1. Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(A\) and \(B\) be two events with \(P(A) = 0.2\), \(P(B) = 0.4\) and \(P(A \cap B) = 0.1\). Find the probability that:
(a) exactly one of the events \(A\) or \(B\) will occur;  
(b) at least one of the events \(A\) or \(B\) will occur;  
(c) none of \(A\) and \(B\) will occur.

2. Suppose that \(n \geq 3\) persons \(P_1, \ldots, P_n\) are made to stand in a row at random. Find the probability that there are exactly \(r\) persons between \(P_1\) and \(P_2\); here \(r \in \{1, \ldots, n - 2\}\).

3. Three numbers are chosen at random from the set \(\{1, 2, \ldots, 50\}\). Find the probability that the chosen numbers are in geometric progression.

4. \textbf{(Matching Problem)} A secretary types \(n\) letters and the \(n\) corresponding envelopes. In a hurry, she places at random one letter in each envelope. What is the probability that at least one letter is in the correct envelope? Find an approximation of this probability for \(n = 50\).

5. In a probability space \((\Omega, \mathcal{F}, P)\), let \(\{E_n\}_{n \geq 1}\) be a sequence of events.
(a) If \(\{E_n\}_{n \geq 1}\) is an increasing sequence (written as \(E_n \uparrow\)), i.e., \(E_n \subseteq E_{n+1}\), \(n = 1, 2, \ldots\), then show that

\[
P(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} P(E_n).
\]

(b) If \(\{E_n\}_{n \geq 1}\) is a decreasing sequence (written as \(E_n \downarrow\)), i.e., \(E_{n+1} \subseteq E_n\), \(n = 1, 2, \ldots\), then show that

\[
P(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \to \infty} P(E_n).
\]

6. \textbf{(Generalized Boole's Inequality)} For a sequence \(\{E_k\}_{k \geq 1}\) of events, in a probability space \((\Omega, \mathcal{F}, P)\), show that

\[
P(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} P(E_k).
\]
Hint: Use Problem 5 and Boole's inequality.

(b). Let \( \{ E_\alpha : \alpha \in \Lambda \} \) be a countable collection of events. Show that:

(i) \( P(E_\alpha) = 0, \forall \alpha \in \Lambda \iff P(\bigcup_{\alpha \in \Lambda} E_\alpha) = 0; \)

(ii) \( P(E_\alpha) = 1, \forall \alpha \in \Lambda \iff P(\bigcap_{\alpha \in \Lambda} E_\alpha) = 1. \)

Hint: Use (a) and monotonicity of probability measures.

7. Consider four coding machines \( M_1, M_2, M_3 \) and \( M_4 \) producing binary codes 0 and 1. The machine \( M_1 \) produces codes 0 and 1 with respective probabilities \( \frac{1}{4} \) and \( \frac{3}{4} \). The code produced by machine \( M_k \) is fed into machine \( M_{k+1} \) (\( k = 1, 2, 3 \)) which may either leave the received code unchanged or may change it. Suppose that each of the machines \( M_2, M_3 \) and \( M_4 \) change the code with probability \( \frac{3}{4} \). Given that the machine \( M_4 \) has produced code 1, find the conditional probability that the machine \( M_1 \) produced code 0.

8. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that the probabilities of the student clearing examinations in these subjects are \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \) and \( \frac{1}{5} \), respectively. Assuming that the performances of the student in four subjects are independent, find the probability that the student will clear examination(s) of

(a) all the subjects;  
(b) no subject;  
(c) exactly one subject;  
(d) exactly two subjects;  
(e) at least one subject.

9. Let \( \{ E_k \}_{k \geq 1} \) be a sequence of events in the probability space \( (\Omega, \mathcal{F}, P) \).

(a) Suppose that \( \sum_{n=1}^{\infty} P(E_n) < \infty \). Show that \( P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = 0. \) Hence conclude that if \( \sum_{n=1}^{\infty} P(E_n) < \infty \) then, with probability one, only finitely many \( E_n \)'s will occur.

Hint: Use Problem 5 and Boole's inequality.

(b) If \( E_1, \ldots, E_n \) are independent, show that \( P(\bigcap_{i=1}^{n} E_i^c) \leq e^{-\sum_{i=1}^{n} P(E_i)}; \)

Hint: \( e^{-x} \geq 1 - x, \forall x \in \mathbb{R}. \)

(c) If \( E_1, E_2, \ldots \) are independent, show that \( P(\bigcap_{i=1}^{\infty} E_i^c) \leq e^{-\sum_{i=1}^{\infty} P(E_i)}; \)

Hint: Use (b) and Problem 5.

