsmooth, have severe boundary layer effect, and possibly be three-dimensional in nature locally. Hence along with a refinement of the mesh, enrichment of the model will also be desired in the shaded regions. Thus, the domain is divided into multiple regions (three regions shown by different shades of grey). The plate model is then fixed for each region. For example, for the domain of Fig. 3(a) *EQ* may be used in the unshaded region; *LM* may be used in the region shaded dark grey.

3.3 Concept of Groups

Let the laminate have *nl* layers. The advantage of one dimensional hierarchic basis functions is exploited to define a generic representation of the group structure. The base two-dimensional mesh T_{2D} (with nel_{2D} number of elements) is made first over the projected two-dimensional surface (see Fig. 3(b) for example). Here meshes of triangles are used. Using this base two-dimensional mesh, the three dimensional mesh T_{3D} of prismatic elements is made over whole domain, layer by layer. Hence, the number of elements is T_{3D} is $nel_{2D}\, X\, nl$. Each two-dimensional element $\tau_{2D} \subseteq T_{2D}$ is assigned the set of all the *nl* three dimensional elements $\tau_{3D} \subseteq T_{3D}$, whose projection on the plane is τ_{2D}. This set is denoted by $P_{\tau 2D}$. For each element τ_{2D}, the type of model to be used through the thickness is then specified. The model is fixed by the region-by-region allocation described above. It may be noted that the neighbouring two-dimensional elements may have the same or different models by this strategy. For the element τ_{2D} we specify the number of groups $ng_{\tau 2D}$. For each group $g_{i,\tau_{2D}}$, $i = 1, 2, \cdots, ng_{\tau 2D}$, the three-dimensional elements $\tau_{3D} \subseteq P_{\tau 2D}$ contained, are specified. Thus, a group will contain one or more three-dimensional elements that are stacked on top of each other. Some of the possible groups are shown in Fig. 4 through frontal view.

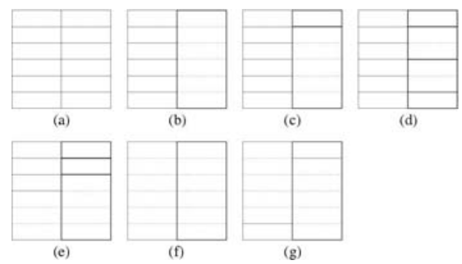


Fig. 4: Grouping Strategies for Region-by-region Model

3.4 Imposition of Constraints

In this section the concept of constrained approximation will be discussed. The ideas are generalization of the concept introduced in [3] and [4]. In order to fix the ideas let us consider a one-dimensional example. Let us take an interval (0, L) with one element, as shown in Fig.5 (a). Let us also assume that piecewise linear basis functions (i.e. *p* = 1) are defined over this mesh.

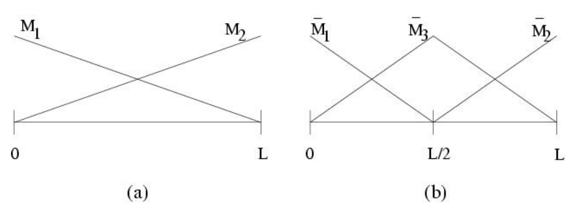


Fig. 5: Constraint Imposition

Let

$$v(z) = \sum_{i=1}^{p+1} a_i M_i(z) \quad (4)$$

be the representation of a function over this domain. Here, $M_i(z)$ are the linear basis functions defined as shown in Fig. 5(a). Let us now subdivide this element into two equal sub-elements, let the function $v(z)$, given above, be represented in terms of the piecewise linear basis functions (as shown in Fig. 5(b)) as

$$v(z) = \sum_{i=1}^{2p+1} \bar{a}_i \bar{M}_i(z) \qquad (5)$$

where, $\bar{M}_i(z)$ are the piecewise linear basis functions defined over the new mesh. Since both Eq. (4) and (5) represent the same function, the coefficients $\bar{a}_i$ can be expressed in terms of the coefficients a_j. It is obvious that

$$\bar{a}_1 = a_1; \quad \bar{a}_3 = \frac{a_1 + a_2}{2}; \quad \bar{a}_2 = a_2 \qquad (6)$$

