1. Let

\[ F(x, y) = \begin{cases} 1, & \text{if } x + 2y \geq 1 \\ 0, & \text{if } x + 2y < 1 \end{cases} \]

Does \( F(\cdot, \cdot) \) define a d.f.?

2. Let

\[ F(x, y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1, & \text{otherwise} \end{cases} \]

Does \( F(\cdot) \) define a d.f.?

3. Let \( X = (X_1, X_2) \) be a bivariate random vector having the d.f.

\[ F(x, y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0 \\ \frac{1+xy}{2}, & \text{if } 0 \leq x < 1, 0 \leq y < 1 \\ \frac{1+x}{2}, & \text{if } 0 \leq x < 1, y \geq 1 \\ \frac{1+y}{2}, & \text{if } x \geq 1, 0 \leq y < 1 \\ 1, & \text{if } x \geq 1, y \geq 1 \end{cases} \]

(a) Verify that \( F \) is a d.f.; (b) Determine whether \( X \) is a discrete or a continuous random vector; (c) Find the marginal distribution functions of \( X_1 \) and \( X_2 \); (d) Find \( P(\frac{1}{2} \leq X_1 \leq 1, \frac{1}{4} < X_2 < \frac{1}{2}), P(X_1 = 1) \) and \( P(X_1 \geq \frac{3}{4}, X_2 < \frac{1}{4}) \); (e) Are \( X_1 \) and \( X_2 \) independent?

4. Let \( X = (X_1, X_2) \) be a bivariate random vector having the d.f.

\[ F(x, y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 1 \\ \frac{y^2-1}{6}, & \text{if } 0 \leq x < 1, 1 \leq y < 2 \\ \frac{1}{2}, & \text{if } 0 \leq x < 1, y \geq 2 \\ \frac{y^2-1}{3}, & \text{if } x \geq 1, 1 \leq y < 2 \\ 1, & \text{if } x \geq 1, y \geq 2 \end{cases} \]

(a) Verify that \( F \) is a d.f.; (b) Determine whether \( X \) is a discrete or a continuous r.v.; (c) Find the marginal distribution functions of \( X_1 \) and \( X_2 \); (d) Find \( P(\frac{1}{2} \leq X_1 \leq 1, \frac{5}{4} < X_2 < \frac{3}{2}), P(X_1 = 1) \) and \( P(X_1 \geq \frac{3}{4}, X_2 < \frac{5}{4}) \); (e) Are \( X_1 \) and \( X_2 \) independent?
5. Let the r.v. $X = (X_1, X_2)'$ have the joint p.m.f.

$$f_X(x_1, x_2) = \begin{cases} c(x_1 + 2x_2), & \text{if } x_1 = 1, 2, x_2 = 1, 2, \\ 0, & \text{otherwise} \end{cases},$$

where $c$ is a real constant. (a) Find the constant $c$; (b) Find marginal p.m.f.s of $X_1$ and $X_2$; (c) Find conditional variance of $X_2$ given $X_1 = x_1$, $x_1 = 1, 2$; (d) Find $P(X_1 < \frac{3}{2})$, $P(X_1 = X_2)$, $P(X_1 \geq \frac{X_2}{2})$ and $P(X_1 + X_2 \leq 3)$; (e) Find $\rho(X_1, X_2)$; (f) Are $X_1$ and $X_2$ independent?

6. Let the r.v. $X = (X_1, X_2)'$ have the joint p.m.f.

$$f_X(x_1, x_2) = \begin{cases} cx_1x_2, & \text{if } x_1 = 1, 2, x_2 = 1, 2, x_1 \leq x_2, \\ 0, & \text{otherwise} \end{cases},$$

where $c$ is a real constant. (a) Find the constant $c$; (b) Find marginal p.m.f.s of $X_1$ and $X_2$; (c) Find conditional variance of $X_2$ given $X_1 = 1$; (d) Find $P(X_1 > X_2)$, $P(X_1 = X_2)$, $P(X_1 < \frac{2}{3}X_2)$ and $P(X_1 + X_2 \geq 3)$; (e) Find $\rho(X_1, X_2)$; (f) Are $X_1$ and $X_2$ independent?

7. Let $(X, Y)$ be a random vector such that the p.d.f. of $X$ is

$$f_X(x) = \begin{cases} 4x(1 - x^2), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and, for fixed $x \in (0, 1)$, the conditional p.d.f. of $Y$ given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} c(x)y, & \text{if } x < y < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $c : (0, 1) \to \mathbb{R}$ is a given function. (a) Determine $c(x), 0 < x < 1$; (b) Find marginal p.d.f. of $Y$; (c) Find the conditional variance of $X$ given $Y = y$, $y \in (0, 1)$; (d) Find $P(X < \frac{1}{2})$, $P(X + Y \geq \frac{3}{4})$ and $P(X = 2Y)$; (e) Find $\rho(X, Y)$; (f) Are $X$ and $Y$ independent?

8. Let $X = (X_1, X_2, X_3)$ be a random vector with joint p.d.f.

$$f_X(x) = \begin{cases} \frac{c}{x_1x_2x_3}, & \text{if } 0 < x_3 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $c$ is a real constant. (a) Find the value of constant $c$; (b) Find marginal p.d.f. of $X_2$; (c) Find the conditional variance of $X_2$ given $(X_1, X_3) = (x, y), 0 < y <$
9. Let $X = (X_1, X_2, X_3)'$ be a random vector with p.m.f.

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \begin{cases} \frac{1}{4}, & \text{if } (x_1, x_2, x_3) \in A \\ 0, & \text{otherwise} \end{cases},$$

where $A = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$. (a) Are $X_1, X_2, X_3$ independent?; (b) Are $X_1, X_2, X_3$ pairwise independent?; (c) Are $X_1 + X_2$ and $X_3$ independent?

10. Let $X = (X_1, X_2, X_3)'$ be a random vector with joint p.d.f.

$$f_X(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)} \left(1 + x_1 x_2 x_3 e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)}\right), \quad -\infty < x_i < \infty,$$

$i = 1, 2, 3$. (a) Are $X_1, X_2, X_3$ independent?; (b) Are $X_1, X_2, X_3$ pairwise independent?; (c) Find the marginal p.d.f.s of $(X_1, X_2)$, $(X_1, X_3)$ and $(X_2, X_3)$.

11. Let $(X, Y, Z)$ have the joint p.m.f. as follows:

<table>
<thead>
<tr>
<th>$(x, y, z)$</th>
<th>$1, 1, 0$</th>
<th>$1, 2, 1$</th>
<th>$1, 3, 0$</th>
<th>$2, 1, 1$</th>
<th>$2, 2, 0$</th>
<th>$2, 3, 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{X, Y, Z}(x, y, z)$</td>
<td>$\frac{2}{15}$</td>
<td>$\frac{4}{15}$</td>
<td>$\frac{3}{15}$</td>
<td>$\frac{1}{15}$</td>
<td>$\frac{1}{15}$</td>
<td>$\frac{4}{15}$</td>
</tr>
</tbody>
</table>

and $f_{X, Y}(x, y) = 0$, elsewhere. (a) Are $X + Y$ and $Z$ independent?; (b) Find $\rho = \text{Corr}(X+Y,Z)$.

12. Let $X = (X_1, X_2, X_3)'$ be a random vector with p.d.f.

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \begin{cases} 2e^{-(x_1^2 + 2x_3)}, & \text{if } 0 < x_1 < 1, x_2 > 0, x_3 > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Are $X_1, X_2, X_3$ independent?; (b) Are $X_1 + X_2$ and $X_3$ independent?; (c) Find marginal p.d.f.s of $X_1, X_2$ and $X_3$; (d) Find conditional p.d.f. of $X_1$ given $X_2 = 2$.

13. Let $X_1, \ldots, X_n$ be $n$ r.v.s with $E(X_i) = \mu_i$, $\text{Var}(X_i) = \sigma_i^2$ and $\rho_{ij} = \text{Corr}(X_i, X_j)$, $i, j = 1, \ldots, n$, $i \neq j$. For real numbers $a_i$, $b_i$, $i = 1, \ldots, n$, define $Y = \sum_{i=1}^{n} a_i X_i$ and $Z = \sum_{i=1}^{n} b_i X_i$. Find $\text{Cov}(Y, Z)$.

14. Let $X$ and $Y$ be jointly distributed random variables with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 2$ and $\text{Cov}(X, Y) = 1/3$. Find $\text{Corr}(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3})$.

15. Let $X_1, \ldots, X_n$ be random variables and let $p_1, \ldots, p_n$ be positive real numbers with $\sum_{i=1}^{n} p_i = 1$. Prove that: (a) $\sqrt{\text{Var}\left(\sum_{i=1}^{n} p_i X_i\right)} \leq \sum_{i=1}^{n} p_i \sqrt{\text{Var}(X_i)} \leq \sqrt{\sum_{i=1}^{n} p_i \text{Var}(X_i)}$; (b) $\text{Var}\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) \leq \frac{1}{n} \sum_{i=1}^{n} \text{Var}(X_i)$.
16. Let \((x_i, y_i) \in \mathbb{R}^2, i = 1, \ldots, n\) be such that \(\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0\). Using a statistical argument show that
\[
\left( \sum_{i=1}^{n} x_i y_i \right)^2 \leq \left( \sum_{i=1}^{n} x_i^2 \right) \left( \sum_{i=1}^{n} y_i^2 \right).
\]

17. Let \((X, Y)\) have the joint p.m.f. as follows:

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>((1, 1))</th>
<th>((1, 2))</th>
<th>((1, 3))</th>
<th>((2, 1))</th>
<th>((2, 2))</th>
<th>((2, 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_{X,Y}(x, y))</td>
<td>(\frac{2}{15})</td>
<td>(\frac{4}{15})</td>
<td>(\frac{3}{15})</td>
<td>(\frac{1}{15})</td>
<td>(\frac{1}{15})</td>
<td>(\frac{4}{15})</td>
</tr>
</tbody>
</table>

and \(f_{X,Y}(x, y) = 0\), elsewhere. Find \(\rho = \text{Corr}(X, Y)\).