(d) Suppose that \( E_1, E_2, \ldots \) are independent and \( \sum_{n=1}^{\infty} P(E_n) = \infty \). Show that \( P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = 1 \). Hence conclude that if \( E_1, E_2, \ldots \) are independent and \( \sum_{n=1}^{\infty} P(E_n) = \infty \) then, with probability one, infinitely many \( E_n \)'s will occur.
10. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$ (power set of $\Omega$). Consider the probability space $(\Omega, \mathcal{F}, P)$, where $P(\{i\}) = \frac{1}{4}, i = 1, 2, 3, 4$. Let $A = \{1, 4\}, B = \{2, 4\}$ and $C = \{3, 4\}$. 
(a) Are $A, B$ and $C$ pairwise independent?;  
(b) Are $A, B$ and $C$ independent?;  
(c) Interpret the findings of (a) and (b) above.

11. Let $A, B$ and $C$ be three events such that $P(B \cap C) > 0$. Prove or disprove each of the following:  
(a) $P(A \cap B|C) = P(A|B \cap C)P(B|C)$;  
(b) (Berkson’s Paradox) $P(A \cap B|C) = P(A|C)P(B|C)$ if $A$ and $B$ are independent events;  
(c) Interpret the finding of (b) above.

12. (Simpson’s Paradox): Trends observed within different groups may disappear or reverse when groups are combined. In a probability space $(\Omega, \mathcal{F}, P)$, let $A$, $B$ and $D$ be three events. Construct an example to illustrate that it is possible to have $P(A|B \cap D) < P(A|B^c \cap D)$ and $P(A|B \cap D^c) < P(A|B^c \cap D^c)$ but $P(A|B) > P(A|B^c)$. (Read the famous example of UC Berkeley’s admission data: https://www.geeksforgeeks.org/probability-and-statistics-simpsons-paradox-uc-berkeleys-lawsuit/)

13. (Monty Hall Problem) There are 3 doors with one door having an expensive car behind it and each of the other 2 doors having a goat behind them. Monty Hall, being the host of the game, knows what is behind each door. A contestant is asked to select one of the doors and he wins the item (car or goat) behind the selected door. The contestant selects one of the doors at random, and then Monty Hall opens one of the other two doors to reveal goat behind it (note that at least one of the other two doors has a goat behind it and Monty Hall knows the door having goat behind it). Monty Hall offers to trade the door that contestant has chosen for the other door that is closed. Should the contestant switch doors if his goal is to win the car? (This problem is based on the American television game show "Let’s Make a Deal" hosted by Monty Hall.)

14. (Gambler’s Ruin Problem) Two gamblers $A$ and $B$ have initial capitals of Rs. $i$ and $N - i$, respectively, for some positive integer $i$. The two gamblers bet on successive and independent flips of a coin that, on each flip, results in a head with probability $p \in (0, 1)$ and a tail with probability $q = 1 - p$. On each flip if heads shows up $A$ wins Rs.1 from $B$ and if tails shows up then $B$ wins Rs.1 from $A$. The game continues until one of the players is bankrupt (ruined of all the capital he/she has).  
(a) Find the probability that $A$ ends up with all the Rs.$N$.  
(b) Show that the probability that either $A$ or $B$ will end up with all the money
is 1 (i.e., the probability that the game will continue indefinitely is 0). (c) For 
\((i, N, p) = (10, 20, 0.49), (50, 100, 0.49), (100, 200, 0.49), (5, 15, 0.5), (5, 15, 0.6),\) find 
the probabilities that \(A\) will end up with all the money. Interpret your findings in 
terms of casino business.
MS0 201A: Probability and Statistics
Assignment - I
Solutions
Problem No. 1

(a) Required probability = \[ P((A \land B) + (A^c \land B)) \]

= \[ P(A \land B) + P(A^c \land B) \]

= \[ (P(A) - P(A \land B)) + (P(B) - P(A \land B)) \]

= \[ P(A) + P(B) - 2P(A \land B) \]

= \[ 0.2 + 0.4 - 2 \times 0.1 = 0.4 \]

(b) Required probability = \[ P(A \cup B) \]

= \[ P(A) + P(B) - P(A \land B) = 0.5 \]

(c) Required probability = \[ P((A \cup B)^c) \]

= \[ 1 - P(A \cup B) = 1 - 0.5 \text{ (using (b))} \]

= \[ 0.5 \]

Problem No. 2

Total number of ways in which \( P_1, \ldots, P_n \) can stand in a row = \( L_n \)

Total number of possible positions for \( P_1 \) and \( P_2 \) such that there are exactly \( r \) persons between \( P_1 \) and \( P_2 \)

\[ = \frac{L_2 \times (n-r-1)}{r} \]

\[ \text{Corresponds to permutations of positions for } P_1 \text{ and } P_2 \]

