Generalizations of the Łoś-Tarski Preservation Theorem

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Introduction

- Preservation theorems have been one of the earliest areas of study in classical model theory.
- A preservation theorem characterizes (definable) classes of structures closed under a given model theoretic operation.
- Preservation under substructures, extensions, unions of chains, homomorphisms, etc.
- Most preservation theorems fail in the finite.
- Some preservation results recovered over special classes of finite structures, like those with bounded degree, bounded tree-width etc. (Dawar et al.)
- Homomorphism preservation theorem is true in the finite (Rossman).
Some assumptions and notations for the talk

Assumptions:

- First Order (FO) logic.
- Arbitrary vocabularies (constants, predicates and functions)
- Arbitrary structures typically, unless stated otherwise explicitly.

Notations:

- $\Sigma_1 = \exists^* (\ldots), \Pi_1 = \forall^* (\ldots)$
- $\Sigma_2 = \exists^* \forall^* (\ldots), \Pi_2 = \forall^* \exists^* (\ldots)$
- $M_1 \subseteq M_2$ means $M_1$ is a substructure of $M_2$. For graphs, $\subseteq$ means induced subgraph.
- $U_M = \text{universe of } M$. 

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A Brief Recap of the Related Talk in CLC 2012
Preservation under Substructures

Definition 1 (Pres. under subst.)

A sentence $\phi$ is said to be preserved under substructures, denoted $\phi \in PS$, if $((M \models \phi) \land (N \subseteq M)) \rightarrow N \models \phi$.

- E.g.: Consider $\phi = \forall x \forall y E(x, y)$ which describes the class of all cliques.
- Any induced subgraph of a clique is also a clique. Then $\phi \in PS$.
- In general, every $\Pi_1$ sentence (i.e. $\forall^*$ sentence) is in $PS$.

Theorem 1 (Łoś-Tarski, 1960s)

A FO sentence in $PS$ is equivalent to a $\Pi_1$ sentence.
Preservation under substructures modulo finite cores

Definition 2

A sentence $\phi$ is said to be preserved under substructures modulo a finite core, denoted $\phi \in PSC_f$, if for each model $M$ of $\phi$, there is a finite subset $C$ of $U_M$ s.t. $((N \subseteq M) \land (C \subseteq U_N)) \rightarrow N \models \phi$.

- The set $C$ is called a core of $M$ w.r.t. $\phi$. If $\phi$ is clear from context, we will call $C$ as a core of $M$.
- For every $\phi \in PS$, for each model $M$ of $\phi$, the empty subset is a core of $M$. Then $PS \subseteq PSC_f$. 

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  - There can be cores that are not witnesses for $x$. 
Example

- Eg. Consider $\phi = \exists x \forall y E(x, y)$.

  $M \models \phi$

  $a$ is the witness for $x$

- Any witness for $x$ is a core. Thus $\phi \in PSC_f$.

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Example

- Eg. Consider $\phi = \exists x \forall y E(x, y)$.

- Any witness for $x$ is a core. Thus $\phi \in PSC_f$.
- There can be cores that are not witnesses for $x$.
- Every model of $\phi$ has a core of size $\leq 1$. 

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Example (Contd.)

\[ M \models \phi \]

- Observe: \( \phi \notin PS \).
Example (Contd.)

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- Observe: $\phi \notin PS$.
- Easy to see: $PS \subseteq PSC_f$. Then $PS \subsetneq PSC_f$.
- In general, $\Sigma_2 \subseteq PSC_f$. In fact, for $\varphi \in \Sigma_2$, each model has a core of size $\leq$ the number of $\exists$ quantifiers of $\varphi$.  

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In general, $\Sigma_2 \subseteq PSC_f$. In fact, for $\varphi \in \Sigma_2$, each model has a core of size $\leq$ the number of $\exists$ quantifiers of $\varphi$.

Interestingly, even for an arbitrary $\phi \in PSC_f$, there exist cores of bounded size in all models!
$PSC_f \equiv \Sigma_2$

**Theorem 2**

A sentence $\phi \in PSC_f$ iff $\phi$ is equivalent to a $\Sigma_2$ sentence.