18. Let the joint m.g.f. of \((Y, Z)\) be \(M_{Y,Z}(t_1, t_2) = e^{\frac{t_1^2 + t_2^2}{2} + \frac{t_1 t_2}{3}}, t_2 < \frac{1}{2}\). (a) Find \(\text{Corr}(Y, Z)\); (b) Are \(Y\) and \(Z\) independent?; (c) Find m.g.f. of \(Y + Z\).

19. Let the joint m.g.f. of \((Y, Z)\) be \(M_{Y,Z}(t_1, t_2) = e^{\frac{t_1^2 + t_2^2 + t_1 t_2}{2}}, (t_1, t_2) \in \mathbb{R}^2\). (a) Find \(\text{Corr}(Y, Z)\); (b) Are \(Y\) and \(Z\) independent?; (c) Find m.g.f. of \(Y - Z\).

20. Let \(X = (X_1, X_2)\) have the joint p.m.f.
\[
f_X(x_1, x_2) = \begin{cases} 
(\frac{2}{3})^{x_1+x_2} (\frac{1}{3})^{2-x_1-x_2}, & \text{if } (x_1, x_2) = (0,0), (0,1), (1,0), (1,1) \\
0, & \text{otherwise}
\end{cases}
\]
(a) Find the joint p.m.f. of \(Y_1 = X_1 - X_2\) and \(Y_2 = X_1 + X_2\); (b) Find the marginal p.m.f.s of \(Y_1\) and \(Y_2\); (c) Find \(\text{Var}(Y_2)\) and \(\text{Cov}(Y_1, Y_2)\); (d) Are \(Y_1\) and \(Y_2\) independent?

21. Let \(X_1, \ldots, X_n\) be a random sample of continuous random variables and let \(X_{1:n} < X_{2:n} < \cdots < X_{n:n}\) be the corresponding order statistics. If the expectation of \(X_1\) is finite and the distribution of \(X_1\) is symmetric about \(\mu \in (-\infty, \infty)\), show that:
(a) \(X_{r:n} - \mu \overset{d}{=} \mu - X_{n-r+1:n}, \ r = 1, \ldots, n\); (b) \(E(X_{r:n} + X_{n-r+1:n}) = 2\mu\); (c) \(E(X_{\frac{n+1}{2}:n}) = \mu\), if \(n\) is odd; (d) \(P(X_{\frac{n+1}{2}:n} > \mu) = 0.5\), if \(n\) is odd.

22. (a) Let \(X_1, \ldots, X_n\) denote a random sample, where \(P(X_1 > 0) = 1\). Show that
\[
E\left( \frac{X_1 + X_2 + \cdots + X_k}{X_1 + X_2 + \cdots + X_n} \right) = \frac{k}{n}, \ k = 1, 2, \ldots, n.
\]
(b) Let \(X_1, \ldots, X_n\) be a random sample and let \(E(X_1)\) be finite. Find the conditional expectation \(E(X_1|X_1 + \cdots + X_n = t)\), where \(t \in \mathbb{R}\) is such that the conditional expectation is defined.
(c) Let \(X_1, \ldots, X_n\) be a random sample of random variables. Find \(P(X_1 < X_2 < \cdots < X_r), \ r = 2, 3, \ldots, n.\)

23. Let \(X_1\) and \(X_2\) be independent and identically distributed random variables with common p.m.f.
\[
f(x) = \begin{cases} \theta(1 - \theta)^{x-1}, & \text{if } x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}
\]
where \(\theta \in (0, 1)\). Let \(Y_1 = \min\{X_1, X_2\}\) and \(Y_2 = \max\{X_1, X_2\} - \min\{X_1, X_2\}\). (a) Find the marginal p.m.f. of \(Y_1\) without finding the joint p.m.f. of \(Y = (Y_1, Y_2)\); (b) Find the marginal p.m.f. of \(Y_2\) without finding the joint p.m.f. of \(Y = (Y_1, Y_2)\); (c) Find the joint p.m.f. of \(Y = (Y_1, Y_2)\); (d) Are \(Y_1\) and \(Y_2\) independent; (e) Using (c), find the marginal p.m.f.s of \(Y_1\) and \(Y_2\).

24. Let \(X = (X_1, X_2, X_3)'\) be a random vector with p.m.f.
\[
f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \begin{cases} \frac{2}{9}, & \text{if } (x_1, x_2, x_3) = (1, 1, 0), (1, 0, 1), (0, 1, 1) \\ \frac{1}{3}, & \text{if } (x_1, x_2, x_3) = (1, 1, 1) \\ 0, & \text{otherwise} \end{cases}
\]
Define \(Y_1 = X_1 + X_2\) and \(Y_2 = X_2 + X_3\). (a) Find the marginal p.m.f. of \(Y_1\) without finding the joint p.m.f. of \(Y = (Y_1, Y_2)\); (b) Find the marginal p.m.f. of \(Y_2\) without finding the joint p.m.f. of \(Y = (Y_1, Y_2)\); (c) Find the joint p.m.f. of \(Y = (Y_1, Y_2)\); (d) Are \(Y_1\) and \(Y_2\) independent; (e) Using (c), find the marginal p.m.f.s of \(Y_1\) and \(Y_2\).

25. Let \(X_1\) and \(X_2\) be independent random variables with p.d.f.s
\[
f_1(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 1, & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}
\]
respectively. Let \(Y = X_1 + X_2\) and \(Z = X_1 - X_2\). (a) Find the d.f. of \(Y\) and hence find its p.d.f.; (b) Find the joint p.d.f. of \((Y, Z)\) and hence find the marginal p.d.f.s of \(Y\) and \(Z\); (c) Are \(Y\) and \(Z\) independent?

26. Let \(X_1\) and \(X_2\) be i.i.d. random variables with common p.d.f.
\[
f(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}
\]
Let \(Y = |X_1| + X_2\) and \(Z = X_2\). (a) Find the d.f. of \(Y\) and hence find its p.d.f.; (b) Find the joint p.d.f. of \((Y, Z)\) and hence find the marginal p.d.f.s of \(Y\) and \(Z\); (c) Are \(Y\) and \(Z\) independent?
Problem No.1

\[ \lim_{x \to \infty} F(x) = 1 = \Phi(z) \quad (x \to \infty) \]

given  \( x \) is not a d.f. (unbounded d.f.)

\( \Rightarrow \) \( F(x) \) is not a d.f.

Problem No.2

For rectangle \( \left( \frac{a}{2}, 0 \right) \times \left( \frac{b}{2}, 1 \right) \)

\[ P \left( \frac{a}{2} < x < \frac{b}{2}, \frac{a}{2} < y < 1 \right) = F(1) - F \left( \frac{a}{2}, 1 \right) - F \left( a, \frac{b}{2} \right) + F \left( \frac{a}{2}, \frac{b}{2} \right) \]

\[ = 1 < 0 \]

\( \Rightarrow \) \( F \) is not a d.f.

Problem No.3

(a) \( \lim_{x \to -\infty} F(x) = \lim_{y \to \infty} F(y) = 0 \)

\( \lim_{x \to \infty} F(x) = 1 \)

For each \( y \in \mathbb{R} \), \( F(x,y) \) is right continuous in \( x \) and

for \( x \in \mathbb{R} \), \( F(y,x) \) is right continuous in \( y \).

For rectangle \( (a_1, b_1) \times (a_2, b_2) \), \( a_1 < b_1, \ a_2 < b_2 \), consider

\[ D = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2) \]

Case I: \( a_1 < 0 \)

\[ D = F(b_1, b_2) - F(b_1, a_2) > 0 \] (since for each \( x \), \( f(y) \uparrow \) \( y \to \infty \))

Case II: \( a_2 < 0 \)

\[ D = F(b_1, b_2) - F(a_1, b_2) > 0 \] (since for each \( x \), \( f(y) \uparrow \) \( y \to \infty \))

Case III: \( 0 < a_1 < b_1, \ 0 < a_2 < b_2, \ 0 \leq b_1 < b_2 < 1 \)

\[ D = \frac{1+b_1-a_2}{2} - \frac{1+b_2-a_1}{2} - \frac{1+a_1-b_2}{2} + \frac{1+a_2-b_1}{2} \]

\[ = \frac{1}{2} \left( (b_1-a_2)(b_2-a_1) \right) > 0 \]

\[ \color{red}{IV \ IV} \]
Case IV: $0 < a_1 < 1, \ 0 < a_2 < 1, \ 0 < b_1 < 1, \ b_2 > 1$

$$
\Delta = \frac{1 + b_1}{2} - \frac{1 + a_1}{2} - \frac{1 + b_1 a_2}{2} + \frac{1 + a_1 a_2}{2}
\quad = \frac{(b_1 - a_1)(1 - a_1)}{2} \geq 0
$$

Case V: $0 < a_1 < 1, \ 0 < a_2 < 1, \ b_1 < 1, \ 0 < b_2 < 1$

$$
\Delta = \frac{1 + b_2}{2} - \frac{1 + a_1 b_1}{2} - \frac{1 + a_2}{2} + \frac{1 + a_1 a_2}{2}
\quad = \frac{(b_2 - a_2)(1 - a_1)}{2} \geq 0
$$

Case VI: $0 < a_1 < 1, \ 0 < a_2 < 1, \ b_1 > 1, \ b_2 < 1$

$$
\Delta = 1 - \frac{1 + a_1}{2} - \frac{1 + a_2}{2} + \frac{1 + a_1 a_2}{2}
\quad = \frac{(1 - a_1)(1 - a_2)}{2} \geq 0
$$

Case VII: $0 < a_1 < 1, \ a_2 > 1, \ 0 < b_1 < 1, \ b_2 > 1$

$$
\Delta = \frac{1 + b_1}{2} - \frac{1 + a_1}{2} - \frac{1 + b_1}{2} + \frac{1 + a_1}{2} \geq 0
$$

Case VIII: $0 < a_1 < 1, \ a_2 > 1, \ b_1 > 1, \ b_2 < 1$

$$
\Delta = 1 - \frac{1 + a_1}{2} - 1 + \frac{1 + a_1}{2} = 0
$$

Case IX: $a_1 > 1, \ 0 < a_2 < 1, \ b_1 > 1, \ 0 < b_2 < 1$

$$
\Delta = \frac{1 + b_1}{2} - \frac{1 + b_1}{2} - \frac{1 + a_1}{2} + \frac{1 + a_1}{2} = 0
$$

Case X: $a_1 > 1, \ 0 < a_2 < 1, \ b_1 > 1, \ b_2 > 1$

$$
\Delta = 1 - 1 - \frac{1 + a_1}{2} + \frac{1 + a_1}{2} = 0
$$

Case XI: $a_1 > 1, \ a_2 > 1, \ b_1 > 1, \ b_2 > 1$

$$
\Delta = 1 - 1 - 1 + 1 = 0
$$

Thus $\Delta \geq 0$

$\Rightarrow f \text{ is a d.b.}$

$2/11$
(b) \( F(0,0) = \frac{1}{2} \neq \lim_{y \to 0} F(x,y) = 0 \)

\( \Rightarrow \) \( F \) is not of AC type.

