Name:

Roll No.:
Problem No. 1:

(a) Three numbers are selected at random, without replacement, from the set \( \{1, 2, \ldots, 50\} \). Find the probability that they form an arithmetic progression.

(b) In a probability space \( (\Omega, \mathcal{F}, P) \), let \( A, B \) and \( C \) be pairwise independent events with \( P(A \cap B) = 0.3 \) and \( P(B \cap C) = 0.2 \). Show that \( P(A \cup C) \geq \frac{11}{36} \).
Problem No. 2: Let $X$ be a random variable having the distribution function

$$F(x) = \begin{cases} 
0, & \text{if } x < -\frac{1}{2} \\
\frac{x+1}{4}, & \text{if } -1 \leq x < 0 \\
\frac{x+1}{3}, & \text{if } 0 \leq x < 1 \\
\frac{x+2}{6}, & \text{if } 1 \leq x < 2 \\
1, & \text{if } x \geq 2 
\end{cases}$$

(a) Show that $X$ is neither a discrete nor a continuous random variable;

(b) Decompose $F$ as $F(x) = \alpha F_d(x) + (1 - \alpha) F_c(x), \ x \in \mathbb{R}$, where $\alpha \in [0,1]$, $F_d$ is a distribution function of some discrete random variable and $F_c$ is a distribution function of some continuous random variable.

[3+5=8 Marks]
Problem No. 3: Let $X$ be a random variable having the probability density function

$$f(x) = \begin{cases} \frac{c|x|}{2}, & \text{if } -2 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $c$ is a real constant. Find the value of $c$ and the variance of $X$. Also derive the distribution function of $X$ and hence verify that

$$E(X) = \int_0^{\infty} P(X > y)dy - \int_{-\infty}^{0} P(X < y)dy.$$
Problem No. 4: Suppose that the random variable \( X \) has the distribution function

\[
F(x) = \begin{cases} 
  a + be^x, & \text{if } x < 0 \\
  \frac{x^2}{2\pi}, & \text{if } 0 \leq x < 2\pi \\
  c + de^{-x}, & \text{if } x \geq 2\pi 
\end{cases}
\]

where \( a, b, c \) and \( d \) are real constants. Find the values of \( a, b, c \) and \( d \). Also derive the probability density/mass function \( X \) and the probability density/mass function of \( Y = \cos X \).

2+3+5=10 Marks
Problem No. 5: Suppose that the random variable $X$ has the moment generating function

$$M(t) = c \sum_{k=-2}^{2} \frac{e^{kt}}{k^2 + 1}, \quad -\infty < t < \infty,$$

where $c$ is a real constant. Find the value of $c$. Derive the probability density/mass function of $Y = X^2 + |X|$ and hence find the distribution function of $Y$.  

[2+3+3=8 Marks]
Problem No. 6:

(a) For any positive real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$, using Jensen’s inequality, show that

$$\sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} \geq (\sum_{i=1}^{n} a_i) \ln \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}.$$ 

(b) The marks scored by students of a college in a test are realizations of a random variable, having mean 120 and standard deviation 5 (or variance 25). According to the declared grading scheme, students securing between 112 and 128 will be awarded $B$ grade. Using Chebyshev’s inequality, find a lower bound on the proportion of students likely to receive $B$ grade.
Problem No. 1

(a) Total number of possible ways to select three numbers from the
set \( \{1, 2, \ldots, 50\} = \binom{50}{3} \)

For \( b \), the selected numbers must be: a card at a, where
1 \leq \text{card} < a + 2d < a + 3d \leq 50 \)
\( d \in \{1, 2, \ldots, 24\}, \quad \text{and} \quad a \in \{1, 2, \ldots, 50 - 24\} \)
Total # of favorable cases = \( \sum_{d=1}^{24} \binom{50-2d}{3} = 600 \) \hspace{1cm} 2 MARKS

Required prob = \( \frac{600}{\binom{50}{3}} = \frac{3}{78} \) \hspace{1cm} 2 MARKS

(b) \( P(\text{AB}) = 0.3 \quad P(\text{BNC}) = 0.2 \quad \text{A, B, C are pairwise independent} \)
\[ P(A) P(B) = 0.3 \quad P(B) P(C) = 0.2 \quad \Rightarrow \quad P(A) P(C) = \frac{3}{5} \]
\( \Rightarrow \quad P(C) = 2 \). Then \( 2 > P(C) = P(B) = 0.2 \Rightarrow P(A, B, C) = \frac{2}{5} \)
\( \Rightarrow \quad 1 - P(A, B, C) = 0.17 \)
\( \Rightarrow \quad \frac{1}{3} \leq x \leq \frac{2}{3} \) \hspace{1cm} 2 MARKS