Similarly, the representation of $v(z)$ over any finer mesh can be obtained in terms of the representation over the coarser mesh, with the new fine mesh coefficients $\bar{a}_j$ constrained by the values of the coefficients a_i for the coarser mesh. This can be easily extended to any *p*-order approximation defined over the coarser and fine meshes. As shown below, the transverse representation of the finite element solution is defined over a group. However, the basic building block in the analysis is the individual three-dimensional element τ_{3D}. Hence, the approach given above will be employed to represent the element degrees of freedom in terms of the group degrees of freedom. The region-by-region model is denoted by the $RRp_{xy} p_z^u p_z^v p_z^w$.

3.5 Finite Element Implementation for Region-by-region Model

The total potential is given as

$$\Pi = \sum_{\tau} \frac{1}{2} \{u^\tau\}^T [K^\tau] \{u^\tau\} - \{u^\tau\}^T \{F^\tau\} \qquad (7)$$

Where, $\{u^\tau\}$ denotes the displacement vector, $[K^\tau]$ denotes the stiffness matrix and $\{F^\tau\}$ denotes the load vector corresponding to element $\tau_{3D}\,(or\ \tau)$. The element degrees of freedom are constrained by group degrees of freedom as $\{u^\tau\} = [A^\tau]\{u^G\}$ where $[A^\tau]$ is the elemental constraint matrix. Thus, the total potential can be re-written as

$$\Pi = \sum_{\tau} \frac{1}{2} \{u^G\}^T [A^\tau]^T [K^\tau][A^\tau]\{u^G\} - \{u^G\}^T [A^\tau]^T \{F^\tau\} \qquad (8)$$

Minimization of Π gives the solution for $\{u^G\}$. Note that this gives the element stiffness matrix $[\bar{K}^\tau]$ and load vector $\{\bar{F}^\tau\}$ in terms of the constraint matrix, as

$$[\bar{K}_\tau] = [A^\tau]^T [K^\tau][A^\tau]; \quad \{\bar{F}^\tau\} = [A^\tau]^T \{F^\tau\} \qquad (9)$$

4.0 MODELING ERROR INDICATOR AND ADPTIVE SELECTION OF MODELS

The total error for the finite element solution of a given plate model can be given (in norm) as

$$\| \mathrm{u}_{3D} - \mathrm{u}_h^M \| \le \| \mathrm{u}_{3D} - \mathrm{u}^M \| + \| \mathrm{u}^M - \mathrm{u}_h^M \| \qquad (10)$$

Where u_{3D} is the exact three dimensional solution of the given system, u_h^M is the finite element solution of the given plate model and u^M is the exact solution of the given plate model. The first part on the right side of the Eq. (10) is the modeling error and the second part is discretisation error. When the discretisation error is controlled, the total error is due to modeling error only.

Here, following [6] explicit type modeling error indicator is given in terms of interelement jumps. The novelty of the proposed indicator over the [6] is that it uses the postprocessed stresses to compute the interelement jumps thus leading to smaller value of the indicator. Here, it is to be noted that since postprocessed stresses are used the jumps at the top and bottom faces of an element are zero. Thus, the contribution to the interelement jumps is only due to jumps at the side faces of the elements.

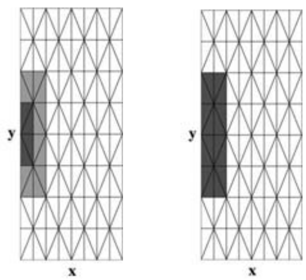
5.0 NUMERICAL RESULTS

Here the key results are given to demonstrate the efficacy of the region-by-region model developed. Also the accuracy of the models and computational cost is comapared. Further, an example with ply level damage is given to elucidate the adaptive selection of the models for the region-by-region modeling approach.