\( F \) is discontinuous at all points of the type \((0,y), y < 0 \) (uncountable number of discontinuities)

\( \Rightarrow F \) is not of discrete type.

(c) 
\[
F_{x_1} (x_1) = \lim_{y \to 0} F(x_1,y) = \begin{cases} 
0, & x_1 < 0 \\
\frac{x_1}{2}, & 0 \leq x_1 < 1 \\
1, & x_1 \geq 1 
\end{cases}
\]

\[
F_{x_2} (x_2) = \lim_{x_1 \to 0} F(x_1,y) = \begin{cases} 
0, & x_2 < 0 \\
\frac{x_2}{2}, & 0 \leq x_2 < 1 \\
1, & x_2 \geq 1 
\end{cases}
\]

(d) \( P\left( \frac{1}{2} \leq x_1, \frac{1}{4} < x_2 < \frac{1}{2} \right) \)

\[= F\left(1, \frac{1}{2}\right) - F\left(\frac{1}{2}, \frac{1}{2}\right) - F\left(\frac{1}{2}, \frac{1}{4}\right) + F\left(\frac{1}{2}, \frac{1}{2}\right) \]

\[= \frac{3}{4} - \frac{5}{8} - \frac{5}{8} + \frac{9}{16} = \frac{1}{16} \]

\( P(x_1 = 1) = F(1,0) - F(1,0) = 1-1 = 0 \)

\( P(x_1, x_2 < \frac{1}{4}) = P(x_1 < \frac{1}{2}) - P(x_1 < \frac{1}{2}, x_2 < \frac{1}{4}) \)

\[= F(\frac{1}{2}, \frac{1}{4}) - F(\frac{1}{2}, \frac{1}{4}) = \frac{5}{8} - \frac{5}{8} = 0 \]

(e) Clearly \( F(x_2, y) \neq F_{x_1}(x_1) F_{x_2}(x_2) \) \& \((x_2) \in \mathbb{R}^2\). Thus \( x_1 \) and \( x_2 \) are not independent.

Problem No. 4

Suggested - Problem No. 2
Problem No. 5

(a) \[ \sum_{(x_1, x_2) \in S_x} \lambda_1 (x_1, x_2) = 1 \]

\[ c = \frac{3 + 5 + 4 + 6}{18} = 1 \Rightarrow c = \frac{1}{18} \]

(b) \[ b_{x_1}(x_1) = \begin{cases} 2c(x_1 + 3) & \lambda_1 = 1, \frac{2c(x_1 + 3)}{0} & \text{otherwise} \\ \frac{0}{0} & \text{otherwise} \end{cases} \]

\[ b_{x_2}(x_2) = \begin{cases} c(1 + 2x_2) & \lambda_2 = 1, \frac{c(1 + 2x_2)}{0} & \text{otherwise} \\ \frac{0}{0} & \text{otherwise} \end{cases} \]

\[ \text{Where } c = \frac{1}{18} \]

(c) For \( x_1 \in \{1, 2\} \)

\[ b_{x_2|x_1}(x_2|x_1) = \frac{b_{x_2}(x_2) b_{x_1}(x_1)}{b_{x_1}(x_1)} = \begin{cases} \frac{\lambda_1 + 2x_2}{2(\lambda_1 + 3)} & \lambda_2 = 1, \frac{0}{0} & \text{otherwise} \end{cases} \]

\[ E(x_2|x_1 = 1) = \sum_{x_2} \lambda_2 b_{x_2|x_1}(x_2|x_1) \]

\[ = \frac{\lambda_1 + 2}{2(\lambda_1 + 3)} + \frac{2(\lambda_1 + 4)}{2(\lambda_1 + 3)} \]

\[ = \frac{3\lambda_1 + 10}{2(\lambda_1 + 3)} \]

\[ E(x_2|x_1 = 2) = \sum_{x_2} \lambda_2 b_{x_2|x_1}(x_2|x_1) \]

\[ = \frac{\lambda_1 + 2}{2(\lambda_1 + 3)} + \frac{4(\lambda_1 + 4)}{2(\lambda_1 + 3)} \]

\[ = \frac{5\lambda_1 + 18}{2(\lambda_1 + 3)} \]

\[ \text{Var}(x_2|x_1 = 1) = E(x_2^2|x_1 = 1) - (E(x_2|x_1 = 1))^2 \]

\[ \frac{9}{14} \]
(d) \[ P \left( x_1 < \frac{x_2}{3} \right) = P \left( x_1 < \frac{1}{3}, x_2 = 1 \right) + P \left( x_1 < \frac{1}{3}, x_2 = 2 \right) = 0 \]
\[ P(x_1 = x_2) = P(x_1 = x_2 = 1) + P(x_1 = x_2 = 2) \]
\[ = 3c + 6c = 9c = \frac{1}{2} \]
\[ P(x_1 > \frac{x_2}{2}) = P(x_1 > \frac{1}{2}, x_2 = 1) + P(x_1 > \frac{1}{2}, x_2 = 2) \]
\[ = P(x_2 = 1) + P(x_2 = 2) = 1. \]
\[ P(x_1 + x_2 \leq 3) = P(x_1 = x_2 = 1) + P(x_1 = x_2 = 2) + P(x_2 = x_1 = 1) \]
\[ = 1 - P(x_1 = x_2 = 2) \]
\[ = 1 - 6c = \frac{2}{3} \]

(e) \[ E(x_1x_2) = c \left[ 1 \times 3 + 2 \times 5 + 2 \times 4 + 4 \times 6 \right] = 45c \]
\[ E(x_1) = 2c \left[ 1 \times 4 + 2 \times 5 \right] = 28c \]
\[ E(x_1^2) = 2c \left[ 1 \times 4 + 4 \times 5 \right] = 48c \]
\[ E(x_2) = c \left[ 1 \times 7 + 2 \times 11 \right] = 29c \]
\[ E(x_2^2) = c \left[ 1 \times 7 + 4 \times 11 \right] = 51c \]
\[ \text{Cov}(x_1, x_2) = E(x_1x_2) - E(x_1)E(x_2) = 45c - 29 \times 28c \]
\[ \text{Var}(x_1) = E(x_1^2) - (E(x_1))^2 = 48c - (28c)^2 \]
\[ \text{Var}(x_2) = E(x_2^2) - (E(x_2))^2 = 51c - (29c)^2 \]
\[ \rho(x_1, x_2) = \frac{\text{Cov}(x_1, x_2)}{\sqrt{\text{Var}(x_1) \text{Var}(x_2)}} \]

Problem No. 6

(a1) \[ \sum_{x \in X} P(x) = 1 \]
\[ 2x = 2 \sum_{x \in X} x_1 \]
\[ = c \sum_{x \in X} x_1 = 1 \]
\[ \Rightarrow c \sum_{x \in X} x_1 = 1 \]
\[ \Rightarrow \frac{c}{2} \sum_{x \in X} x_1 = 1 \]
\[ \Rightarrow c = \frac{1}{7} \]

[5/14]
(b) \[ b_{x_1}(x_1) = \sum_{x_2} b_{x_1 x_2}(x_1, x_2) \]
\[ b_{x_2}(x_2) = \sum_{x_1} b_{x_1 x_2}(x_1, x_2) \]

\[ b_{x_1}(x_1) = \begin{cases} \frac{3}{7}, & x_1 = 1 \\ \frac{1}{7}, & x_1 = 2 \\ 0, & \text{otherwise} \end{cases} \]
\[ b_{x_2}(x_2) = \begin{cases} \frac{4}{7}, & x_2 = 1 \\ \frac{2}{7}, & x_2 = 2 \\ 0, & \text{otherwise} \end{cases} \]

(c) \[ b_{x_1|1}(x_1|1) = \frac{b_{x_1 x_2}(1, x_2|1)}{b_{x_1}(1)} \]

\[ E(x_2|x_1=1) = \sum_{x_2} b_{x_1 x_2}(x_1, x_2) = 1 \times \frac{1}{3} + 2 \times \frac{2}{3} = \frac{5}{3} \]

\[ E(x_2^2|x_1=1) = \sum_{x_2} x_2^2 b_{x_1 x_2}(x_1, x_2) = 1 \times \frac{1}{3} + 4 \times \frac{2}{3} = \frac{3}{3} = 1 \]

\[ \text{Var}(x_2|x_1=1) = E(x_2^2|x_1=1) - (E(x_2|x_1=1))^2 = \frac{3}{3} - \frac{5}{9} = \frac{2}{9} \]