\( P(A | C) = \frac{P(A \cap C)}{P(C)} \)
\[ = \frac{\frac{2}{2} - \frac{2}{2}}{\frac{3}{2}} = h(\lambda) \quad \lambda \in \left[ \frac{1}{3}, \frac{2}{3} \right] \quad \text{LHS}. \]
\[ h(\lambda) = \frac{\lambda}{2} - 3\lambda > 0 \quad \forall \lambda \in \left[ \frac{1}{3}, \frac{2}{3} \right] \]
\[ = h(\lambda) \in [h(\frac{1}{3}), h(\frac{2}{3})] = [\frac{11}{25}, 17] \]
\[ = P(A | C) \geq \frac{11}{25} \] \hspace{1cm} 2 MARKS

Y6
Problem No. 2

(a) Let $D$ be the set of discontinuity points of $f$. Then

\[ D = \{ x \mid 2^x \neq 0 \} \Rightarrow x \text{ is not continuous} \]

\[ \sum_{x \in D} |f(x^+) - f(x^-)| = \left( \frac{1}{3} - \frac{1}{4} \right) + (1 - \frac{1}{2}) = \frac{1}{4} = 1 \]

\[ \Rightarrow x \text{ is not differentiable} \]

(b) \[ \sum_{x \in D} |f(x^+) - f(x^-)| = \frac{1}{4} \Rightarrow \alpha = \frac{1}{4} \]

\[ \alpha \text{, } f_d(x) = \begin{cases} 0, & x < 0 \ \frac{1}{12}, & 0 \leq x < 2 \ \frac{1}{4}, & x \geq 2 \end{cases} \Rightarrow f_d(x) = \begin{cases} 0, & x < 1 \ \frac{x + 1}{2}, & 1 \leq x < 2 \ 0, & x \geq 2 \end{cases} \]

\[ \frac{1}{4} f_c(x) = f_c(x) - \alpha f_d(x) = \begin{cases} 0, & x < 1 \ \frac{x + 1}{2}, & 1 \leq x < 2 \ \frac{w(x^2)}{4}, & 0 \leq x < 1 \ \frac{2x + 5}{12}, & 1 \leq x < 2 \ \frac{2}{4}, & x \geq 2 \end{cases} \]

\[ f_c(x) = \begin{cases} 0, & x < 1 \ \frac{x + 1}{2}, & -1 \leq x < 0 \ \frac{w(x^2)}{4}, & 0 \leq x < 1 \ \frac{2x + 5}{12}, & 1 \leq x < 2 \ \frac{2}{4}, & x \geq 2 \end{cases} \]
(a) \[ \int_{-\infty}^{\infty} bx \, dx = 1 \Rightarrow c \int_{-\infty}^{\infty} x \, dx = 1 \Rightarrow c = \frac{2}{3} \] .. INARIC

\[ E(x) = \int_{-\infty}^{\infty} x^2 \, dx = \frac{2}{5} \left( -\frac{\frac{1}{2}}{\frac{1}{2}} \right) = -\frac{14}{15} \] INMARK

\[ E(x^2) = \int_{-\infty}^{\infty} x^4 \, dx = \frac{2}{5} \left( -\frac{\frac{1}{2}}{\frac{1}{2}} \right) = \frac{17}{10} \] 2MARKS

\[ \text{Var}(x) = E(x^2) - (E(x))^2 = \frac{333}{450} \] 3MARKS

\[ F(x) = \int_{-\infty}^{x} b \, dx = \begin{cases} 0, & x < -2 \\ \frac{2}{5} x^2 + c_1, & -2 \leq x < 2 \\ 1, & x \geq 2 \\ \end{cases} \] 2MARKS

\[ \int_{-\infty}^{\infty} P(x > 1) \, dx = \int_{-\infty}^{\infty} P(x < 1) \, dx = \int_{-\infty}^{0} \left[ 1 - \frac{1}{3} (4-x^2) \right] \, dx - \int_{0}^{\infty} \frac{1}{5} (4-x^2) \, dx \]

\[ = -\frac{14}{15} = E(x) \] 2MARKS
Problem No. 4

\[ f(0) = 0 \Rightarrow f(\infty) = a + b e^{\lambda} = 0, \quad \lambda < 0 \]

\[ \Rightarrow a = b = 0 \]

\[ f(2\pi) = 1 \Rightarrow f(x) = c + d e^{x} = 1, \quad x \geq 2\pi \Rightarrow c = 1 \text{ and } d = 0 \]

\[ a = b = d = 0 \text{ and } c = 1 \quad \text{2 marks} \]

\[ f(x) = \begin{cases} 
0, & x < 0 \\
\frac{x^2}{2\pi^2}, & 0 \leq x < 2\pi \\
1, & x \geq 2\pi 
\end{cases} \]

\[ f'(x) = \begin{cases} 
\frac{2x}{2\pi^2}, & 0 \leq x < 2\pi \\
0, & \text{otherwise (wherever } f' \text{ exists)} 
\end{cases} \]

\[ \int_{0}^{2\pi} f'(x) \, dx = \frac{1}{2\pi^2} \int_{0}^{2\pi} 2x \, dx = 1 \]

\[ x \text{ is continuous with a lab } f(x) = \begin{cases} 
\frac{2x^2}{2\pi^2}, & 0 \leq x < 2\pi \\
0, & \text{otherwise} 
\end{cases} \quad \text{3 marks} \]