**Corollary 3 (Finite core implies bounded core)**

If $\phi \in PSC_f$, there exists $k \in \mathbb{N}$ such that every model of $\phi$ has a core of size at most $k$.

*Proof:* Take $k$ to be the number of $\exists$ quantifiers in the equivalent $\Sigma_2$ sentence guaranteed by Theorem 2.
Preservation under substructures modulo Bounded Cores

Definition 3 (Pres. under subst. modulo bounded cores)

A sentence $\phi$ is said to be preserved under substructures modulo a core of size $k$, denoted $\phi \in PSC(k)$, if $\phi \in PSC_f$ and each model $M$ of $\phi$ has a core of size at most $k$.

- Observe that $PSC(0) = PS$.
- Easy to see that $PSC(l) \subsetneq PSC(k)$ for $l < k$. Consider $\phi$ which says that there are at least $k$ distinct elements in any model. Then $\phi \in PSC(k) \setminus PSC(l)$.
- Let $PSC = \bigcup_{k \geq 0} PSC(k)$.
Towards a Syntactic Characterization of $PSC(k)$

Since finite core implies bounded core, we have

**Lemma 4**

\[ PSC = PSC_f. \]

- A $\Sigma_2$ sentence $\phi$ with $k \exists$ quantifiers is in $PSC(k)$.
- In the converse direction, $\phi \in PSC(k)$ has an equivalent $\Sigma_2$ sentence.

**Question**: For $\phi \in PSC(k)$, is there an equivalent $\Sigma_2$ sentence having $k \exists$ quantifiers?
A sentence is in $PSC(k)$ iff it is equivalent to a $\Sigma_2$ sentence having $k$ existential quantifiers.

- The proof uses the notion of saturations from classical model theory.
- Theorem 5 works over arbitrary vocabularies and over any class of structures definable by FO theories.
- The case of $k = 0$ is exactly the Łoś-Tarski theorem for sentences.
Preservation Properties Dual to $PSC(k)$ and $PSC_f$
Preservation under Extensions

Definition 4

A sentence $\phi$ is said to be preserved under extensions, denoted $\phi \in PE$, if $(\models (M \models \phi) \land (M \subseteq N)) \rightarrow N \models \phi$.

E.g.: Let $\phi = \exists x \exists y E(x, y)$. Easy to see that $\phi \in PE$.

Following is a duality lemma.

Lemma 6

A sentence $\phi$ is in $PS$ iff $\neg \phi$ is in $PE$.

Theorem 7 (Łoś-Tarski, 1960s)

A FO sentence in $PE$ is equivalent to a $\Sigma_1$ sentence.
An Alternate Form of Łoś-Tarski Theorem

**Definition 5**

A structure $M$ is said to be an extension of a collection $R$ of structures, denoted $R \subseteq M$, if for each $N \in R$, we have $N \subseteq M$.

- Easy to check: Preservation under extensions of single structures $\equiv$ Preservation under extensions of collections of structures.
- Then $PE$ can be defined to be preservation under extensions of collections of structures and the Łoś-Tarski theorem statement would still be true.
**Definition 6**

For $k \in \mathbb{N}$, a structure $M$ is said to be a $k$-ary covered extension of a non-empty collection $R$ of structures if (i) $M$ is an extension of $R$, and (ii) for every $A \subseteq U_M$ s.t. $|A| \leq k$, there is a structure in $R$ that contains $A$. We call $R$ a $k$-ary cover of $M$. 
Definition 6

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**Diagram:**

- **$G_2$**: A simple graph with two nodes $a$ and $b$ connected by an edge.
- **$M$**: A larger graph with nodes $a$, $b$, $c$, and $d$, connected in a specific pattern.

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\textbf{$k$-ary Covered Extensions}

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[Diagram of $G_1$ and $M$]
\(k\)-ary Covered Extensions

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$k$-ary Covered Extensions

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![Diagram](G1.png)

![Diagram](M.png)
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$R$ is a 2-ary cover of $M$
Definition 6

For $k \in \mathbb{N}$, a structure $M$ is said to be a $k$-ary covered extension of a non-empty collection $R$ of structures if (i) $M$ is an extension of $R$, and (ii) for every $A \subseteq U_M$ s.t. $|A| \leq k$, there is a structure in $R$ that contains $A$. We call $R$ a $k$-ary cover of $M$.