(d) \[ \rho(x_1, x_2) = \sum_{x_1, x_2} b_{x_1 x_2}(x_1, x_2) = 0 \]

\[ p(x_1, x_2) = \sum_{x_2} b_{x_1}(x_2) = c_1 [1 x_1 + 2 x_2] = \frac{5}{7} \]

\[ p(x_1 < \frac{3}{5} x_2) = p(x_1 < \frac{3}{5}, x_2=1) + p(x_1 < \frac{3}{5}, x_2=2) \]
\[ = p(x_1 < \frac{3}{5}, x_2=2) = 2c = 2/7 \]

\[ p(x_1 + x_2 > 3) = 1 - p(x_1 + x_2 \leq 2) \]
\[ = 1 - p(x_1 < \frac{3}{5}, x_2=2) = 1 - c = \frac{6}{7} \]

(e) \[ E(x_1 x_2) = \sum_{x_1, x_2} x_1 x_2 b_{x_1 x_2}(x_1, x_2) \]

\[ = c_1 \left[ 1 x_1 + 2 x_2 \cdot x_1 x_2 \right] \]
\[ c_1 = 2\rho = 3 \]

\[ E(x_1) = 1 \times \frac{3}{7} + 2 \times \frac{4}{7} = \frac{11}{7} \]

\[ E(x_1^2) = 1 \times \frac{3}{7} + 4 \times \frac{4}{7} = \frac{19}{7} \]

\[ E(x_2) = 1 \times \frac{1}{7} + 2 \times \frac{6}{7} = \frac{13}{7} \]

\[ E(x_2^2) = 1 \times \frac{1}{7} + 4 \times \frac{6}{7} = \frac{25}{7} \]

[6/14]
\[ \text{Cov}(x_1, x_2) = E(x_1 x_2) - E(x_1) E(x_2) \]
\[ = 3 - \frac{14}{9} = \frac{4}{9} \]
\[ \text{Var}(x_i) = E(x_i^2) - (E(x_i))^2 = \frac{19}{7} - \frac{121}{99} = \frac{12}{99} \]
\[ \text{Var}(x_2) = E(x_2^2) - (E(x_2))^2 = \frac{25}{7} - \frac{169}{99} = \frac{6}{99} \]
\[ p(x_1, x_2) = \frac{\text{Cov}(x_1, x_2)}{\sqrt{\text{Var}(x_1) \text{Var}(x_2)}} = \frac{4}{\sqrt{72}} \]

(b) \( p(x_1, x_2) \neq 0 \) \( \iff \) \( x_1 \) and \( x_2 \) are not independent.

**Problem No. 7**

(a) \( F_{x_1}(x_1) = \int_{-\infty}^{x_1} f_{x_1}(x_1) \, dx = 1 \) \( \Rightarrow \) \( c(x_1) \int_{-\infty}^{x_1} \, dx = 1 \) \( \Rightarrow \) \( c(x_1) = \frac{1}{1-x^2} \).

(b) \( b_{x_1}(x_1) = b_{x_1}(x_1) \, dx \, dx = \begin{cases} 8x^3, & 0 < x < 1 \\ 0, & \text{o.w.} \end{cases} \)

\[ b_{y_1}(y_1) = \int_{-\infty}^{\infty} b_{x_1}(x_1) \, dx_1 = \begin{cases} 4y^2, & 0 < y < 1 \\ 0, & \text{o.w.} \end{cases} \]

(c) \( F_{x_1|x_2}(x_1|x_2) = \frac{b_{x_1}(x_1|x_2)}{b_{y_1}(y_1)} = \begin{cases} \frac{2x}{y}, & 0 < x < y \leq 1 \\ 0, & \text{o.w.} \end{cases} \)

\[ E(x_1|x_2 = y) = \frac{2}{y^2} \int_0^y x \, dx = \frac{2}{3} \]

\[ E(x_1^2|x_2 = y) = \frac{2}{y^2} \int_0^y x^2 \, dx = \frac{y}{2} \]

\[ \text{Var}(x_1|x_2 = y) = E(x_1^2|x_2 = y) - (E(x_1|x_2 = y))^2 = \frac{y^2}{2} - \frac{y^4}{2} = \frac{y^2}{18} \]

(d) \( p(x < \frac{x_2}{2}) = \int_{\frac{x_2}{2}}^{x_2} b_{x_1}(x_1) \, dx_1 \, dx_2 = \begin{cases} 2^2, & 1 \frac{3}{2} \leq \frac{3}{2} \leq 1 \\ \int_0^{\frac{3}{2}} \int_0^{\frac{3}{2}} \, dx_1 \, dx_2 = \frac{1}{4} \end{cases} \]
\[ P(x+y \geq \frac{3y}{2}) = \int_{0}^{3y/2} \int_{0}^{2-x} g(x,y) \, dx \, dy + \int_{3y/2}^{1} \int_{0}^{1} g(x,y) \, dx \, dy \]

\[ P(x = 2y) = \int_{0}^{2y} b_{x=2y} \, dx \, dy = 0 \]

(c) \[ E(x+y) = \int_{0}^{1} \int_{0}^{y} g(x+y) \, dx \, dy = \frac{8}{9} \]
\[ E(x) = \int_{0}^{1} \int_{0}^{y} g(x) \, dx \, dy = \frac{8}{15} \]
\[ E(x^2) = \int_{0}^{1} \int_{0}^{y} g(x^2) \, dx \, dy = \frac{1}{3} \]
\[ E(y) = \int_{0}^{1} \int_{0}^{y} g(y) \, dx \, dy = \frac{1}{3} \]
\[ E(y^2) = \int_{0}^{1} \int_{0}^{y} g(y^2) \, dx \, dy = \frac{2}{3} \]

\[ \text{Cov}(x,y) = E(x,y) - E(x)E(y) = \frac{4}{9} - \frac{32}{15} = \frac{4}{22.5} \]
\[ \text{Var}(x) = \frac{1}{3} \left( \frac{64}{225} - \frac{1}{225} \right) = \frac{22}{225} \]
\[ \text{Var}(y) = \frac{2}{3} \left( \frac{16}{25} - \frac{2}{25} \right) = \frac{2}{75} \]

\[ P(x,y) = \frac{\text{Cov}(x,y)}{\sqrt{\text{Var}(x) \text{Var}(y)}} \]

(b) \[ P(x,y) \neq 0 \Rightarrow x \text{ and } y \text{ are not independent.} \]

**Problem 15.9**

(a) \[ \sum_{x=0}^{2} \sum_{y=0}^{2} b_{x,y} \, dx \, dy = 1 \]
\[ = c \int_{0}^{2} \int_{0}^{2} \frac{1}{x+y} \, dx \, dy = 1 \Rightarrow c = 1 \]

(b) \[ b_{x,y}(x,y) = \int_{0}^{2} \int_{0}^{2} b(x,x,y) \, dx \, dy \]
\[ = \begin{cases} \frac{2}{x+y} \frac{1}{x+y} \, dx \, dy, & 0 < x < 1, \quad 0 < y < 1, \\ 0, & \text{otherwise} \end{cases} \]

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(c) For $0 < y < x < 1$,
\[
\frac{f_{x_2 | x_1, x_3}(x_2 | x_1, x_3)}{f_{x_2, x_1, x_3}(x_1, x_2, x_3)} = \begin{cases} \frac{\alpha (x_2, y)}{\lambda x_2}, & 0 < x_2 < 1 \\ \frac{\alpha (x_2, y)}{\lambda x_2}, & 0 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}
\]

\[
E(x_2 | (x_1, x_3) = (x_1, y)) = \frac{1}{\ln(2/3)} \int_0^2 \frac{2}{2} \frac{1}{x_2} \, dx_2 = \frac{x_1 - 1}{\ln(2/3)}
\]

\[
E(x_2^2 | (x_1, x_3) = (x_1, y)) = \frac{1}{\ln(2/3)} \int_0^2 \frac{2}{2} \frac{1}{x_2} \, dx_2 = \frac{x_1 - 1}{2 \ln(2/3)}
\]

\[
\text{Var}(x_2 | (x_1, x_3) = (x_1, y)) = E(x_2^2 | (x_1, x_3) = (x_1, y)) - (E(x_2 | (x_1, x_3) = (x_1, y)))^2
\]

(d) \[
P(x_2 < \frac{x_1}{2}) = \int_0^{2/2} \frac{1}{2} dx_2 \frac{1}{3 x_2} = \frac{1}{6}
\]

\[
P(x_2 = 2 x_1 > \frac{x_1}{2}) = 0
\]

(e) \[
E(x_1) = \int_0^2 \frac{1}{2} dx_1 \frac{1}{2} dx_2 \frac{1}{2} dx_3 = \frac{1}{2}
\]

\[
E(x_1^2) = \int_0^2 \frac{1}{2} dx_2 \frac{1}{2} dx_3 \frac{1}{2} dx_1 = \frac{1}{3}
\]

\[
E(x_2) = \int_0^2 \frac{1}{2} dx_1 \frac{1}{2} dx_2 \frac{1}{2} dx_3 = \frac{1}{4}
\]

\[
E(x_2^2) = \int_0^2 \frac{1}{2} dx_1 \frac{1}{2} dx_2 \frac{1}{2} dx_3 = \frac{1}{9}
\]

\[
\text{Var}(x_1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{12}; \quad \text{Var}(x_2) = \frac{1}{4} - \frac{1}{16} = \frac{7}{144}
\]

\[
\text{Cov}(x_2, x_3) = E(x_2 x_3) - E(x_2) E(x_3) = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}
\]

\[
P(x_2, x_3) = \frac{\text{Cov}(x_2, x_3)}{\sqrt{\text{Var}(x_2) \text{Var}(x_3)}} = \frac{1}{24} \times \sqrt{12 \times 12}
\]

(4) $P(x_2, x_3) 
eq 0 \Rightarrow x_1$ and $x_2$ are not independent.

\[
\Rightarrow x_1, x_2, x_3 \text{ are not independent.}
\]

\[9/10\]
Problem No. 9

(a) (b) \( P((x_1 \cdot x_2) = (0, 0)) = P((x_1 \cdot x_3) = (0, 0)) = P((x_2 \cdot x_3) = (0, 0)) \)

\[ = P((x_1 \cdot x_2) = (0, 0)) = \frac{1}{4} \]

Also, \( (x_1 \cdot x_2) \neq (x_2 \cdot x_3) \neq (x_1 \cdot x_3) \)

\[ P(x_1 = 0) = P(x_1 = 1) = \frac{1}{2}, \quad \forall i \in \{1, 2, 3\} \]

Thus \( x_1, x_2, \) and \( x_3 \) are pairwise independent.

\[ P(x_1 = 0, x_2 = 0, x_3 = 0) = \frac{1}{8} \neq P(x_1 = 0) \cdot P(x_2 = 0) \cdot P(x_3 = 0) = \frac{1}{8} \]

\( \Rightarrow x_1, x_2, \) and \( x_3 \) are not independent.

