MTH 412: Stochastic Processes
SE 359: Applied Stochastic Processes
Assignment No. 7: Poisson Process

1. Let $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ be two independent Poisson processes having rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Let $S_n^{(i)},\ i = 1, 2$, be the time of occurrence of the n-th event in the process $\{N_i(t) : t \geq 0\},\ i = 1, 2$. Show that

$$P(S_n^{(1)} < S_m^{(2)}) = \sum_{j=n}^{m} \binom{n+m-1}{j} (\frac{\lambda_1}{\lambda_1 + \lambda_2})^j (\frac{\lambda_2}{\lambda_1 + \lambda_2})^{n+m-1-j}.$$

2. Customers arrive at a service station in accordance with a PP having rate $\lambda > 0$. Upon arrival, customer is immediately served by one of infinite number of servers. Suppose that the service times are iid with continuous distribution and that the service and inter-arrival times are independent. Let $X(t) =$ number of customers that have completed service by time $t \geq 0$ and $Y(t) =$ number of customers that are being served at time t.
(a) Find the distributions of $X(t)$ and $Y(t)$.
(b) Find the joint distribution of $Y(t)$ and $Y(t+s),\ s, t \geq 0$.
(c) Find $\text{Cov}(Y(t), Y(t+s))$.

3. Let $\{N(t) : t \geq 0\}$ be a PP and let $S_i,\ i = 1, \ldots, n$, denote the occurrence of the i-th event. Show that, given $S_n = t$, the conditional distribution $S_1, \ldots, S_{n-1}$ is the same as that of order statistics based on a random sample of size $n - 1$ from $\text{U}(0, t)$ distribution.

4. Let $\{N(t) : t \geq 0\}$ be a nonhomogeneous PP having intensity function $\lambda(\cdot)$ and let S_n denote the time of occurrence of the n-th event. Show that the pdf of S_n is

$$f_{S_n}(t) = \frac{\lambda(t) e^{-m(t)} (m(t))^{n-1}}{(n-1)!},$$

where $m(t) = \int_0^t \lambda(u) du,\ t \geq 0$.

5. Let $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ be two independent Poisson processes having rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, and let $N(t) = N_1(t) + N_2(t),\ t \geq 0$. Show that $\{N(t) : t \geq 0\}$ is also a PP having rate $\lambda_1 + \lambda_2$. Find the probability that in the process $\{N(t) : t \geq 0\}$ the first event comes from the process $\{N_1(t) : t \geq 0\}$.

6. If $\{N(t) : t \geq 0\}$ is a PP and $s, t \geq 0$, then show that

$$\text{Corr}(N(s), N(t)) = \frac{\min(s, t)}{\sqrt{st}}.$$

7. Suppose that an event is occurring over a period of time and let $N(t)$ denote the number of times this event occurred in $[0, t],\ t \geq 0$. If the inter arrival times between the occurrences of event are iid exponential random variables having rate $\lambda > 0$, show that $\{N(t) : t \geq 0\}$ is a PP having rate λ.

8. Let $\{N(t) : t \geq 0\}$ be a PP. Given that only one event has occurred by epoch T, show that the time of occurrence of this event is $\text{U}(0, T)$.

9. Consider two independent sequences of events E and F occurring in accordance with PP's having rates λ_1 and λ_2, respectively. Show that the number M of occurrences of E between two consecutive occurrences of F has a geometric distribution. Relate it with the M_js in the coupon collecting problem.

10. Buses arrive at a certain stop according to a PP with rate λ. If you take the bus from stop then it takes a time R to arrive home. If you walk from the bus stop then it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.
(a) Compute the expected time from when you arrive at the bus stop until you reach home.
(b) For what value of s, the expected time in (a) is minimized.

11. Suppose that shocks occur according to a PP with rate λ and suppose that each shock, independently, causes the system to fail with probability p. Let N denote the number of shocks that it takes for the system to fail and let T denote the time of failure. Find $P(N = n|T = t)$.

12. Bus loads of customers arrive at an infinite server queue at a Poisson rate λ. Let $G(\cdot)$ denote the service time d.f.. A bus contains j customers with probability α_j, $j = 1, 2, \ldots$. Let $X(t)$ denote the number of customers that have been served by time t. Find $E(X(t))$. Is $X(t)$ Poisson distributed?

13. Let $T_1, T_2, \ldots$ denote the inter arrival times of a nonhomogeneous PP having intensity function $\lambda(t)$.
(a) Are T_is independent?
(b) Are T_is identically distributed?
(c) Find the distributions of T_1 and T_2.

14. (a) Let $\{N(t) : t \geq 0\}$ be a nonhomogeneous PP with mean value function $m(t)$. Given $N(t) = n$, show that unordered set of arrival times has the same distribution as that of n iid r.v.s having common d.f. $F(x) = m(x)/m(t)$, if $x \leq t$, $= 1$, if $x > t$.
(b) Suppose that workers incur accidents in accordance with nonhomogeneous PP with mean value function $m(t)$. Suppose further that each injured person is out of work for a random amount of time having d.f. $F(\cdot)$. Let $X(t)$ be the number of workers who are out of work at time t. Compute $E(X(t))$ and $\text{Var}(X(t))$.

15. Let $\{N(t) : t \geq 0\}$ be a nonhomogeneous PP with intensity function $\lambda(t)$. However, suppose that one starts observing the process at a random time τ having d.f. $F(\cdot)$. Let $N^*(t) = N(\tau+t) - N(\tau)$ denote the number of events that occur in the first t time units of observation.
(a) Does the process $\{N^*(t) : t \geq 0\}$ possess independent increments?
(b) Repeat (a) when $\{N(t) : t \geq 0\}$ is a homogeneous PP.

16. Let $\{X(t) : t \geq 0\}$ be a compound PP with $X(t) = \sum_{i=1}^{N(t)} X_i$ and suppose that X_i can assume only finite set of possible values. Argue that, for large t, the distribution of $X(t)$ is approximately normal.

17. $\{X(t) : t \geq 0\}$ be a compound PP with $X(t) = \sum_{i=1}^{N(t)} X_i$ and suppose that $\lambda = 1$ and $P(X_i = j) = j/10$, $j = 1, 2, 3, 4$. Calculate $P(X(4) = 20)$.

18. For a compound PP, compute $\text{Cov}(X(s), X(t))$.