The total number of ways in which \( P_1, \ldots, P_n \) can stand in a row such that there are exactly \( r \) persons between \( P_1 \) and \( P_2 \)

\[ = (L_2 \times (n-r-1)) \times \frac{(n-2)}{(n-1)} \]

\[ \text{Corresponds to permutations of } \]

\[ \frac{(n-2)}{n-1} \text{ persons other than } P_1 \text{ and } P_2 \]

Required probability = \[ \frac{(L_2 \times (n-r-1)) \times \frac{(n-2)}{(n-1)}}{L_n} = \frac{2(n-r-1)}{n(n-1)} \]
Problem No. 3. For $a, b, c$ (all $b, c > 0$) to be in $A_p$ we must have $b = ar$ and $c = ar^k$ for some $r > 1$ and $a, b, c \in \{1, \ldots, 50\}$. Thus we have

$$1 \leq a < ar < ar^k \leq 50, \ a, ar, \text{ and } ar^k \text{ are integers}$$

$$\Rightarrow 1 \leq a \leq \frac{50}{\sqrt[1/k]{r^k}}, \ 1 < r \leq \sqrt[1/k]{50}$$

(For $r^k = 50$) Also $r$ is rational (as $r = \frac{1}{\sqrt[k]{n}}$, $a, b \in \{1, \ldots, 50\}$)

The following cases arise.

Case I. $r$ is an integer

$$1 < r \leq \sqrt[1/k]{50} \Rightarrow r \in \{2, 3, \ldots, 7\}$$

For each $r \in \{2, 3, \ldots, 7\}$:

$$1 \leq a \leq \left[\frac{50}{r}\right], \text{ maximum integer contained in } \frac{50}{r}$$

Thus total number of favorable cases with $r$ as an integer

$$= \sum_{r=2}^{7} \left[\frac{50}{r}\right] = 12 + 5 + 3 + 2 + 1 + 1 = 24$$

Case II. $r = \frac{m}{n}$, where $m$ and $n$ are co-primes, $m > n > 1$.

We have

$$1 \leq a < \frac{m}{n} \ a < \frac{m^k}{n^k} \ a \leq 50, \ a \ \text{and } \frac{m}{n} \text{ and } \frac{m^k}{n^k} \text{ are integers}$$

Thus $a$ is an integer and $a$ is a multiple of $n^k$ (as $m$ and $n$ are co-primes).

Thus for each fixed $r = \frac{m}{n}$ ($m > n$) (where $m$ and $n$ co-primes) we have

$$1 \leq a \leq \frac{50n^k}{m^k} \ and \ a \ is \ a \ multiple \ of \ n^k$$

i.e., $1 \leq a \leq \left[\frac{50n^k}{m^k}\right]$ and $a$ is a multiple of $n^k$.
<table>
<thead>
<tr>
<th>Range of $a$ s.t. $1 \leq a \leq \frac{50}{3}$</th>
<th>Possible $a's$ that are multiples of $n^2$</th>
<th># of Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left{ 1, 9 \right}$</td>
<td>$\left{ 4, 8, 12, 16, 20 \right}$</td>
<td>5</td>
</tr>
<tr>
<td>$\left{ 1, 18 \right}$</td>
<td>$\left{ 9, 18 \right}$</td>
<td>2</td>
</tr>
<tr>
<td>$\left{ 1, 28 \right}$</td>
<td>$\left{ 9, 18 \right}$</td>
<td>3</td>
</tr>
<tr>
<td>$\left{ 1, 32 \right}$</td>
<td>$\left{ 16, 32 \right}$</td>
<td>2</td>
</tr>
<tr>
<td>$\left{ 1, 6 \right}$</td>
<td>$\left{ 16 \right}$</td>
<td>1</td>
</tr>
<tr>
<td>$\left{ 1, 34 \right}$</td>
<td>$\left{ 25 \right}$</td>
<td>1</td>
</tr>
<tr>
<td>$\left{ 1, 25 \right}$</td>
<td>$\left{ 25 \right}$</td>
<td>1</td>
</tr>
<tr>
<td>$\left{ 1, 36 \right}$</td>
<td>$\left{ 36 \right}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>20</strong></td>
</tr>
</tbody>
</table>

The total # of cases within $r$ fractioned = 20

⇒ total # of favorable cases (Case I + Case II) = 24 + 20 = 44

Required probability = $\frac{44}{\binom{50}{3}}$
Problem No. 4

Define events

$E_k$: k-th letter is in right envelope, $k=1...n$

Then

Required probability $= P \left( \bigcup_{k=1}^{n} E_k \right)$

$= P_1 - P_2 + P_3 + ... + (-1)^{n+1} P_n$

(Inclusion-Exclusion Principle)

where

$P_{kn} = \sum \sum P(E_{i_1} \cap E_{i_2} \cap ... \cap E_{i_k})$, $1 \leq i_1 < i_2 < ... < i_k \leq n$