(c) \( P(x_1 \cdot x_2 = 0, x_3 = 1) = P(x_1 \cdot x_3 = 2, x_2 = 1) = \frac{1}{4} \)

\[ P(x_1 \cdot x_2 = 2, x_3 = 0) = \frac{1}{2} \cdot P(x_1 \cdot x_2 = 2) = \frac{1}{4} \]

\[ P(x_1 = 1) = \frac{1}{2}, \quad P(x_1 = 2) = \frac{1}{4} \]

(clearly)

\[ P(x_1 \cdot x_2 = 0, x_3 = 1) = \frac{1}{4} \neq P(x_1 \cdot x_2 = 0) \cdot P(x_3 = 1) = \frac{1}{2} \]

\( \Rightarrow x_1 \cdot x_2, \) and \( x_3 \) are not independent.

Problem No. 10

(a) (b) \( b_{x_1 \cdot x_2}(x_1, x_2) = \int_0^\infty b_{x_1 \cdot x_2}(x_1, x_2, x_3) \, dx_3 \)

\[ = \frac{1}{24} e^{-\frac{x_1^2 + x_2^2}{2}} \cdot \quad \forall x_1, x_2 < 0 \]

\( b_{x_1}(x_1) = \int b_{x_1 \cdot x_2}(x_1, x_2) \, dx_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \cdot \quad \forall x_1 < 0 \)

B) \( b_{x_i \cdot x_j}(x_i, x_j) = \frac{1}{24} e^{-\frac{x_i^2 + x_j^2}{2}} \cdot \quad \forall x_i < x_j < 0 \quad (\forall i) \)

\( b_{x_i}(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} \cdot \quad \forall x_i < 0 \quad (\forall i) \)

\( \) (Note \( \frac{1}{24} \int_0^\infty e^{-\frac{x^2}{2}} \, dx = 1 \) and \( \int_0^\infty e^{-\frac{x^2}{2}} \, dx = 0 \).)
Thus, \( x_1, x_2, x_3 \) are pair-wise independent but not independent.

\( (c) \) Joint pdf of \((x_1, x_2)\) is (see above)

\[
\begin{align*}
\text{pdf } (x_1, x_2) & (x < 1) = \frac{1}{2}\exp\left(-\frac{2\sqrt{x_1 x_2}}{1}\right), & 0 < x_1, x_2 < 1.
\end{align*}
\]

**Problem No. 11**

(a) Let \( W = X + Y \). The joint p.m.f. of \((W, Z)\) is

\[
\begin{array}{c|ccc|c|c}
(W, Z) & (2, 0) & (2, 1) & (4, 0) & (4, 1) \\
\hline
b_{W,Z} & \frac{1}{15} & \frac{1}{3} & \frac{4}{15} & \frac{4}{15} \\
\end{array}
\]

The p.m.f. of \( W \) is

\[
b_W(w) = \begin{cases} 
\frac{1}{15}, & w = 2 \\
\frac{1}{3}, & w = 3 \\
\frac{4}{15}, & w = 4 \\
0, & \text{otherwise}
\end{cases}
\]

The p.m.f. of \( Z \) is

\[
b_Z(z) = \begin{cases} 
\frac{2}{3}, & z = 0 \\
\frac{1}{3}, & z = 1 \\
0, & \text{otherwise}
\end{cases}
\]

Clearly, \( b_{W,Z}(w, z) \neq b_W(w) b_Z(z) \), \( W, Z \) are not independent

\[
\Rightarrow W = X + Y \text{ and } Z \text{ are not independent}
\]

(b) \( E(WZ) = 0 \times \frac{2}{15} + 3 \times \frac{1}{5} + 0 \times \frac{4}{15} + 5 \times \frac{4}{15} = \frac{7}{3} \)

\[
E(W) = 2 \times \frac{2}{15} + 3 \times \frac{1}{5} + 4 \times \frac{4}{15} + 5 \times \frac{4}{15} = \frac{11}{5}
\]

\[
E(W^2) = 4 \times \frac{2}{15} + 9 \times \frac{1}{5} + 16 \times \frac{4}{15} + 25 \times \frac{4}{15} = \frac{217}{15}
\]

\[
E(2^Z) = 0 \times \frac{2}{15} + 1 \times \frac{1}{5} + 0 \times \frac{4}{15} + 1 \times \frac{4}{15} = \frac{3}{5}
\]

\[
E(2^W) = 0 \times \frac{2}{15} + 1 \times \frac{1}{5} + 0 \times \frac{4}{15} + 1 \times \frac{4}{15} = \frac{3}{5}
\]

\[
\frac{\sqrt{11}}{3}
\]
\[
\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{217}{15} - \frac{121}{9} = \frac{44}{45}
\]

\[
\text{Var}(Z) = E(Z^2) - [E(Z)]^2 = \frac{3}{5} \left( 1 - \frac{3}{5} \right) = \frac{6}{25}
\]

\[
\text{Cov}(W, Z) = E(WZ) - E(W)E(Z) = \frac{7}{3} - \frac{33}{15} = \frac{2}{15}
\]

\[
P(W, Z) = \frac{\text{Cov}(W, Z)}{\sqrt{\text{Var}(W) \cdot \text{Var}(Z)}} = \frac{2}{15} \frac{\sqrt{25 \times 45}}{\sqrt{6 \times 45}}
\]

**Problem 12.12**

(a) Through observation we have

\[
x_1 \cdot x_2 \cdot x_3 = x_1 x_1 x_1, x_2 x_2 x_2, x_3 x_3 x_3, \quad \text{as} \quad x_1, x_2, x_3 \in \mathbb{R}^3
\]

where

\[
x_1 \sim \mathcal{N}(0, 1), \quad x_2 \sim \mathcal{N}(2, 1), \quad x_3 \sim \mathcal{N}(2, 1)
\]

\[
x_1 x_2 x_3 \sim \mathcal{N}(0, 2)
\]

\[
x_1, x_2, x_3 \text{ are independent.}
\]

(b) \( x_1, x_2, x_3 \) are independent

\[
\Rightarrow (x_1 x_2) \text{ and } x_3 \text{ are independent}
\]

\[
\Rightarrow x_1 x_2 \text{ and } x_3 \text{ are independent}
\]

(c) See (a)

(d) Since \( x_1 \) and \( x_2 \) are independent

\[
x_1 | x_2 \sim \mathcal{N}(0, 1)
\]

\[
x_1 | x_2 \sim \mathcal{N}(0, 1)
\]

\[
\text{Problem 13.12}
\]

\[
\text{Cov}(X_i, X_j) = \sigma \text{ and } P_i, \quad \text{as}\]

\[
E(X_i X_j) = \mu_i \mu_j + \rho \sigma_i \sigma_j
\]

\[
\text{Cov}(Y, Z) = E((Y - E(Y))(Z - E(Z))) = E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \right] \left( X_i - \mu_i \right) \left( X_j - \mu_j \right)
\]

\[
= E \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \text{ Cov}(X_i, X_j)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \cdot \sigma_i \cdot \sigma_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \text{ Cov}(X_i, X_j)
\]

\[
12/12
\]
Problem No. 14

\[ \text{Var}(x^2) = E(x^4) - [E(x^2)]^2 = 2 = \text{Var}(x) \]

\[ \text{Cov} \left( \frac{x}{3} + \frac{2x}{5}, \frac{2x}{3} + \frac{x}{5} \right) = \frac{2}{9} \text{Var}(x) + \frac{8}{9} \text{Cov}(x, x) \]

\[ = \frac{8}{9} + \frac{5}{9} \text{Cov}(x, x) = \frac{8}{9} \times \frac{2}{3} \times 1 \times \sqrt{2} \times \sqrt{2} \]

\[ \text{Var} \left( \frac{x}{3} + \frac{2x}{5} \right) = \text{Var} \left( \frac{2x}{3} + \frac{x}{5} \right) = \frac{\text{Var}(x)}{9} + \frac{5}{9} \text{Var}(x) \]

\[ = \frac{10}{9} \times \frac{8}{27} = \frac{32}{27} \]

\[ P \left( \frac{x}{3} + \frac{2x}{5}, \frac{2x}{3} + \frac{x}{5} \right) = \frac{32}{38} \]

Problem No. 15

(a) \[ \text{Var} \left( \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} \text{Var}(x_i) + 2 \sum_{i=1}^{n} \sum_{j>i} \text{Cov}(x_i, x_j) \]

\[ \leq \sum_{i=1}^{n} \left( \frac{1}{15} \text{Var}(x_i) \right)^{\frac{2}{3}} + \sum_{i=1}^{n} \left( \frac{2}{15} \text{Var}(x_i) \text{Var}(x_j) \right)^{\frac{2}{3}} \]

\[ = \left( \sum_{i=1}^{n} \frac{1}{15} \text{Var}(x_i) \right)^{\frac{2}{3}} \]

For proving the other inequality consider a B.V. \( Y \) A.T.

\[ P(Y = a_i) = \frac{1}{n}, \text{ for some positive real constants } a_1, \ldots, a_n. \]

\[ \text{Th. } E(Y) \geq \left( E(Y^2) \right)^{\frac{1}{2}} \text{ (Jensen's inequality)} \]

\[ = \left( \sum_{i=1}^{n} a_i x_i \right)^{\frac{1}{2}} \geq \left( \sum_{i=1}^{n} a_i \right)^{\frac{1}{2}} \]

Now taking \( a_i = \text{Var}(x_i) \), \( \sum a_i \), we get the result.