$R$

(is a 1-ary cover of $M_1$ and $M_2$)
\(k\)-ary Covered Extensions

**Definition 6**

For \(k \in \mathbb{N}\), a structure \(M\) is said to be a \(k\)-ary covered extension of a non-empty collection \(R\) of structures if (i) \(M\) is an extension of \(R\), and (ii) for every \(A \subseteq U_M\) s.t. \(|A| \leq k\), there is a structure in \(R\) that contains \(A\). We call \(R\) a \(k\)-ary cover of \(M\).

\(R\) has no extension!
**Definition 6**

For \( k \in \mathbb{N} \), a structure \( M \) is said to be a \( k \)-ary covered extension of a non-empty collection \( R \) of structures if (i) \( M \) is an extension of \( R \), and (ii) for every \( A \subseteq U_M \) s.t. \( |A| \leq k \), there is a structure in \( R \) that contains \( A \). We call \( R \) a \( k \)-ary cover of \( M \).
Definition 7

Given $k \in \mathbb{N}$, a sentence $\phi$ is said to be preserved under $k$-ary covered extensions, denoted $\phi \in PCE(k)$, if for each collection $R$ of models of $\phi$, $(M$ is a $k$-ary covered extension of $R) \rightarrow M \models \phi$. 

Preservation under $k$-ary Covered Extensions ($PCE(k)$)
Definition 7

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E.g.: $\phi = \forall x \forall y \exists z ((x = y) \lor E(x, y) \lor (E(x, z) \land E(z, y)))$. 
Definition 7

Given $k \in \mathbb{N}$, a sentence $\phi$ is said to be *preserved under $k$-ary covered extensions*, denoted $\phi \in PCE(k)$, if for each collection $R$ of models of $\phi$, $(M$ is a $k$-ary covered extension of $R) \rightarrow M \models \phi$.

E.g.: $\phi = \forall x \forall y \exists z \left( (x = y) \lor E(x, y) \lor (E(x, z) \land E(z, y)) \right)$.

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- E.g.: $\phi = \forall x \forall y \exists z ((x = y) \vee E(x, y) \vee (E(x, z) \land E(z, y)))$.

![Diagram](attachment:G1-M.png)
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E.g.: $\phi = \forall x \forall y \exists z ((x = y) \lor E(x, y) \lor (E(x, z) \land E(z, y)))$.

![Diagram](attachment:diagram.png)
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- E.g.: $\phi = \forall x \forall y \exists z \left( (x = y) \lor E(x, y) \lor (E(x, z) \land E(z, y)) \right)$.

$G_1 \models \left\{ E(b, c), E(c, d) \right\}$

$M \models \left\{ E(b, c), E(c, d) \right\}$
Preservation under $k$-ary Covered Extensions ($PCE(k)$)

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- E.g.: $\phi = \forall x \forall y \exists z ((x = y) \vee E(x, y) \vee (E(x, z) \wedge E(z, y)))$.

![Diagram](image-url)
Definition 7

Given \( k \in \mathbb{N} \), a sentence \( \phi \) is said to be preserved under \( k \)-ary covered extensions, denoted \( \phi \in PCE(k) \), if for each collection \( R \) of models of \( \phi \), \((M \text{ is a } k\text{-ary covered extension of } R) \rightarrow M \models \phi\).

E.g.: \( \phi = \forall x \forall y \exists z \left( (x = y) \lor E(x, y) \lor (E(x, z) \land E(z, y)) \right) \).

\[ G_3 \models \begin{cases} E(a, c) \\ E(c, d) \end{cases} \]

\[ M \]
Definition 7

*Given* $k \in \mathbb{N}$, a sentence $\phi$ is said to be *preserved under* $k$-*ary covered extensions*, denoted $\phi \in PCE(k)$, if for each collection $R$ of models of $\phi$, $(M$ is a $k$-*ary covered extension of* $R) \rightarrow M \models \phi$.