5.1 Comparison of Plate Models

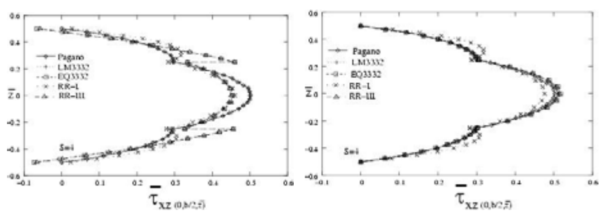
Here, [150/-150/150] laminate under cylindrical loading of the type $q = q_0 \sin(\pi x / a)$ is considered here. Here, *a* is the *x*-dimension and *b* is the *y*-dimension of the laminate. The laminate is made of high modulus Graphite/Epoxy composite material. The properties can be seen in [5]. The length to thickness ratio of the laminate is 4. The laminate is infinite in *y* direction. The thickness of outer laminae is 1 *mm* each and middle lamina is 2 *mm*. The through thickness variation of transverse shear stress τ_{xz} at point (0,b/2) is given in Fig 7. The stress is nondimensionalized as $\bar{\tau}_{xz} = \tau_{xz} / q_0$. The stress obtained directly from the finite element data and use of constitutive equation is given in Fig. 7(a) and that obtained by the use of equilibrium based postprocessing is given in Fig. 7(b). The region-by-region modeling strategies *RR-I* and *RR-II* for this numerical example are given in Fig. 6(a) and (b), respectively. In *RR-I* the darker region shows the *LM3332* model, the light shaded region shows *LM3112* model and in the rest of the region *EQ3112* model is used. In *RR-II* model, the darker region uses *EQ3332* model while rest of the domain uses *EQ3112* model. The results are also obtained using *LM3332* and *EQ3332* models in the whole domain. The results are compared with exact three-dimensional elasticity solution given in [5].

The computational cost for each of the models is given in Table 1.



(a) (b)

Fig. 6: Region-by-region Schemes for [150/-150/150] Laminate Under Cylindrical Bending (a) *RR-I* (b) *RR-II*



(a) (b)

Fig. 7: Through Thickness Variation of Transverse Shear Stress in [150/-150/150] Laminate Under Cylindrical Bending (a) Direct Stresses (b) Postprocessed Stresses

Here, it is to be noted that the meshes shown in Fig. 6 are truncated in *y*-direction.

Table 1: Number of Unknowns for Models Used

Models	**LM3332**	**EQ3332**	**RR-I**	**RR-II**
Unknowns	11875	6875	5239	4715

From the Fig. 7 it can be seen that

- The *LM3332* model accurately predicts the stress components both by direct and postprocessed approach.
- The direct stresses obtained using *EQ3332* model shows jumps at interlaminar interface whereas with postprocessing the stresses are accurate.
- The stresses obtained by *RR-I* and *RR-II* models by direct and postprocessed approach are very close to the exact one.
- The stresses obtained by the *RR-II* model are very close to those obtained by the *EQ3332* model.

From the Table 1 it can seen that

- The *LM3332* model is computationally very expensive.
- The RR-I and RR-II strategies are computationally very economical. The savings in computational cost using region-by-region models over layerwise can be upto 50%.

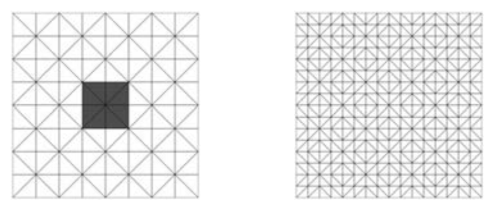
5.2 Modeling Error Estimation and Control

Here, [0/90] square laminate of dimensions *a*=*b*=100 *mm* with all edges clamped and under uniform transverse load of intensity 0.0001 N/mm^2 is considered. The laminate is made of M55J/M18 material. The thickness of each lamina is 0.1 *mm*. The thickness of interface between the layers is 0.01 *mm* and is made of M18 Epoxy. The bottom layer has a square damage of size 25X25 *mm* in the fibre fracture mode.