(b) Take \( \frac{1}{n} \), \( c_1, \ldots, c_n \) in (c).
Problem No. 16

\[ P \left( x \mid y \right) = P \left( x \mid x = x_0 \right) = \frac{1}{x}, \quad x = 2, \ldots, n. \]

\[ P \left( x \mid y \right) = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad E(x) = \frac{1}{n} \sum_{i=1}^{n} x_i = 0 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 = E(y) \]

\[ E(x^4) = \frac{1}{n} \sum_{i=1}^{n} x_i^4, \quad E(y^4) = \frac{1}{n} \sum_{i=1}^{n} y_i^4 \]

\[ = \text{Var}(x) = \text{Var}(y) \]

\[ P \left( x \mid y \right) \leq 1 \Rightarrow \text{Cov}(x,y) \leq \text{Var}(x) \text{Var}(y) \]

\[ = \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \leq \left( \frac{1}{n} \sum_{i=1}^{n} x_i^2 \right) \left( \frac{1}{n} \sum_{i=1}^{n} y_i^2 \right) \]

\[ = \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \leq \left( \frac{E(x)}{n} \right) \left( \frac{E(y)}{n} \right). \]

Problem No. 17

\[ E(x^4) = \frac{47}{15}, \quad E(x) = \frac{7}{5}, \quad E(x^4) = \frac{11}{5}, \quad \text{Var}(x) = \frac{6}{25} \]

\[ E(y) = \frac{34}{15}, \quad E(y^4) = \frac{86}{25}, \quad \text{Var}(y) = \frac{134}{225}, \quad \text{Cov}(x,y) = \frac{7}{45} \]

\[ \rho = \frac{\text{Cov}(x,y)}{\sqrt{\text{Var}(x) \text{Var}(y)}} = \frac{7}{\sqrt{801}} \]

Problem No. 18

(a) \[ \Psi_1 (z) = \frac{1}{1 - 2x}, \quad \Psi_2 (z) = \frac{\text{ln}(1-2x)}{1 - 2x} \]

\[ \frac{\partial}{\partial x} \Psi_1 (z) = \frac{2 - \Psi_2 (z)}{1 - 2x}, \quad \frac{\partial}{\partial x} \Psi_2 (z) = \frac{2}{1 - 2x} \]

\[ \frac{\partial^2}{\partial x^2} \Psi_1 (z) = \frac{4 \Psi_1 (z) - 4 \Psi_2 (z)}{(1 - 2x)^2} \]

\[ \text{Cov} (\Psi_1, \Psi_2) = \left[ \frac{\partial^2}{\partial x^2} \Psi_1 (z) \right]_{z=\Psi_2} = 0 \]

\[ \Rightarrow \text{Cov} (\Psi_1, \Psi_2) = 0 \]

(b) \[ \Pi_1 (t) = \Psi_2 (0, 0) = e^t, \quad t \in \mathbb{R} \]

\[ \Pi_2 (t) = \Psi_2 (0, 0) = \frac{1}{1 - 2t}, \quad t \in \mathbb{R} \]

\[ \Pi_2 (t) = \Psi_2 (0, 0) = \frac{1}{1 - 2t}, \quad t \in \mathbb{R} \]

\[ \Pi_2 (t) = \Psi_2 (0, 0) = \frac{1}{1 - 2t}, \quad t \in \mathbb{R} \]

\[ \Pi_2 (t) = \Psi_2 (0, 0) = \frac{1}{1 - 2t}, \quad t \in \mathbb{R} \]

\[ \Pi_2 (t) = \Psi_2 (0, 0) = \frac{1}{1 - 2t}, \quad t \in \mathbb{R} \]
\[ \Rightarrow \text{ } Y \text{ and } Z \text{ are not independent (although } \text{Cov}(Y, Z) = 0) . \]

(c) \[ \Pi_{Y|Z}(\pm \frac{4}{3}) = E \left( e^{+Y|Z}(\pm \frac{4}{3}) \right) \]
\[ = \Pi_Y Z(\pm \frac{4}{3}) + \frac{\pm \frac{4}{3}}{1 - \frac{1}{2}} \]
\[ = \frac{e^{\pm \frac{4}{3} - \frac{4}{3}}}{1 - \frac{1}{2}} + \frac{\pm \frac{4}{3}}{1 - \frac{1}{2}} . \]

**Problem No. 1.9**

(a) \[ \psi_Y Z(\pm) = \ln \Pi_Y Z(\pm) = \frac{t + \frac{4}{3} + t + \frac{4}{3}}{2} - \frac{1}{2} \]
\[ \frac{\partial^2}{\partial t^2} \psi_Y Z(\pm) = 2 \frac{(t + \frac{4}{3})}{2} , \quad \frac{\partial^2}{\partial t \partial \frac{4}{3}} \psi_Y Z(\pm) = 1 , \quad \frac{\partial^2}{\partial \frac{4}{3}^2} \psi_Y Z(\pm) = \frac{1}{2} \]
\[ \Rightarrow \text{Cov}(Y, Z) = \left[ \frac{\partial^2}{\partial t \partial \frac{4}{3}} \psi_Y Z(\pm) \right] \frac{4}{3} = \frac{1}{2} \]
\[ \mathrm{Var}(Y) = \left( \frac{\partial}{\partial t} \psi_Y Z(\pm) \right) \frac{4}{3} = 2 \Rightarrow \mathrm{Var}(Z) \]
\[ \Rightarrow \text{P}(Y, Z) = \frac{1}{2} \]

(b) \[ \Pi Y - Z \neq 0 \Rightarrow \text{Y and Z are not independent.} \]

(c) \[ \Pi_{Y|Z}(\pm) = E \left( e^{+Y|Z}(\pm) \right) = \Pi_Y Z(\pm) \frac{4}{3} - \frac{1}{2} \text{ term}. \]

**Problem No. 2.0**

(a) \[ S_2 = \{10, 01, 01, 11, 10 \} \]
\[ b_{y_1}(y_2|y_2) = 1 \left( x_1 - x_2 \geq 7, \quad x_1 + x_2 \geq 7 \right) \]
\[ = 1 \left( x_1 \geq \frac{y_2 + 7}{2} , \quad x_2 = \frac{y_2 - 7}{2} \right) \]
\[ = \begin{cases} \left( \frac{2}{3} \right)^{y_1} \left( \frac{1}{3} \right)^{y_2 - y_1} , & y_1 \in S_2 \\ 0 , & \text{otherwise} \end{cases} \]

\[ \frac{15/14}{} \]
\[
\alpha_k(\tau) = \sum_{j \leq \tau} \alpha_j(\tau) = \begin{cases} 
\frac{1}{q}, & \tau = 0 \\
\frac{2}{q}, & \tau = 1 \\
\frac{1}{q}, & \tau > 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\beta_k(\tau) = \sum_{j \leq \tau} \beta_j(\tau) = \begin{cases} 
\frac{1}{q}, & \tau = 0 \\
\frac{2}{q}, & \tau = 1 \\
\frac{1}{q}, & \tau > 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
E(\tau) = 0 \times \frac{1}{q} + 1 \times \frac{2}{q} + 2 \times \frac{1}{q} = \frac{1}{3}
\]

\[
E(\tau)^2 = 0 \times \frac{1}{q} + 1 \times \frac{2}{q} + 4 \times \frac{1}{q} = \frac{20}{2}
\]

\[
\text{Var}(\tau) = E(\tau^2) - (E(\tau))^2 = \frac{4}{9}
\]

\[
E(\tau_1, \tau_2) = 0 \times (\frac{2}{3})^0 (\frac{1}{3})^2 + (1)(\frac{2}{3})' (\frac{1}{3})' \frac{1}{3} = (\frac{2}{3})' (\frac{1}{3})' \frac{1}{3}
\]

\[
E(\tau_1) = - \frac{2}{q} + \frac{2}{q} = 0
\]

\[
E(\tau_1, \tau_2) = E(\tau_1) E(\tau_2) = 0
\]

\[
P(\tau_1 > 0, \tau_2 = 0) = \frac{1}{q} \neq P(\tau_1 > 0) P(\tau_2 = 0) > \frac{5}{9}
\]

\[
\tau_1 \text{ and } \tau_2 \text{ are not independent (although } \text{Cov} (\tau_1, \tau_2) = 0).
\]

**Problem 4.21** (a)

\[x_1, \ldots, x_n \text{ is a random sample and } x_i - \mu \overset{d}{=} \mu - x_i\]

\[\Rightarrow (x_{\mu+1}, \ldots, x_{n-\mu}) \overset{d}{=} (\mu - x_{\mu+1}, \ldots, \mu - x_n)\]

\[\Rightarrow \text{ } x_{\mu+1}, \ldots, x_{n-\mu} \overset{d}{=} x_{\mu+1} \ldots, x_{n-\mu}\]
\[ Y_i : x_{i:n} - \mu \equiv \mu - x_{i:n} : \eta, \quad \eta = \ldots \]

(b) \quad \text{If} \quad \eta = \frac{1}{2}

\[ E(\frac{X_{i:n} - \mu}{\eta}) = E(\frac{\mu - X_{i:n}}{\eta}) \]

\[ = E(X_{i:n} + X_{i:n}) = 2\mu, \quad \eta = \ldots \]

(c) \quad \text{Taking} \quad \eta = \frac{1}{2} \quad \text{in} \quad (b) \quad \text{we get}

\[ E(X_{i:n} : \eta) = \mu \]