- E.g.: $\phi = \forall x \forall y \exists z ((x = y) \lor E(x, y) \lor (E(x, z) \land E(z, y)))$. 

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Preservation under $k$-ary Covered Extensions ($PCE(k)$)

Definition 7

Given $k \in \mathbb{N}$, a sentence $\phi$ is said to be preserved under $k$-ary covered extensions, denoted $\phi \in PCE(k)$, if for each collection $R$ of models of $\phi$, $(M$ is a $k$-ary covered extension of $R) \rightarrow M \models \phi$.

E.g.: $\phi = \forall x \forall y \exists z \left( (x = y) \lor E(x, y) \lor (E(x, z) \land E(z, y)) \right)$.

$G_1 \models \phi$, $G_2 \models \phi$, $G_3 \models \phi$, $R \models \phi$, $M \models \phi$.

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The Duality of $PSC(k)$ and $PCE(k)$

Lemma 8

A sentence $\phi$ is in $PSC(k)$ iff $\neg \phi$ is in $PCE(k)$.

Proof Sketch:
(We prove the ‘If’ direction; the ‘Only If’ is by a dual argument. Below, $A \subseteq_k B$ means $A \subseteq B$ and $|A| \leq k$. )

- Suppose $M \models \phi$ and there is no $k$-core in $M$.
- Then for each $A \subseteq_k U_M$, there exists $N_A \subseteq M$ containing $A$ s.t. $N_A \models \neg \phi$.
- Then $R = \{ N_A \mid A \subseteq_k U_M \}$ forms a $k$-ary cover of $M$. Since $\neg \phi \in PCE(k)$, we get $M \models \neg \phi$ – a contradiction.
A Syntactic Characterization of $PCE(k)$

**Theorem 9**

A sentence $\phi$ is in $PCE(k)$ iff $\phi$ is equivalent to a $\Pi_2$ sentence having $k$ universal quantifiers.

**Proof Sketch:**

- Let $\Gamma = \{ \psi \mid \psi = \forall^k \exists^*(\ldots), \phi \rightarrow \psi \}$. Clearly, $\phi \rightarrow \Gamma$.
- Show that $\Gamma \rightarrow \phi$ holds over the class $C$ of $\alpha$-saturated structures, where $\alpha \geq \omega$.
- Use the fact that every structure has an elementarily equivalent structure in $C$ to show that $\Gamma \rightarrow \phi$ holds over all structures.
- Finally, by Compactness theorem, the result follows.
A Generalization of the Łoś-Tarski Theorem

Theorem 9 and the $PSC(k)$-$PCE(k)$ duality imply the following.

**Theorem 5**

A sentence $\phi$ is in $PSC(k)$ iff $\phi$ is equivalent to a $\Sigma_2$ sentence with $k$ existential quantifiers.

- Theorem 5 gives us exactly the substructural version of Łoś-Tarski theorem for $k = 0$.
- Theorem 9 gives us exactly the extensional form of the Łoś-Tarski theorem for $k = 0$. 
Preservation under Finitary Covered Extensions ($PCE_f$)

- **Finitary covered extension** – replace ‘$k$-ary’ in the definition of $k$-ary covered extension with ‘finitary’.

- **Preservation under finitary covered extensions**, denoted $PCE_f$, – replace ‘$k$-ary’ with ‘finitary’ in the $PCE(k)$ defn.

**Lemma 10**

A sentence $\phi$ is in $PSC_f$ iff $\neg \phi$ is in $PCE_f$.

**Theorem 11**

A sentence $\phi$ is in $PCE_f$ iff $\phi$ is equivalent to a $\Pi_2$ sentence.

**Corollary 12**

$PCE_f = \bigcup_{k \geq 0} PCE(k)$.
Comparison with Semantic Characterizations of $\Sigma_2$ and $\Pi_2$ in the Literature

- Define $PSC = \bigcup_{k \geq 0} PSC(k)$ and $PCE = \bigcup_{k \geq 0} PCE(k)$. Theorems 5 and 9 give new semantic characterizations of $\Sigma_2$ and $\Pi_2$ via $PSC$ and $PCE$ respectively.