We have \( S_x = \{(x, 0) \mid 0 \leq x \leq \pi \} \Rightarrow S_x = S_{S_x} \cup S_{S_x}^{(2\pi)} . \)

\[ h(x) \text{ is monotone in each } S_{S_x} \text{ or } S_{S_x}^{(2\pi)} . \]

\[ S_{S_x} = \{(x, 0) \mid 0 \leq x \leq \pi \} \]

\[ S_{S_x}^{(2\pi)} = \{ (2\pi, 0) \} \]

\[ h'(x) = \cot^{-1}(x) \]

\[ h''(x) = -\frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx} \left( h''(x) \right) = \frac{1}{\sqrt{1-x^2}} \]

\[ h(1) = (1) \quad \text{2 marks} \]

Thus the lab of \( y = \text{con} v \)

\[ y(1) = x (h'(1)) \frac{d}{dx} (h''(1)) + h'(1) (h''(1)) + \int \frac{h''(1)}{dx} (x - 1) \]

\[ = \frac{1}{\sqrt{1-1}} \left( \frac{1}{\sqrt{1-1}} \right) + \frac{2\pi - \cot^{-1}(1)}{2\pi} \left( \frac{1}{\sqrt{1-1}} \right) \]

\[ = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{1-1}}, & -1 < x < 1 \\
0, & \text{otherwise} 
\end{array} \right. \quad \text{3 marks} \]

\[ 9/6 \]
Problem 12.5

\[ \text{M_x(10)} = 1 \Rightarrow c \sum_{k=2}^{1} \frac{1}{k^2 - 2} = 1 \Rightarrow c = \frac{5}{12}. \]

The p.m.f. of \( X \) is

\[ f_X(x; 1) = P(X=x) = \begin{cases} \frac{5}{12(2-x)} & 2 \leq x \leq 5, \\ 0 & \text{otherwise}. \end{cases} \]

Clearly, \( S_X = \{0, 2, 4, 6\} \) and the p.m.f. of \( Y \) is

\[ b_Y(y; 1) = P(Y=y) = P(x^2 + x + 1 = y) = \begin{cases} P(x=0), & y = 0 \\ P(x=1), & y = 2 \\ P(x=2), & y = 6 \\ 0 & \text{otherwise}. \end{cases} \]

The c.d.f. of \( Y \) is

\[ F_Y(Y; 1) = P(Y \leq y) = \begin{cases} 0, & y < 0 \\ \frac{5}{12}, & 0 \leq y < 2 \\ \frac{5}{6}, & 2 \leq y < 6 \\ 1, & y \geq 6. \end{cases} \]

5/6.
Problem No. 6  

(a) Consider a RV $X$ having the p.m.f.

$$ b_{x_1x_2} = \begin{cases} \frac{a_c}{\sum_{j=1}^n a_j}, & \text{if } x = \frac{b_c}{a_c}, \ c = 1, \ldots, n \\ 0, & \text{otherwise} \end{cases} $$

Clearly, $b_{x_1x_2}$ is a proper p.m.f. Let

$$ h(x) = -\ln(x), \ x > 0, $$

so that $h$ is a convex function. On applying the Jensen inequality we get

$$ E(h(X)) \geq h(E(X)) \quad \cdots \quad 2 \text{ marks} $$

$$ \Rightarrow \sum_{i=1}^n (-\ln \frac{b_i}{a_i}) x_i \geq -\ln \left( \sum_{i=1}^n \frac{b_i}{a_i} \right) \sum_{i=1}^n \frac{a_i}{b_i} \geq \left( \sum_{i=1}^n a_i \right) \ln \left( \frac{\sum_{i=1}^n \frac{a_i}{b_i}}{\sum_{i=1}^n b_i} \right) \quad \cdots \quad 2 \text{ marks} $$

(b) $X$: Mark obtained by a typical student

$\mu = E(X) = 120, \ \sigma^2 = \text{Var}(X) = 25$

By Chebyshev's inequality,

$$ P\left( |X-\mu| < \frac{1}{4}\sigma \right) \geq 1 - \frac{1}{16}, \ \forall \ k > 0 \quad 2 \text{ marks} $$

Thus,

$$ P\left( 112 < X < 128 \right) = P\left( -8 < X-\mu < 8 \right) $$

$$ = P\left( -1.6 < \frac{X-\mu}{\sigma} < 1.6 \right) $$

$$ > 1 - \frac{1}{(1.6)^2} = \frac{39}{64} $$

Proportion of students likely to get $B$ grade = $\frac{39}{64} \times 100 = 60.13\% \quad 2 \text{ marks} $