$
\Rightarrow \text{this has } \binom{n}{k} \text{ terms}
$

We have, for $1 \leq i_1 < i_2 < ... < i_k \leq n$

$E_{i_1} \cap E_{i_2} \cap ... \cap E_{i_k} = \text{lethwa } i_1...i_k \text{ go to right envelope}$

$\Rightarrow \text{the number of favorable cases to } E_{i_1} \cap E_{i_2} \cap ... \cap E_{i_k} = \frac{h-k}{h}$

$= P_{kn} = \binom{n}{k} \frac{h-k}{h} = \frac{1}{h}, \ k=1...n$

Required probability $= \frac{1}{h} - \frac{1}{h^2} + \frac{1}{h^3} + ... + \frac{(-1)^{n-1}}{h^n}$

For large $n$ ($n \geq 50$)

Required prob. $\approx 1 - \frac{1}{h} - \frac{1}{h^2} + ... = 1 - e^{-1} = 0.632$

54/1
Problem No. 5

Define events
\[ B_1 = E_1, \quad B_n = E_n - E_{n-1}, \quad n = 2, 3, \ldots \]
Then \( B_n \) are disjoint, \( \bigcup_{k=1}^{n} B_k = E_n, \quad n = 1, 2, \ldots \)
and
\[ \bigcup_{k=1}^{n} B_k = E_n \]

\[ P\left( \bigcup_{n=1}^{\infty} E_n \right) = P\left( \bigcup_{n=1}^{\infty} B_n \right) \]
\[ = \sum_{n=1}^{\infty} P(B_n) \quad \text{(} B_n \text{ are disjoint} \text{)} \]
\[ = \lim_{n \to \infty} \sum_{k=1}^{n} P(B_k) \]
\[ = \lim_{n \to \infty} P\left( \bigcup_{k=1}^{n} B_k \right) = \lim_{n \to \infty} P(E_n) \]

(b) \( E_n \downarrow \Rightarrow E_n^c \uparrow \). Thus by (a)

\[ P\left( \bigcap_{n=1}^{\infty} E_n^c \right) = \lim_{n \to \infty} P(E_n^c) \]

\[ 1 - P\left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} \left[ 1 - P(E_n) \right] \]

\[ P\left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} P(E_n) \quad \text{(} \text{De Morgan's law} \text{)} \]

\[ \left( \bigcup_{a \in F} E_a \right)^c = \bigcap_{a \in F} E_a^c \]
Problem No. 6
(a) Define

\[ B_h = \bigcup_{k=1}^{\infty} E_k, \quad h = 1, 2, \ldots \]

Then \( B_h \uparrow \) and \( \lim_{h \to \infty} B_h = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} E_k \).

By continuity of probability measures (Problem 5)

\[ P\left( \lim_{h \to \infty} B_h \right) = \lim_{h \to \infty} P(B_h) \]

\[ P\left( \bigcup_{k=1}^{\infty} E_k \right) = \lim_{k \to \infty} P\left( \bigcup_{k=1}^{\infty} E_k \right) \quad \cdots \quad (1) \]

But using Borel's (neglects)

\[ P\left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} P(E_k) \quad \forall h = 2, \ldots \]

\[ \Rightarrow \lim_{h \to \infty} P\left( \bigcup_{k=1}^{\infty} E_k \right) \leq \lim_{h \to \infty} \sum_{k=1}^{\infty} P(E_k) = \sum_{k=1}^{\infty} P(E_k) \]

\[ \Rightarrow P\left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} P(E_k) \quad (\text{using (1)}). \]

(b)(i) Clearly \( \forall \varepsilon \in \mathbb{R} \), \( P\left( \bigcup_{\varepsilon=0}^{\infty} E_\varepsilon \right) = 0 \) then

\[ 0 \leq P(E_\varepsilon) \leq P\left( \bigcup_{\varepsilon=0}^{\infty} E_\varepsilon \right) = 0 \quad \forall \varepsilon \in \mathbb{R} \]

\[ \Rightarrow P(E_\varepsilon) = 0 \quad \forall \varepsilon \in \mathbb{R}. \]

Conversely, if \( P(E_\varepsilon) = 0 \quad \forall \varepsilon \in \mathbb{R} \) then \( \lim_{\varepsilon \to \infty} P(\alpha) \)

\[ 0 \leq P\left( \bigcup_{\varepsilon=0}^{\infty} E_\varepsilon \right) \leq \sum_{\varepsilon=0}^{\infty} P(E_\varepsilon) = 0 \]

\[ \Rightarrow P\left( \bigcup_{\varepsilon=0}^{\infty} E_\varepsilon \right) = 0. \]