- Existing characterizations in the literature for $\Sigma_2$ and $\Pi_2$ are via unions of ascending chains, intersections of descending chains, Keisler’s 1-sandwiches, etc. None of these relate the count of the quantifiers to any model-theoretic properties, and hence do not generalize the Łoś-Tarski theorem.

- The $PSC$ and $PCE$ conditions are combinatorial in nature unlike any of the above literature notions.

- All of the above literature notions become trivial in the finite. However, there are sentences inside and outside of $PSC$ and $PCE$ in the finite.
Our Preservation Theorems over Finite Structures

- The failure of Łoś-Tarski theorem in the finite implies the failure of Theorems 5 and 9. In fact, the failure is stronger.

**Theorem 13**

$PSC(k)$, resp. $PCE(k)$, is strict semantic superset of the class of $\exists^k\forall^*$ sentences, resp. $\forall^k\exists^*$ sentences, for each $k \in \mathbb{N}$.

- However, for each $k$, the example witnessing the strict subsumption of $\exists^k\forall^*$ sentences by $PSC(k)$, is a $\exists^{k+1}\forall^*$ sentence – which is therefore in $PSC(k+1)$.

- This raises the possibility that $PSC(k)$ is semantically subsumed by the class of $\exists^l\forall^*$ sentences for some $l > k$.

- If so, then $PSC \equiv \Sigma_2$ and $PCE \equiv \Pi_2$ over the class of finite structures as well!
A Quick Note on Further Generalizations

- For $n \geq 1$, let $\Sigma_n(k_1, k_2, *, k_4, *, \ldots)$ be the subset of $\Sigma_n$ in which each sentence has $k_1$ quantifiers in the first block and $k_2, k_4, \ldots$ quantifiers in the even indexed blocks. Likewise define $\Pi_n(k_1, k_2, *, k_4, *, \ldots)$.

- We have semantic characterizations for $\Sigma_n(k_1, k_2, *, k_4, *, \ldots)$ and $\Pi_n(k_1, k_2, *, k_4, *, \ldots)$ for each $n \geq 1$ and each $k_1, k_2, k_4, \ldots \in \mathbb{N}$ via variants of the $PSC(k)$ and $PCE(k)$ notions.

- These give us new and much finer characterizations of $\Sigma_n$ and $\Pi_n$ compared to those in the literature via unions of ascending $\Sigma_n$-chains and Keisler’s $n$-sandwiches.
Directions for Future Work
Future Work

Over arbitrary structures:
- Semantic characterizations of $\Sigma_n$ and $\Pi_n$ sentences in which the number of quantifiers in each block is given.
- A syntactic characterization of theories in $PSC(k)$ and $PCE(k)$.

Over finite structures:
- Investigating if $PSC = \Sigma_2$ and $PCE = \Pi_2$ over the class of all finite structures.
- Characterizing $PSC(k)$ and $PCE(k)$ over interesting classes of finite structures like equivalence relations, partial orders, acyclic graphs, graphs of bounded degree, bounded tree-width, bounded split-width, etc.
References I


Thank you!
Appendix
An Intuitive but Incorrect Attempt at Characterizing $PSC(k)$

- Let $\phi \in PSC(k)$, $S = \text{Models}(\phi)$, $\text{Vocab}(\phi) = \tau$, $\tau_k = \tau \cup \{c_1, \ldots, c_k\}$.
- Let $Z$ be the class of models of $\phi$ expanded with their core elements. Formally, $Z = \{(M, a_1, \ldots, a_k) \mid M \in S \text{ and } a_1, \ldots, a_k \text{ forms a core in } M\}$.
- Clearly $Z$ is pres. under substr. Then by Łoś-Tarski theorem, $Z$ is captured by a $\Pi_1$ sentence. Replace $c_1, \ldots, c_k$ with fresh variables $x_1, \ldots, x_k$ and existentially quantify out the latter.
- **Error**: $Z$ is assumed FO definable.
- The above proof attempt fails for as simple a sentence as $\phi = \exists x \forall y E(x, y)$. (In fact, $Z$ in this case is not definable by any FO theory too!)