On Axiomatizations of Dynamic Epistemic Logic

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(based on joint work with Guillaume Aucher)
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### Motivation

### Alternative Axiomatization

### Conclusions and future work
Background

Two modal logic approaches handling knowledge and actions:

- *Epistemic* Temporal Logic (**ETL**): knowledge in distributed systems based on *temporal logic*. [Fagin et al., 1995, Parikh and Ramanujam, 1985]

Background

They are both semantics-driven two-dimensional modal logics:

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<tr>
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<tr>
<td>DEL</td>
<td>K+events</td>
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\[ \neg Kp \land EF Kp \]

\[
\begin{array}{c}
\bullet p \\
\downarrow \\
p
\end{array}
\]

\[ \neg Kp \land [!p]Kp \]

\[
\begin{array}{c}
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### Background

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\[-Kp \land EF \, Kp\]

\[-Kp \land [!p]Kp\]

Dynamic semantics:
The **meaning** of an event is the **change** it brings to the knowledge states (dates back to [Stalnaker, 1978]).
Bridging the two

**An earlier insight**: Iterated updating epistemic structures generates special ETL-style “super models” [van Benthem et al., 2009].

**Our approach**: relate the two via **axioms**.


**In this work:**

- New axiomatization of DEL using ETL-style axioms
- ETL-style completeness proof method for DEL-style logics
- Characterization results of product update and DEL-generatable ETL models.
Dynamic Epistemic Language (**LDEL**)

\[ \phi ::= \top | p | \neg \phi | \phi \land \phi | \square \phi | [e] \phi \]

where \( p \in P \) and \( e \in \Sigma \).

An event model \( \mathcal{U} \) is a tuple \((\Sigma, \rightarrow, Pre)\) where:

- \( \Sigma \) is a non-empty (countable) set of events.
- \( \rightarrow \subseteq \Sigma \times \Sigma \) is a binary relation on \( \Sigma \) (image finite).
- \( Pre : \Sigma \rightarrow LDEL \) is a function assigning each event a precondition (an \( LDEL \) formula).

Given an (epistemic) model \( M = (S, \rightarrow, V) \) and a **fixed** event model \( \mathcal{U} \), the semantics is as follows ([Baltag et al., 1998]):

\[
\begin{align*}
M, s \models \square \psi & \iff \forall t : s \rightarrow t \text{ implies } M, t \models \psi \\
M, s \models [e] \phi & \iff M, s \not\models Pre(e) \text{ implies } M \otimes \mathcal{U}, (s, e) \not\models \phi
\end{align*}
\]
## Redefinition-to-static-based axiomatization

**System $\mathcal{DE}$ (without Uni. sub.)**

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<tr>
<td>UCON</td>
<td>$[e](\phi \land \chi) \leftrightarrow ([e] \phi \land [e] \chi)$</td>
</tr>
<tr>
<td>UK</td>
<td>$[e] \Box \phi \leftrightarrow (\text{Pre}(e) \rightarrow \bigwedge_{f:e \rightarrow f} \Box[f] \phi)$</td>
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### Proof of completeness via reduction to basic modal logic $K$:

$$\vdash \phi \iff \vdash t(\phi) \implies \vdash_K t(\phi) \implies \vdash_{\mathcal{DE}} t(\phi) \implies \vdash_{\mathcal{DE}} \phi.$$ 

It does not come free. Be careful! [Wang and Cao, 2013]
New axiomatization

System $\mathcal{DEN}$

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<tr>
<td><strong>DISTU</strong></td>
<td>$[e](\phi \rightarrow \chi) \rightarrow ([e]\phi \rightarrow [e]\chi)$</td>
</tr>
<tr>
<td><strong>INV</strong></td>
<td>$(p \rightarrow [e]p) \land (\neg p \rightarrow [e]\neg p)$</td>
</tr>
<tr>
<td><strong>PRE</strong></td>
<td>$\langle e \rangle \top \leftrightarrow Pre(e)$</td>
</tr>
<tr>
<td><strong>NM</strong></td>
<td>$\Diamond\langle f \rangle\phi \rightarrow [e]\Diamond\phi$ (if $e \rightarrow f$ in $\mathcal{U}$)</td>
</tr>
<tr>
<td><strong>PR</strong></td>
<td>$\langle e \rangle\Diamond\phi \rightarrow \bigvee_{f:e \rightarrow f} \Diamond\langle f \rangle\phi$</td>
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New axiomatization

PR is in the shape of $\langle a \rangle \diamond \phi \to \diamond \langle a \rangle \phi$ (or $\Box [a] \phi \to [a] \Box \phi$).

NM is in the shape of $\diamond \langle a \rangle \phi \to [a] \diamond \phi$ (or $\langle a \rangle \Box \phi \to \Box [a] \phi$).

No Learning (NL) in ETL: $\diamond \langle a \rangle \phi \to \langle a \rangle \diamond \phi$ (or $[a] \Box \phi \to \Box [a] \phi$).