(ii) \( P(E_\varepsilon) = 1 \quad \forall \varepsilon \in \mathbb{R} \) \( \implies P(E_\varepsilon^c) = 0 \quad \forall \varepsilon \in \mathbb{R} \)

\[ \implies P\left( \bigcup_{\varepsilon=0}^{\infty} E_\varepsilon^c \right) = 0 \quad (\text{using (i)}) \]

\[ \implies P\left( \bigcup_{\varepsilon=0}^{\infty} E_\varepsilon \right) = 1 \]

\[ \implies P\left( \bigcup_{\varepsilon=0}^{\infty} E_\varepsilon \right) = 1. \]
Problem No. 7

Define events

\( E_1: \) The machine produced code 1, \( i = 1 \) \( \frac{3}{4} \)

Required probability = \( P(E_1 | E_\text{u}) = 1 - P(E_1 | E_\text{u}) \)

We have \( P(E_1) = \frac{3}{4} \). By Bayes' Theorem

\[
P(E_1 | E_\text{u}) = \frac{P(E_\text{u} | E_1) P(E_1)}{P(E_\text{u} | E_1) P(E_1) + P(E_\text{u} | E_\text{u}) P(E_\text{u})}
\]

\[
P(E_\text{u} | E_1) = P(\text{machines } M_2, M_3 \text{ and my error make no code change or make 2 code changes})
\]

\[
= \left( \frac{1}{4} \right)^2 + \left( \frac{3}{4} \right) \left( \frac{3}{4} \right) \cdot \frac{1}{4} = \frac{7}{16}
\]

\[
P(E_\text{u} | E_\text{u}) = P(\text{machines } M_2, M_3 \text{ and my error make 1 code change or make 3 code changes})
\]

\[
= \left( \frac{3}{4} \right) \left( \frac{3}{4} \right) \left( \frac{1}{4} \right) + \left( \frac{3}{4} \right)^3 = \frac{9}{16}
\]

Thus

Required probability = \( 1 - \frac{\frac{7}{16} \times \frac{3}{4}}{\frac{7}{16} \times \frac{3}{4} + \frac{9}{16} \times \frac{1}{4}} = \frac{3}{10} \)

Problem No. 8

Define the events

\( B: \) Student clears Biology examination

\( C: \) Student clears Chemistry examination

\( P: \) Student clears Physics examination

\( M: \) Student clears Mathematics examination

Then

\( P(B) = \frac{1}{2}, \ P(C) = \frac{1}{3}, \ P(P) = \frac{1}{4}, \ P(M) = \frac{1}{4} \) and

\( B, C, P \text{ and } M \) are independent events.
(a) Required probability = \( P(B \cap c \cap n \cap n) \)
\[ = P(B) P(c) P(n) P(n) \quad \text{(independence)} \]
\[ = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} = \frac{1}{120} \]

(b) Required probability = \( P(B \cap c \cap p \cap n \cap n) \)
\[ = P(B) P(c) P(p) P(n) P(n) \quad \text{(independence)} \]
\[ = \left( \frac{1}{2} \right) \left( \frac{1}{4} \right) \left( \frac{1}{3} \right) \left( \frac{1}{5} \right) \left( \frac{1}{5} \right) = \frac{1}{5} \]

(c) Required probability = \( P(B \cap c \cap n \cap n \cap n) + P(B \cap c \cap n \cap p \cap n) + P(B \cap c \cap n \cap p \cap n) \)
\[ = \frac{1}{2} \times \left( \frac{1}{3} \right) \times \left( 1 - \frac{1}{4} \right) \times \left( 1 - \frac{1}{5} \right) + \left( \frac{1}{3} \right) \times \left( \frac{1}{5} \right) \times \left( 1 - \frac{1}{4} \right) \times \left( 1 - \frac{1}{5} \right) \]
\[ + \left( \frac{1}{2} \right) \times \left( 1 - \frac{1}{3} \right) \times \left( 1 - \frac{1}{4} \right) \times \left( 1 - \frac{1}{5} \right) \times \left( 1 - \frac{1}{5} \right) \times \frac{1}{5} \]
\[ = \frac{5}{12} \quad \text{(using independence)} \]