The starting model is *EQ3333* in the whole domain.

Note: The discretisation and modeling error tolerance specified for this study is 5% each.

The initial mesh with the location of damage is shown in Fig. 8(a). The discretisation error for this mesh is 58%. The discretisation error is controlled by controlling error in the energy norm using smoothening based discretisation error estimator and control methodology developed by authors in [1]. The Fig. 8(b) shows the adapted mesh. The discretisation error for this mesh is 0.9%.



(a) (b)

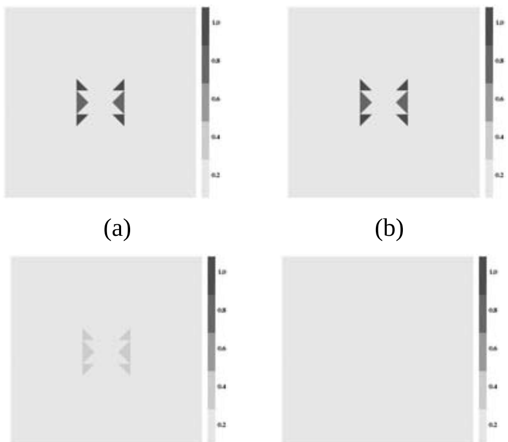
Fig. 7: (a) Initial Mesh with Location and Size of Damage and (b) Final Adapted Mesh

The densities of energy of the error for the sequences of the models adapted are shown in Fig. 8. Here, it is to be noted that the density of energy of the error is nondimensionalized by the maximum density of energy of the error for the lowest order (*EQ3333*) model.

The key points of this study are:

- The modeling error for *EQ3333* model everywhere in the domain is 16% (see Fig. 8(a)). The region near the damage front shows the highest error in the density of energy of error.
- The second sequence is *LM3333* model in the region of highest error (see Fig. 8(b)). In the rest of the domain *EQ3333* model is present. The modeling error for this sequence is 16%.

- The next model is same as above with two sublayers in the bottom lamina. The modeling error for this sequence is 8% (see Fig. 8(c)).
- Finally, the model same as in above with additional two sublayers (total four sublayers) in the bottom lamina gives the modeling error of 4%, which is below the specified tolerance.



(a) (b) (c) (d)

Fig. 8: Densitiy of Energy of Error for Sequences of Models Adapted (a) *EQ3333* Model Everywhere (16%) (b) *EQ3333* + *LM3333* Model (16%) (c) *EQ3333* + *LM3333* Model with 2 sublayers in bottom lamina (8%) and (d) *EQ3333* + *LM3333* Model with 4 sublayers in bottom lamina (4%)

The models adapted in the first step of the sequences of the adapted models are shown in Fig. 9. The darker region shows the *LM3333* model while rest of the region has *EQ3333* model.

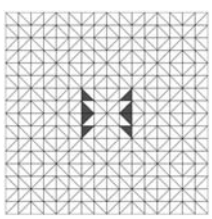


Fig. 9: The Models Adapted for the First Step in the Sequence: in the Darker region LM3333 Model elsewhere *EQ3333* Model

2.0 CONCLUSION

In this paper a families of plate models for layered composites have been proposed. The novel concept of the region-by-region modeling approach has been successfully implemented. Further, a reliable explicit type modeling error indicator using postprocessed transverse stresses has been proposed for the adaptive selection of models in the region-by-region strategy. The key points from this study that can be concluded are:

- The layerwise modeling approach is very accurate pointwise in predicting the state of stress obtained both with direct and postprocessing approach but computationally very expensive.
- The equivalent model is also accurate with postprocessing approach and computationally very economical.
- The region-by-region modeling approach is as accurate as layerwise model and computationally very economical. The saving in computational cost can be as high as 25-50% over the layerwise model.
- The proposed modeling error indicator is reliable.
- The layerwise model reduces the modeling error.
- The concept of sublayer is very effective in reducing the modeling error.
- The damaged front region requires the highest order models with sublayers to capture the effects accurately.

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