(d) \quad \text{Using} \quad (a) \quad \text{for} \quad \eta = \frac{1}{2} \quad \text{we get}

\[ X_{i:n} : \mu - \mu \equiv \frac{1}{2} \mu - \frac{1}{2} X_{i:n} \]

\[ \Rightarrow \quad P(X_{i:n} > \mu) = P(\frac{X_{i:n}}{\frac{1}{2}} < \frac{\mu}{\frac{1}{2}}) \]

\[ \Rightarrow \quad P(\frac{X_{i:n} - \mu}{\frac{1}{2}} > 0) = P(\frac{\mu - X_{i:n}}{\frac{1}{2}} < 0) \]

\[ \Rightarrow \quad P(\frac{X_{i:n} - \mu}{\frac{1}{2}} > 0) = \frac{1}{2} \quad (\text{as} \quad P(\frac{X_{i:n} - \mu}{\frac{1}{2}} = 0) = 0) \]

\[ \text{and} \quad X_{i:n} \text{ are AC} \]

\[ \text{Problem 10.21} \]

(c)

\[ (x_1, x_2, \ldots, x_{\infty}, x_{\infty}, x_{\infty}, x_{\infty}, \ldots) \overset{d}{=} (x_1, x_2, \ldots, x_{\infty}, x_{\infty}, x_{\infty}, \ldots) \]

\[ \Rightarrow E\left(\frac{x_1}{x_1 + x_2 + \cdots + x_n}\right) = E\left(\frac{x_1}{x_1 + x_2 + \cdots + x_n}\right) \]

\[ = E\left(\frac{x_1}{x_1 + x_2 + \cdots + x_n + x_1 + x_2 + \cdots + x_n}\right), \quad \eta = \ldots \]

\[ E(\frac{x_1}{\sum_{j=1}^{\infty} x_j}) = E(\frac{x_1}{\sum_{j=1}^{\infty} x_j}) = c \]

\[ \sum_{i=1}^{n} E\left(\frac{x_i}{\sum_{j=1}^{\infty} x_j}\right) = n \cdot c = \Rightarrow E\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{j=1}^{\infty} x_j}\right) = n \cdot c \]

\[ \boxed{n/1} \]
\( E(\frac{X_i}{\sum_{j=1}^{n} x_j}) = \frac{1}{n}, \quad i = 1, \ldots, n \)

\[ \sum_{i=1}^{n} E(\frac{X_i}{\sum_{j=1}^{n} x_j}) = \frac{k}{n} \]

(b) \( \text{An u (a)} \)

\( (x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots, x_k) \equiv (x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots, x_k) \)

\[ E(x_i \mid x_1 + x_2 + \ldots + x_{n-1} + x_n + \ldots + x_k = t) = E(x_i \mid x_1 + x_2 + \ldots + x_{n-1} + x_n + \ldots + x_k = t) \]

\[ E(x_i \mid \sum_{j=1}^{n} x_j = t) = E(x_i \mid \sum_{j=1}^{n} x_j = t) = c(t, n) \]

\[ \sum_{i=1}^{n} E(x_i \mid \sum_{j=1}^{n} x_j = t) = nc(n) \]

\[ E(\sum_{i=1}^{n} x_i \mid \sum_{j=1}^{n} x_j = t) = nc(n) \]

\[ t = nc(n) \Rightarrow c(n) = \frac{t}{n} \]

\[ E(x_i \mid \sum_{j=1}^{n} x_j = t) = \frac{t}{n} \]

(c) \( x_1, x_2, \ldots, x_n \) is a random sample

\( \Rightarrow x_1, \ldots, x_n \) is a random sample

\( x_1, \ldots, x_n \sim N(\mu, \sigma^2) \)

\[ P(x_1 \leq \ldots \leq x_n) = \int_{x_1}^{x_n} \int_{x_2}^{x_n} \ldots \int_{x_{n-1}}^{x_n} f(x_1, \ldots, x_n) \ dx_1 \ldots dx_n \]

\( \text{Since} \quad x_1, \ldots, x_n \sim \text{N}(\mu, \sigma^2) \)

\[ \sum_{i=1}^{n} P(x_{a_i} < \ldots < x_{a_n}) = 1 \quad \Rightarrow P(x_{(1)} < \ldots < x_{(n)}) = \frac{1}{n!}. \]
Problem No. 23

The set \( S \) of \( X = (x_1, x_2) \) is

\[
x^2(x) = b(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2), (x_2, x_1) \in S \\
0.5 & \text{otherwise}
\end{cases}
\]

\( S \times S = \{(x_1, x_2), (x_2, x_1) \mid x_1, x_2 \in S\} \)

(a) \( b_1(y) = \sum_{x \in S} b_x(x) \)

\[
= \sum_{x_1, x_2 \in S} b_x(x) + \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_2, x_1}
\]

\[
= \theta^{(2-y)} \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_2, x_1}
\]

\[
= \theta^{(2-y)} \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_2, x_1}
\]

\( b_1(y) = \begin{cases} \theta^{(2-y)} \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_2, x_1}, & y \in \{2, \ldots, 7\} \\
0, & \text{otherwise}
\end{cases} \)

(b) \( b_2(y) = \sum_{x \in S} b_x(x) \)

\[
\text{Case } i \quad y = 0
\]

\[
b_2(y) = \sum_{x_1, x_2 \in S} b_{x_1, x_2} = \sum_{x_1, x_2 \in S} \theta^{(1-y) (2^{x_1} - 2^{x_2})} = \frac{\theta}{2 - \theta}
\]

\[
\text{Case } ii \quad y \in \{1, 2, \ldots, 7\}
\]

\[
b_2(y) = \sum_{x_1, x_2 \not\in S} b_{x_1, x_2} + \sum_{x_1, x_2 \not\in S} b_{x_2, x_1}
\]

\[
\text{Case } ii
\]

\[
b_2(7) = \sum_{x \in S} b_x(x)
\]
\[ \sum_{k=1}^{n} \theta (1-\theta)^{-2k+2} \sum_{k=1}^{n} \theta^k (1-\theta)^{2k-2} \]

\[ = \frac{20 (1-\theta)}{2-\theta} \]

Thus

\[ f_{\theta}(y) = \begin{cases} \frac{\theta}{2-\theta}, & y = 0 \\ \frac{20 (1-\theta)}{2-\theta}, & y \in \{1, 2, \ldots, n\} \\ 0, & \text{otherwise} \end{cases} \]

(a) \[ f_2(x, y) = p(\min\{x_i, x_n\} = y_i, \max\{x_i, x_n\} = x_i + y_i) \]

Case I: \( y_1 \in \{1, 2, \ldots, y_2 = 0 \}

\[ f_2(x, y) = p(\min\{x_i, x_n\} = y_i, \max\{x_i, x_n\} = x_i + y_i) \]

Case II \( y_1 \in \{2, 3, \ldots, y_2 \in \{2, 3, \ldots, n\} \}

\[ f_2(x, y) = p(\min\{x_i, x_n\} = y_i, \max\{x_i, x_n\} = x_i + y_i) + p(x_i = y_i + y_2, x_i = y_i) \]

\[ = 20 \left(1-\theta \right)^{2y_1 + y_2 - 2} \]

Thus

\[ f_2(y) = \begin{cases} 0, & y \in 1\times2 \times 2 \times \ldots \times n \\ 20 \left(1-\theta \right)^{2y_1 + y_2 - 2}, & y \in \{1, 2, \ldots, n\} \end{cases} \]

(d) Clearly \( b_2(y) = b_3(y) b_3(y_2), y \in 1\times2 \times \ldots \times n \)

\[ \Rightarrow \text{if and } y \in 1\times2 \]

\[ \overline{20} \]

(1)
\( \begin{align*}
\delta y_1(j_2) &= \sum_{j_1=2}^{2} \delta y_2(j_1, j_2) \\
&= \delta (2-\delta) (1-\delta)^{j_2-2} + \sum_{j_1=2}^{2} (2-\delta) (1-\delta)^{j_2-2} \\
&= \delta (2-\delta) (1-\delta)^{j_2-2}, \quad j_1 \in \{1, \ldots, 2\} \\
&= \delta (2-\delta) (1-\delta)^{j_2-2}, \quad j_1 \in \{1, \ldots, 2\} \\
\end{align*} \)

Similarly,
\( \begin{align*}
\delta y_2(j_2) &= \sum_{j_1=2}^{2} \delta y_2(j_1, j_2) \\
&= \sum_{j_1=2}^{2} \delta (2-\delta) (1-\delta)^{j_2-2}, \quad j_2 = 0 \\
&= \delta \frac{2-\delta}{2-\delta}, \quad j_2 = 0 \\
\end{align*} \)

For \( j_2 \in \{1, 2, \ldots \} \)
\( \begin{align*}
\delta y_2(j_2) &= \sum_{j_1=2}^{2} (2-\delta) (1-\delta)^{j_2-2}j_2+2 = 2(2-\delta)^{j_2} \\
&= \frac{2(2-\delta)^{j_2}}{2-\delta}, \quad j_2 = 0 \\
&= \delta (2-\delta)^{j_2}, \quad j_2 \in \{1, 2, \ldots \} \\
\end{align*} \)

\( \begin{align*}
\text{Problem No. 24 (a)} \\
\text{(a)} \quad P(y_2 = 1) &= P(x_1+x_2 = 1) = \begin{cases} 
\frac{5}{9}, & j_2 = 1 \\
\frac{5}{9}, & j_2 = 2 \\
0, & 0.5. 
\end{cases} \\
\text{(b)} \quad P(y_2 = 1) &= P(x_1+x_2 = 2) = \begin{cases} 
\frac{3}{9}, & j_2 = 1 \\
\frac{5}{9}, & j_2 = 2 \\
0, & 0.5. 
\end{cases} 
\end{align*} \)
(c) \( f(x, y) = f_1(x) f_2(y) = 9(x=x=1, x_2=x_3=3) \)

\[
\begin{cases} 
\frac{2}{9}, & f_1(x_1) = 1, f_2(x_2) = 1, f_2(x_3) = 1 \\
\frac{1}{3}, & f_2(x_2) = 2, f_2(x_3) = 2 \\
0, & \text{otherwise}
\end{cases}
\]

(d) \( P(\bar{x}, \bar{y}) = \frac{2}{9} + P(\bar{x}, 2y) P(2x, \bar{y}) \)

\( x_1 \) and \( x_2 \) are not independent.