(d) Required probability = \( P(B \cap c \cap n \cap n \cap n) + P(B \cap n \cap c \cap p \cap n \cap n) + P(B \cap n \cap c \cap p \cap n \cap n) \)
\[ + P(B \cap n \cap c \cap n \cap p \cap n) + P(B \cap n \cap c \cap n \cap p \cap n) \]
\[ = \frac{1}{2} \times \frac{1}{3} \times \left( 1 - \frac{1}{4} \right) \times \left( 1 - \frac{1}{5} \right) + \frac{1}{2} \times \left( 1 - \frac{1}{3} \right) \times \frac{1}{4} \times \left( 1 - \frac{1}{5} \right) \]
\[ + \frac{1}{2} \times \left( 1 - \frac{1}{3} \right) \times \left( 1 - \frac{1}{4} \right) \times \frac{1}{5} \times \left( 1 - \frac{1}{5} \right) \]
\[ + \left( 1 - \frac{1}{2} \right) \times \frac{1}{3} \times \left( 1 - \frac{1}{4} \right) \times \frac{1}{5} \times \left( 1 - \frac{1}{5} \right) \times \frac{1}{5} \times \frac{1}{5} \]
\[ = \frac{7}{24} \]

(e) Required probability = \( 1 - P(\text{no audits in cleared}) \)
\[ = 1 - \frac{1}{5} = \frac{4}{5} \quad \text{(using (b))} \]
Problem No. 9 (a) Let $B_n = \bigcap_{k=n}^{\infty} E_k$, $n = 1, 2, \ldots$. Then $B_{\infty}$, 

$$P(\bigcap_{n=1}^{\infty} B_n) = \lim_{n \to \infty} P(B_n)$$

(Boole's Inequality)

$$= P(\bigcap_{n=1}^{\infty} E_k) = \lim_{n \to \infty} P(\bigcup_{k=n}^{\infty} E_k)$$

(Problem 5 (b) \footnote{Independence})

$$\leq \lim_{n \to \infty} \sum_{k=n}^{\infty} P(E_k)$$

(Boole's Inequality)

$$= 0 \quad (\lim_{n \to \infty} \sum_{k=n}^{\infty} P(E_k) = 0)$$

$$= P(\bigcap_{n=1}^{\infty} E_k) = 0$$

$$= P(\left(\bigcap_{n=1}^{\infty} E_k\right)^c) = 1$$

$$= P(\bigcap_{n=1}^{\infty} E_k^c) = 1$$

Note that $\bigcap_{n=1}^{\infty} E_k = \{\omega \in \Omega : \text{there exists an } n \geq 1 \text{ such that } \omega \in E_k \text{ for all } k \geq n\}$

$$= \{\omega \in \Omega : \text{\omega belongs to only finitely many } E_k\}$$

(5) $P(\bigcap_{n=1}^{\infty} E_k^c) = \prod_{n=1}^{\infty} P(E_k^c)$ \footnote{Independence}

$$= \prod_{n=1}^{\infty} (1 - P(E_k))$$

$$\leq \prod_{n=1}^{\infty} e^{-P(E_k)}$$

$$= e^{-\sum_{n=1}^{\infty} P(E_k)}$$

$$= e^{-\sum_{k=1}^{\infty} P(E_k)}$$

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(c) Let \( B_n = \bigcap_{i=1}^{n} E_i \), \( n \geq 1 \), ... Then \( B_n \downarrow \)

\[
P(B_n) = \lim_{n \to \infty} P(D_n)
\]

(Problem 5 (c1))

\[
P(\bigcap_{i=1}^{n} E_i^c) = \lim_{n \to \infty} P(\bigcap_{i=1}^{n} E_i^c)
\]

Using (c1)

\[
\lim_{n \to \infty} e^{-\sum_{i=1}^{n} P(E_i)}
\]

\[
= e^{-\sum_{i=1}^{\infty} P(E_i)}
\]

\[
(\text{we will show that } P(\bigcap_{k=1}^{\infty} E_k) = 0 \text{, i.e.})
\]

Let \( D_n = \bigcap_{i=1}^{n} E_i \), \( n \geq 1 \), ... Then \( D_n \downarrow \)

\[
P(D_n) = \lim_{n \to \infty} P(D_n)
\]

(Problem 5(c1))

\[
P(\bigcup_{k=1}^{\infty} E_k) = \lim_{n \to \infty} P(\bigcap_{i=1}^{n} E_i)
\]

\[
= \lim_{n \to \infty} e^{-\sum_{i=1}^{n} P(E_i)}
\]

\[
= e^{0} \quad (\text{as } \sum_{i=1}^{\infty} P(E_k) = 0, \quad k \geq 1)
\]

\[
P(\bigcup_{k=1}^{\infty} E_k) = 0
\]

\[
= 1 - P(\bigcap_{k=1}^{\infty} E_k)
\]

\[
= 1 - 0 = 1
\]

Note that \( \bigcap_{k=1}^{\infty} E_k \).

