MTH 418a: Inference-I
Assignment No. 1: Sampling Distributions

1. For a positive integer ν, let $X \sim t_\nu$. We know that $X \stackrel{d}{=} \frac{Z}{\sqrt{\frac{Y}{\nu}}}$, where $Z \sim N(0,1)$ and $Y \sim \chi^2_\nu$ are independently distributed. Using the aforementioned representation, find the mean and variance of t_ν. Also show that $t^2_\nu \sim f_{1,\nu}$.

2. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where $\mu \in (-\infty, \infty)$ and $0 < \sigma < \infty$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Find means and variances of r.v.s $S_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ and $T = \frac{\overline{X}_n}{S_n}$.

3. For positive integers ν_1 and ν_2, let $X \sim f_{\nu_1,\nu_2}$. We know that $X \stackrel{d}{=} \frac{Y_1/\nu_1}{Y_2/\nu_2}$, where $Y_1 \sim \chi^2_{\nu_1}$ and $Y_2 \sim \chi^2_{\nu_2}$ are independently distributed. Using this representation of f_{ν_1,ν_2}, find its mean and variance. Also show that $T = \frac{1}{f_{\nu_1,\nu_2}} \sim f_{\nu_2,\nu_1}$.

4. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, where $\mu_i \in (-\infty, \infty)$ and $0 < \sigma_i < \infty, i = 1, 2$. Let $\overline{X} = \frac{1}{m}\sum_{i=1}^m X_i$, $\overline{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$, $S_X^2 = \frac{1}{m}\sum_{i=1}^m (X_i - \overline{X})^2$ and $S_Y^2 = \frac{1}{n}\sum_{i=1}^n (Y_i - \overline{Y})^2$. Find means and variances of following r.v.s:

   (a) $T_1 = \frac{\overline{X} - \overline{Y}}{\sqrt{S_X^2 + S_Y^2}}$, when $m = n$ and $\sigma_1^2 = \sigma_2^2$;

   (b) $T_2 = \frac{S_X}{S_Y}$.

   Also, for $m = n$, find $\lim_{n\to\infty} P(U_n \le 1)$, where $U_n = \frac{\sqrt{n}(\overline{X} - \overline{Y} - (\mu_1 - \mu_2))}{\sqrt{\sigma_1^2 + \sigma_2^2}}$.

5. Let $Z_1, \ldots, Z_{12}$ be a random sample from $N(0,1)$. Find the p.d.f.s of the following random variables:

   (a) $Y_1 = \frac{Z_1}{\sqrt{\sum_{i=2}^{11} Z_i^2}}$;

   (b) $Y_2 = \frac{Z_1 + Z_2}{|Z_1 - Z_2|}$;

   (c) $Y_3 = \frac{Z_1 + Z_2}{Z_1 - Z_2}$;

   (d) $Y_4 = \frac{Z_1^2 + Z_2^2}{\sum_{i=3}^{11} Z_i^2}$;

   (e) $Y_5 = \frac{Z_1^2 + Z_2^2 + Z_3^2}{\sum_{i=1}^{11} Z_i^2}$;

   (f) $Y_6 = \sum_{i=1}^9 (Z_i - \overline{Z}_9)^2 + Z_{10}^2$, where $\overline{Z}_9 = \frac{1}{9}\sum_{i=1}^9 Z_i$;

   (g) $Y_7 = \frac{3(Z_{11} + Z_{12})}{\sqrt{Y_6}}$;

   (h) $Y_8 = \frac{Z_{11}^2 + Z_{12}^2}{2Y_6}$.

6. Let $X_1, X_2, \ldots$ be a sequence of r.v.s such that the r.v. X_n follows the Poisson distribution with mean n. Find the values of $\lim_{n\to\infty} P(\frac{9n}{10} \leq X_n \leq \frac{11n}{10})$, $\lim_{n\to\infty} P(X_n \leq n + 2\sqrt{n})$ and $\lim_{n\to\infty} e^{-n} \sum_{j=0}^{[n-\sqrt{n}]} \frac{n^j}{j!}$, where, for any real number x, $[x]$ denotes the largest integer not exceeding x. .

Honors Problems

7. Let $X_1, \ldots, X_n$ be a random sample from a population having d.f. F and the Lebegue p.d.f. f. Show that the d.f. of $X_{(r)}$ $(r = 1, \ldots, n)$ is

$$G_r(y) = \sum_{i=r}^{n} \binom{n}{i} (F(y))^i (1 - F(y))^{n-i} = \frac{n!}{(r-1)!(n-r)!} \int_0^{F(y)} u^{r-1}(1-u)^{n-r} du.$$

Hence derive the expression for the p.d.f. of $X_{(r)}$ $(r = 1, \ldots, n)$. Using the above identities, derive the relationship between the d.f.s of binomial and beta distributions.

8. Let $X_1, \ldots, X_n$ be a random sample from $Exp(\theta)$ $(0 < \theta < \infty)$, and let $X_{(1)}, \ldots, X_{(n)}$ be the corresponding order statistics. Define $Y_i = (n - i + 1)(X_{(i)} - X_{(i-1)}), i = 1, \ldots, n$, where $X_{(0)} = 0$ (Y_is are called normalized spacings). Show that $Y_1, \ldots, Y_n$ are i.i.d. $Exp(\theta)$. Hence, show that $E(X_{(r)}) = \theta \sum_{i=n-r+1}^{n} \frac{1}{i}$, $Var(X_{(r)}) = \theta^2 \sum_{i=n-r+1}^{n} \frac{1}{i^2}$, $r = 1, \ldots, n$, and $Cov(X_{(r)}, X_{(s)}) = \theta^2 \sum_{i=n-r+1}^{n} \frac{1}{i^2}, 1 \leq r < s \leq n$. Further, show that $T = nX_{(1)} \sim Exp(\theta)$.

9. Let $X_i s$ be as defined in Problem 8. Let $U_{(1)}, \ldots, U_{(n-1)}$ be the order statistics based on a random sample of size $n - 1$ from $U(0, 1)$. Show that

$$(U_{(1)}, \ldots, U_{(n-1)}) \stackrel{d}{=} (\frac{X_1}{\sum_{i=1}^{n} X_i}, \frac{X_1 + X_2}{\sum_{i=1}^{n} X_i}, \ldots, \frac{\sum_{i=1}^{n-1} X_i}{\sum_{i=1}^{n} X_i}).$$