(2) \( f_2(y) \) :

\[
\begin{cases} 
\sum_{x_2} f_1(x) f_2(y) = 2 \times \frac{2}{9}, & y_1 = 1 \\
\times \frac{1}{3}, & y_1 = 2 \\
0, & \text{otherwise}
\end{cases}
\]

By symmetry:

\[
f_{1,2}(y_2) = \begin{cases} 
\frac{2}{9}, & y_2 = 1 \\
\frac{1}{3}, & y_2 = 2 \\
0, & \text{otherwise}
\end{cases}
\]

**Problem No. 25**

The joint p.d.f. of \( X = (X_1, X_2) \) is

\[
f_{x_1, x_2} = f_1(x_1) f_2(x_2) = \begin{cases} 
\frac{1}{6}, & 0 \leq x_1 \leq 3, 0 \leq x_2 \leq 2 \\
0, & \text{otherwise}
\end{cases}
\]

(a) \( S_{y1} = (1, 3) \). Clearly for \( y < 1 \), \( F_{y1}(y) = 0 \) and for \( y \geq 3 \), \( F_{y1}(y) = 1 \). For \( 1 \leq y < 3 \):

\[
F_{y1}(y) = P(x_1 + x_2 \leq y)
\]

**Case 1:** \( 1 \leq y < \frac{3}{2} \)

\[
F_{y1}(y) = \int_0^y \int_0^{y-x_1} \text{ d}x_2 \text{ d}x_1 = \frac{(y-1)^2}{2}
\]

**Case 2:** \( \frac{3}{2} \leq y < 3 \)

\[
F_{y1}(y) = 1 - \int_{y-2}^y \int_0^{y-x_1} \text{ d}x_2 \text{ d}x_1 = 1 - \frac{(3-y)^2}{2}
\]

\[22/14\]
\[ F_{Y}(y) = \begin{cases} 
0 & y < 1 \\
\frac{(y-1)^2}{2} & 1 \leq y < 2 \\
1 - \frac{(3-y)^2}{2} & 2 \leq y < 3 \\
1 & y \geq 3
\end{cases} \]

Then the pdf of \( Z \) is

\[ f_{Z}(z) = \begin{cases} 
y-1 & 1 \leq y < 2 \\
2 - y & 2 \leq y < 3 \\
0 & \text{o.w.}
\end{cases} \]

(b) \( S_{Y}^{0} = \{(x, y) : x \geq 0, y \geq 0, 0 \leq x+y \leq 2, 2 \leq x-3 \leq y\} \)

The transformation \( h_{2} = (h_{1}, h_{2}) : S_{Z}^{0} \rightarrow \mathbb{R}^{2} \)

\( h_{1}(z_{1}, z_{2}) = \frac{z_{1}z_{2}}{2}, \quad h_{2}(z_{1}, z_{2}) = \frac{y-3}{2} \)

\[ J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2} \]

\( S_{Y}^{0} = \{(y, z) : 0 \leq y+z \leq 2, 2 \leq y-3 \leq z\} \)

The Jacobian is \( h_{1,1} \phi(z_{1}, z_{2}) = \frac{1}{2} \left( \frac{y-3}{2} - \frac{z_{2}}{2} \right) \)

\[ f_{Y}(y) = \int f_{Z}(z) \, dz = \int_{0}^{\min(2-y, y-3)} \frac{1}{2} \, dz, \quad 1 \leq y < 3 \]

\[ f_{Y}(y) = \begin{cases} 
\frac{1}{6} & 0 < y < 1 \\
\frac{1}{6} & 1 \leq y < 2 \\
0 & \text{o.w.}
\end{cases} \]
\[
\begin{align*}
\frac{y-2}{2} \quad & 1 < y < 4 \\
-\frac{2}{5} \ln(2) \quad & 1 < y < 2 \\
\frac{d}{2} \quad & 2 < y < 3 \\
eg \frac{1}{2} \quad & 0.5 \\
0 \quad & 0.5
\end{align*}
\]

\[b_2(y) = \int_{1}^{y} 2 \ln(3) \, dy = \int_{\max(5,3)}^{y} \frac{dy}{2}, \quad -2 \leq y < 0\]

\[\begin{align*}
\begin{cases}
\frac{2e^y}{1} & -2 \leq y < -1 \\
-3 & -1 \leq y < 0 \\
0 & 0.5
\end{cases}
\end{align*}\]

(c) Clearly, \(b_{y=2}(1.3) = b_{y=7}(1.3) = 2 \times 1.3 > 0\)

= \(y\) and \(z\) are not independent.

Problem No. 2.6

The joint p.d.f. of \(X = (x_1, x_2)\) is

\[f_X(x_1, x_2) = f_{1,2}(x_1, x_2) = \begin{cases}
\frac{1}{4} & -1 < x_1, x_2 < 1 \\
0 & 0.5
\end{cases}\]

\[S_{X} = (-1, 1) \times (-1, 1).\]
(a) $S^0 = (-\infty, 2)$. Thus for $\eta < -1$, $F_{\eta}(\eta) > 0$ and for $\eta \geq 2$, $F_{\eta}(\eta) > 1$. For $-1 \leq \eta < 2$,

$$F_{\eta}(\eta) = \mu \left( 1 + \frac{1}{\eta} x \mathbb{1}_{\{x \leq \eta \}} \right)$$

$$= \mu \left( x_2 - x_1 \mathbb{1}_{\{x_1 < \eta \}} + x_1 \mathbb{1}_{\{x_1 \leq \eta \}} \right)$$

$$= \mu \left( x_2 - x_1 \mathbb{1}_{\{x_1 < \eta \}} + 0 \mathbb{1}_{\{x_1 \leq \eta \}} \right) + \mu \left( x_1 \mathbb{1}_{\{x_1 \leq \eta \}} \right)$$

$$= \mu \left( x_2 - x_1 \mathbb{1}_{\{x_1 < \eta \}} \right) + \mu \left( x_1 \mathbb{1}_{\{x_1 \leq \eta \}} \right)$$

**Case I**

$$-1 \leq \eta < 0$$

$$F_{\eta}(\eta) = \int_{y_1}^{y} \int_{x_1}^{y} \frac{1}{y} dx_1 dx_2 + \int_{y_1}^{y} \int_{-1}^{x_1} \frac{1}{y} dx_1 dx_2$$

Note that

$$x_2 < y-1 \Rightarrow y-x_2 > 1$$

$$y-x_2 \leq 1$$

**Case II**

$$0 \leq \eta < 1$$

$$F_{\eta}(\eta) = \int_{y_1}^{y} \int_{x_1}^{y} \frac{1}{y} dx_1 dx_2 + \int_{y_1}^{y} \int_{-1}^{x_1} \frac{1}{y} dx_1 dx_2$$

$$= \int_{y_1}^{y} \int_{x_1}^{y} \frac{1}{y} dx_1 dx_2 + \int_{y_1}^{y} \int_{-1}^{x_1} \frac{1}{y} dx_1 dx_2$$

$$+ \int_{y_1}^{y} \int_{-1}^{x_1} \frac{1}{y} dx_1 dx_2 + \int_{y_1}^{y} \int_{y-1}^{x_1} \frac{1}{y} dx_1 dx_2$$
\[ f_1(\theta) = \begin{cases} 
0, & y < -1 \\
\frac{(y+1)^2}{4}, & -1 \leq y < 0 \\
\frac{2y+1}{4}, & 0 \leq y < 1 \\
\frac{y^2 - 2y}{4}, & 1 \leq y < 2 \\
1, & y \geq 2 
\end{cases} \]
(b) \( S^0 = (e, 0) \times (e, -1) \) and \( S_1^0 = (e, 1) \times (e, -1) \)

We can take \( S^2 = S^1 \cup S_1^0 \). Let \( h = (h_1, h_2) \),

\[ h_1(x, y, z) = 1211 + x_1, \quad h_2(x, y, z) = 12. \]

On \( S^1 \), \( h^{-1} \) is 1-1 with inverse function

\[ h_1^{-1}(y, z) = y - 2, \quad h_2^{-1}(y, z) = z. \]

Jacobian \( J_1 = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \)

\[ h(S_1) = \{(y, z) \mid 1 < y < 0, \ -1 < z < 1\}. \]

On \( S_2 \), \( h^{-1} \) is 1-1 with inverse function

\[ h_1^{-1}(y, z) = y - 2, \quad h_2^{-1}(y, z) = z. \]

Jacobian \( J_2 = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 2 \)

\[ h(S_2) = \{(y, z) \mid 1 < y - 3 < 1, \ -1 < z < 1\}. \]

Then the depth \( h_{d, 2} \) of \((1, 2)\) is

\[ h_{d, 2}(1, 2) = \begin{cases} \frac{1}{2}, & 0 < y - 3 < 1, -1 < z < 1 \\ 0, & \text{otherwise} \end{cases}, \]

\[ h_{d, 2}(1, 2) = \int_{-1}^{1} \int_{-1}^{1} h_{d, 2}(1, 2) \, dz \, dy = \begin{cases} \frac{1}{2}, & 0 < y - 3 < 1 \\ 0, & \text{otherwise} \end{cases}. \]

\[ h_{d, 2}(1, 2) = \int_{-1}^{1} \int_{-1}^{1} h_{d, 2}(1, 2) \, dz \, dy = \begin{cases} \frac{1}{2}, & 0 < y - 3 < 1 \\ 0, & \text{otherwise} \end{cases}. \]