\[\begin{align*}
\bigcap_{k=1}^{\infty} E_k &= \{ \omega \in \Omega : \forall n \geq 1, \exists k \geq n \text{ such that } \omega \in E_k \} \\
&= \{ \omega \in \Omega : \omega \text{ belongs to infinitely many } E_i \}
\end{align*}\]
Problem No. 10
(a) \( P(A) = P(B) = \frac{1}{4} \Rightarrow \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \)
\( P(\text{A} \cap \text{B}) = P(\text{A} \cap \text{c}) = P(\text{B} \cap \text{c}) = P(\text{c}) = \frac{1}{4} \)

Claim: \( P(\text{A} \cap \text{B} \cap \text{c}) = P(\text{A} | \text{B} \cap \text{c}) \cdot P(\text{B} | \text{c}) \cdot P(\text{c}) = \frac{1}{8} \neq \frac{1}{4} = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{8} \)
\( \Rightarrow \text{A, B, and C are pairwise independent.} \)

(b) \( P(\text{A} \cap \text{B} \cap \text{c}) = \frac{1}{4} \neq P(\text{A}) \cdot P(\text{B}) \cdot P(\text{c}) = \frac{1}{8} \)
\( \Rightarrow \text{A, B, and C are not independent.} \)

(c) Pairwise independence \( \neq \) independence.

Problem 16.11
(a) \( P(\text{A} \mid \text{B} \cap \text{c}) = \frac{P(\text{A} \cap \text{B} \cap \text{c})}{P(\text{c})} = \frac{P(\text{A} | \text{B} \cap \text{c}) \cdot P(\text{B} | \text{c}) \cdot P(\text{c})}{P(\text{c})} = P(\text{A} | \text{B} \cap \text{c}) \cdot P(\text{B} | \text{c}) \)

(b) Consider the example of Problem 10.
\( P(\text{A} \mid \text{B} \cap \text{c}) = \frac{P(\text{A} \cap \text{B} \cap \text{c})}{P(\text{c})} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \)
\( P(\text{A} \mid \text{c}) = \frac{P(\text{A} \cap \text{c})}{P(\text{c})} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \)
\( P(\text{B} \mid \text{c}) = \frac{P(\text{B} \cap \text{c})}{P(\text{c})} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \)

(Notice) \( P(\text{A} \cap \text{B} \mid \text{c}) \neq P(\text{A} \mid \text{c}) \cdot P(\text{B} \mid \text{c}) \).

(c1) Even if A and B may be independent in fact but given any other event C they may not be independent.
Let \( P(A|B \land B^c) = \alpha_1 \), \( P(A|B^c \land D) = \beta_1 \), \( P(A|B \land D^c) = \alpha_2 \), \( P(A|B^c \land D^c) = \beta_2 \), \( P(D|B) = \alpha \), and \( P(D^c|B^c) = \beta \). Then

\[
P(A|D) = P(A|D \land D^c) = P(A|B \land D) + P(A|B^c \land D)
\]

\[
= P(A|D \land B) P(B|D) + P(A|D \land B^c) P(B^c|D)
\]

\[
= \beta_1 \alpha + (1-\beta) \alpha_2.
\]

\[
P(A|D^c) = \beta_2 \beta + (1-\beta) \alpha_1.
\]

We have to choose real numbers \( \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_1 \), and \( \beta_2 \) such that \( 0 < \alpha_1 < \beta_1 < 1, \ 0 < \alpha_2 < \beta_2 < 1, \ 0 < \beta_1 < 1, \ 0 < \beta_2 < 1 \)

\[
\beta_1 \alpha + (1-\beta_1) \alpha_2 > \beta_2 \beta + (1-\beta_2) \alpha_1.
\]

I.e., \( \alpha_1 + (\alpha_2-\alpha_1) \beta_2 > \beta_2 + (\beta_1-\beta_2) \alpha_1 \).

Let us take \( \alpha_1 = 0.2, \ \alpha_2 = 0.6, \ \beta_1 = 0.4 \) and \( \beta_2 = 0.8 \). Then

\[
0.2 - 0.4 \beta_2 > 0.8 - 0.6 \beta_1 \Rightarrow \beta_2 - \beta_1 > \frac{1}{2}
\]

Thus one may take, for example, \( \alpha_1 = 0.2, \ \alpha_2 = 0.6, \ \beta_1 = 0.4, \ \beta_2 = 0.8, \ \beta_1 = 0.2 \) and \( \beta_2 = 0.8 \).
Problem No. 13

Let the doors be numbered 1 to N and assume that car is behind door no. 1. Define events:

\( D_i \): Contestant chooses door no. \( i \), \( i = 2, 3, \ldots, N \).

\( W \): 'A' wins the car.

