Negation fragment of graded consequence: an algebraic study

Classically, explosiveness condition, which states for any α, $\{\alpha, \neg\alpha\} \vdash \beta$ for any β, and reasoning by cases, that means for any α, $X \cup \{\alpha\} \vdash \beta$ and $X \cup \{\neg\alpha\} \vdash \beta$ imply $X \vdash \beta$, are two very important properties of negation ($\neg$). In the context of the theory of graded consequence, the graded version of these two properties are given as follows.
- There is some $k > 0$ such that $\inf_{\alpha,\beta} gr(\{\alpha, \neg\alpha\} \mid\sim \beta) = k$.
- There is some $c > 0$ such that $gr(X \cup \{\alpha\} \mid\sim \beta) * gr(X \cup \{\neg\alpha\} \mid\sim \beta) * c \leq gr(X \mid\sim \beta)$.
$\mid\sim$ is a fuzzy relation between $P(F)$ and F representing the graded counterpart of the two-valued notion of consequence $\vdash$.

An algebraic structure for a graded consequence relation satisfying the above two properties happens to be a complete residuated lattice where other than the top and the least elements two more elements, may be called the threshold for the law of contradiction and the law of excluded middle, need to be distinguished. In this presentation we shall explore some properties of this algebraic structure.