Note the difference between NM and NL:

$\diamond \langle a \rangle \phi \to [a] \diamond \phi$ (NM) vs. (NL) $\diamond \langle a \rangle \phi \to \langle a \rangle \diamond \phi$

NL is too strong: if you consider possible that an event is executable then it must be executable (take $\phi$ to be $\top$).
New proof method

Basic idea [WC12]: treat [e] as a **standard** modality interpreted on the following ETL models:

$$(S, \rightarrow, \{\overset{e}{\rightarrow}| e \in \Sigma\}, V)$$

$$\vdash \phi \iff \forall t : s \overset{e}{\rightarrow} t \text{ implies } M, t \vDash \phi$$

Proof strategy: find a class of ETL-style models $C$ and show the following:

$$\vdash \phi \implies C \vDash \phi \implies \vdash_{DE\Sigma N} \phi.$$ 

It works for other DEL-like logics.
The class $C$: normal ETL models

$\text{PRE: } \langle e \rangle \top \leftrightarrow Pre(e)$,

$\text{INV: } (p \rightarrow [e]p) \land (\neg p \rightarrow [e]\neg p)$,

$\text{PR: } \langle e \rangle \Diamond \phi \rightarrow \bigvee_{f: e \leftrightarrow f} \langle f \rangle \phi$,

$\text{NM: } \Diamond \langle f \rangle \phi \rightarrow [e] \Diamond \phi$ (if $e \leftrightarrow f$).

$\text{Pre}$. $s$ has $e$-successors iff $N, s \models Pre(e)$.

$\text{Inv}$. if $s \xrightarrow{e} t$ then for all $p \in P : t \in V(p) \iff s \in V(p)$.

$\text{Nm}$. if $s \rightarrow s'$ and $s' \xrightarrow{f} t'$ then for all $e$ and $t$ such that $s \xrightarrow{e} t$ and $e \leftrightarrow f$, we have $t \rightarrow t'$.

$\text{Pr}$. if $s \xrightarrow{e} t$ and $t \rightarrow t'$ then there exists an $s'$ such that $s \rightarrow s'$ and $s' \xrightarrow{f} t'$ for some $f$ such that $e \leftrightarrow f$ in $U$.

\[
\begin{array}{ccccccc}
    s & \rightarrow & s' & \quad \text{NM} \quad & s & \rightarrow & s' & \quad \text{PR} & s \\
    e & \rightarrow & f & \quad \Rightarrow & e & \rightarrow & f & \quad \Leftarrow & e \\
    t & \rightarrow & t' & & t & \rightarrow & t' & & t & \rightarrow & t'
\end{array}
\]
Same language, two logics: $\langle \text{LDEL}, \mathbb{M}, \vDash \rangle$ and $\langle \text{LDEL}, \mathbb{C}, \models \rangle$.

**Step 1 (Flatten the dynamics):**
If $w \rightarrow_e v$ in a normal model $\mathcal{N}$, then $\mathcal{N}^{-} \otimes \mathcal{U}, (w, e) \leftrightarrow \mathcal{N}^{-}, v$.

**Step 2:**
For any $\phi$ and any normal $\mathcal{N}, s$: $\mathcal{N}, s \models \phi \iff \mathcal{N}^{-}, s \vDash \phi$.

**Step 3:** $\vDash \phi \iff \mathbb{C} \vDash \phi$ (actually: $\vDash \phi \iff \mathbb{C} \models \phi$).  

**Step 4:** $\mathbb{C} \vDash \phi \iff \Vdash_{\text{DEN}} \phi$.

Finally: $\vDash \phi \iff \mathbb{C} \vDash \phi \iff \Vdash_{\text{DEN}} \phi$.  

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Application 1: Axiomatization of DEL with protocols

Enrich the epistemic model with state-dependent protocols:

\[ \mathcal{M}, \rho, s \models [e] \phi \iff \mathcal{M}, \rho, s \models \text{Pre}(e) \text{ and } e \in \rho(s) \]

implies \((\mathcal{M}, \rho) \odot \mathcal{U}, (s, e) \models \phi\)

We can axiomatize it by \(\mathcal{DE}N\) without \(\text{PRE}\) but the following:

\[ \text{PPRE} : \langle e \rangle \top \rightarrow \text{Pre}(e) \quad \text{and} \quad \text{DET} : \langle e \rangle \langle h \rangle \top \rightarrow [e] \langle h \rangle \top \]

Reduction (to epistemic logic) is not possible.
The proof system is equivalent to the system of [Hoshi and Yap, 2009] based on an ETL semantics.
Application 2: Characterization theorems

**Theorem**
\[ \text{NM, PR, INV and PRE characterize the update product operation.} \]

Similar result: [van Benthem, 2011, Ch 3.8] on PAL.

**Theorem**
\[ \text{Nm,Pr,Inv and Pre characterize the product update generatable image-finite ETL models.} \]

Similar results:
[van Benthem and Liu, 2004, van Benthem et al., 2009] (tree-like models and arbitrary event model)
Conclusions

- Event-model-based DEL can be viewed as a special case of step-wise ETL in terms of axioms.
- Our proof method does not rely on the reduction to static logic thus can be used for various DEL-like logics.
- Based on the axiomatization, various characterization results become transparent and simple.
- The new axioms are more meaningful (they even give new readings to red. axioms).
### Reduction-to-static-based axiomatization

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\[
\frac{\phi, \phi \rightarrow \psi}{\psi}
\]

\[
\frac{\phi}{\square \phi}
\]

\[
\frac{\phi \leftrightarrow \chi}{\psi \leftrightarrow \psi[\chi/\phi]}
\]
Future work

- DEL with common knowledge
- A Gentzen-style proof system
- DEL without INV nor PRE [Wang and Li, 2012] but iterations

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Searching for the right logic:
Desired ETL properties of agents

\[\implies\text{the corresponding dynamics (if possible)}\]
\[\implies\text{search for logics with the right computational properties}\]
The reduction axioms are boring, let’s have a (systematic) prison break!
Motivation

Alternative Axiomatization

Conclusions and future work


Distributed processes and the logic of knowledge.

Logics of public communications.

Assertion.

Logical Dynamics of Information and Interaction. Cambridge University Press.


Not all those who wander are lost: dynamic epistemic reasoning in navigation.