**Case I:** Contestant decides to switch

\[
P(W | D_i) = 0, \quad P(W | D_i) = 1 \quad (i = 2, 3, \ldots, N).
\]

\[
P(W) = \sum_{i=1}^{N} P(W | D_i) P(D_i)
\]

\[
= \frac{1}{N} \left[ 0 + \frac{N-1}{N} \right] = \frac{N-1}{N}.
\]

For \( N=3 \), \( P(W) = \frac{2}{3} \)

**Case II:** Contestant decides not to switch

\[
P(W | D_i) = 1, \quad P(W | D_i) = 0 \quad (i = 2, 3, \ldots, N)
\]

\[
P(W) = \sum_{i=1}^{N} P(W | D_i) P(D_i)
\]

\[
= \frac{1}{N}.
\]

For \( N=3 \), \( P(W) = \frac{1}{3} \)

Thus the contestant should **switch** the door as the probability of winning doubles by doing so (in case of \( N=3 \)) and increases the same in case of \( N \) doors.

**Remark:** In light of new additional information it is advisable to update prior probabilities

\[ \rightarrow \text{Bayesian Approach.} \]
Problem No. 14

(a) Let

\[ p_c = P(A \text{ wins all the money}), \quad c = 1, 2, \ldots, N \]

Here the probability of win depends on the initial capital
'c' available with 'A'. Then by theorem of total probability
(on the venet of first flip)

\[ p_c = P(A \text{ wins all the money} \mid \text{first flip is head}) \times p + \]
\[ P(A \text{ wins all the money} \mid \text{first flip is tail}) \times (1-p) \]

\[ \Rightarrow p_c = b \cdot p \cdot p_{c+1} + q \cdot p_{c-1}, \quad c = 2, 3, \ldots, N-1 \] \quad \text{(I)}

\[ p_1 = b \cdot p \Rightarrow p_{2-k} = \frac{q}{b} \cdot p \]

\[ p_{N+1} = 1 \]

\[ p_{c+1} - b = \frac{q}{b} (p_{c} - b) \]

\[ p_{2-k} = \frac{q}{b} \]

\[ p_{3-k} = \frac{q}{b} \left( p_{2-k} - b \right) = \left( \frac{q}{b} \right)^2 p \]

\[ p_{4-k} = \frac{q}{b} \left( p_{3-k} - b \right) = \left( \frac{q}{b} \right)^3 p \]

... 

\[ p_{N-k} = \left( \frac{q}{b} \right)^{c-1} p \]

Summing the last \((c-1)\) terms, we get

\[ p_c - b = \left[ \frac{q}{b} + \left( \frac{q}{b} \right)^2 + \ldots + \left( \frac{q}{b} \right)^{c-1} \right] \]

\[ p_c = p_1 \left( 1 + \frac{q}{b} + \left( \frac{q}{b} \right)^2 + \ldots + \left( \frac{q}{b} \right)^{c-1} \right) = \left\{ \begin{array}{cl}
\frac{1 - \left( \frac{q}{b} \right)^c}{1 - \left( \frac{q}{b} \right)} & \text{if } c = 1, 2, \ldots, N \quad \text{(I)} \\
\frac{1}{2} & \text{if } c = 1 \end{array} \right. \]

We have \( p_N = 1 \). Thus

\[ p_c = \left\{ \begin{array}{cl}
\frac{1 - \left( \frac{q}{b} \right)^c}{1 - \left( \frac{q}{b} \right)^N} , & c = 1, 2, \ldots, N \\
\frac{1}{2} & c = 1 \end{array} \right. \]

\((q/b + 1) \Rightarrow 1 = (q/b) + (2) \quad \text{(II)}\)
(b) Let \( q_i \) be the probability that B will win all the money.

By symmetry,

\[
q_i = \begin{cases} 
\frac{1 - (1/4)^{N-i}}{1 - 1/4}, & (i = 0, 1, \ldots, N-1) \\
\frac{N-i}{64}, & (i = N) 
\end{cases}
\]

Clearly, \( p_i + q_i = 1 \) for \( i = 0, \ldots, N-1 \).

(c) For \( c = 10, \ n = 20 \) \( p = 0.49, \ p_i = 0.4, \ q_i = 0.6 \)

For \( c = 50, \ n = 100 \) \( p = 0.49, \ p_i = 0.12, \ q_i = 0.88 \)

For \( c = 100, \ n = 200 \) \( p = 0.49, \ p_i = 0.02, \ q_i = 0.98 \)

In Casino even if the game may look fair (\( p = 0.49 \)) the gambler is bound to be ruined.

For \( c = 5, \ n = 15 \) and \( p = 0.5 \) \( p_i = \frac{1}{3}, \ q_i = \frac{2}{3} \)

For \( c = 5, \ n = 15 \) and \( p = 0.6 \) \( p_i = 0.87, \ q_i = 0.13 \)

Small variations (\( p \) effect is insignificant)