MTH 418a: Inference-I
Assignment No. 4: Basu's Theorem

1. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2),\ \mu \in \mathbb{R}, \sigma > 0$.

   (i) Show that $\sum_{i=1}^n (X_i - \mu)^2$ and $U = \frac{\overline{X} - \mu}{\sqrt{\sum_{i=1}^n (X_i - \mu)^2}}$ are independent. Are $\sum_{i=1}^n (X_i - \mu)^2$ and $V = \frac{\overline{X} - \mu}{\sqrt{\sum_{i=1}^n (X_i - \overline{X})^2}}$ also independent? Find the mean and variance of U^2.

   (ii) Show that $\overline{X}$ and $\underline{T} = (X_1 - \overline{X}, \ldots, X_n - \overline{X})$ are independent. In particular, show that $\overline{X}$ and $W = \sum_{i=1}^n (X_i - \overline{X})^2$ are independent.

   (iii) Find the conditional distribution function, conditional density and conditional expectation of of X_1, given $\overline{X} = t, t \in \mathbb{R}$.

   (iv) Using the m.g.f. and findings in (ii), find the distribution of $W = \sum_{i=1}^n (X_i - \overline{X})^2$.

2. Let $X_1, \ldots, X_n$ be a random sample from $Exp(\mu, \sigma),\ \mu \in \mathbb{R}, \sigma > 0$.

   (i) Show that $\sum_{i=1}^n (X_i - \mu)$ and $U = \frac{X_{(1)} - \mu}{\sum_{i=1}^n (X_i - \mu)}$ are independent. Are $\sum_{i=1}^n (X_i - \mu)$ and $V = \frac{X_{(1)} - \mu}{\sum_{i=1}^n (X_i - X_{(1)})}$ also independent? Find the mean and variance of U.

   (ii) Show that $X_{(1)}$ and $\underline{T} = (X_1 - X_{(1)}, \ldots, X_n - X_{(n)})$ are independent. In particular, show that $X_{(1)}$ and $\sum_{i=1}^n (X_i - X_{(1)})$ are independent.

   (iii) Find the conditional distribution function, conditional density and conditional expectation of X_1, given $X_{(1)} = t, t > \mu$.

   (iv) Using the m.g.f. and findings in (ii), find the distribution of $W = \sum_{i=1}^n (X_i - X_{(1)})$.

   (v) Let $Z_i = \frac{X_{(n)} - X_{(i)}}{X_{(n)} - X_{(n-1)}}, i = 1, \ldots, n-2$. Show that $\underline{T} = (X_{(1)}, \sum_{i=1}^n (X_i - X_{(1)}))$ and $\underline{Z} = (Z_1, \ldots, Z_{n-2})$ are independent.

3. Let $X_1, \ldots, X_n$ be a random sample from $U(\theta_1, \theta_2),\ -\infty < \theta_1 < \theta_2 < \infty$. Let $Z_i = \frac{X_{(i)} - X_{(1)}}{X_{(n)} - X_{(1)}}, i = 2, \ldots, n-1$. Show that $\underline{T} = (X_{(1)}, X_{(n)})$ and $\underline{Z} = (Z_2, \ldots, Z_{n-1})$ are independent.

4. Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta),\ \theta > 0$.

   (i) Show that $X_{(n)}$ and $\underline{T} = (\frac{X_1}{X_{(n)}}, \ldots, \frac{X_n}{X_{(n)}})$ are independent.

   (ii) Find $E(\frac{X_1}{X_{(n)}})$. Also find the conditional distribution function, conditional density and conditional expectation of X_1, given $X_{(n)} = t,\ 0 < t < \theta$.

   (iii) Find $E(\frac{X_{(r)}}{X_{(n)}}), r = 1, \ldots, n-1$.

5. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, $-\infty < \mu_i < \infty, \sigma_i > 0, i = 1, 2$. Show that $\underline{T} = (\overline{X}, \overline{Y}, \sum_{i=1}^n (X_i - \overline{X})^2, \sum_{i=1}^n (Y_i - \overline{Y})^2)$ and

$$V(\underline{X}) = \frac{\sum_{i=1}^n (X_i - \overline{X})(Y_1 - \overline{Y})}{\sqrt{\sum_{i=1}^n (X_i - \overline{X})^2 \sum_{i=1}^n (Y_i - \overline{Y})^2}}$$

are statistically independent.