MTH 418a: Inference-I
Assignment No. 6: Unbiased Estimation, UMVUE, Lehmann-Scheffe Theorem

1. Let $X_1, \ldots, X_n$ be a random sample from p.m.f./p.d.f. $f_\theta, \theta \in \Theta$. In each of the following cases, find the UMVUE of $\psi(\theta)$ if it exists.

   (i) $X_1 \sim \mathrm{N}(\theta, \sigma_0^2), \Theta = \mathbb{R}, \psi(\theta) = \theta^r, r = 1, 2, 3, 4, \sigma_0$ is a known positive constant;

   (ii) $X_1 \sim \mathrm{N}(\mu_0, \theta^2), \Theta = (0, \infty), \psi(\theta) = \theta^r, r > -n, \mu_0$ is a known real constant;

   (iii) $X_1 \sim \mathrm{Exp}(\mu_0, \theta), \Theta = (0, \infty), \psi(\theta) = \theta^r, r > -n, \mu_0$ is a known real constant;

   (iv) $X_1 \sim \mathrm{Exp}(\mu_0, \theta), \Theta = (0, \infty), \psi(\theta) = P_\theta(X_1 \leq t_0)$, where μ_0 and t_0 are fixed real constants;

   (v) $X_1 \sim \mathrm{Exp}(\theta, \sigma_0), \Theta = \mathbb{R}, \psi_r(\theta) = \theta^r, r = 1, 2, 3, 4$, where σ_0 is a fixed positive constant;

   (vi) $X_1 \sim \mathrm{U}(\mu_0 - \theta, \mu_0 + \theta), \Theta = (0, \infty), \psi(\theta) = \theta^r, r > -n, \mu_0$ is a known real constant;

   (vii) $f_\theta(x) = \alpha_0 \frac{x^{\alpha_0 - 1}}{\theta^{\alpha_0}} I(0 < x < \theta), \Theta = (0, \infty), \psi(\theta) = \theta^r, r > -n\alpha_0, \alpha_0$ is a known positive constant;

   (viii) $f_\theta(x) = \alpha_0 \frac{\theta^{\alpha_0}}{x^{\alpha_0 + 1}} I(x > \theta), \Theta = (0, \infty), \psi(\theta) = \theta^r, r < n\alpha_0, \alpha_0$ is a known positive constant;

   (ix) $X_1 \sim \mathrm{Bin}(m_0, \theta), \Theta = (0, 1), \psi(\theta) = \theta^r$, r is a non-zero integer, m_0 is a known positive integer;

   (x) $X_1 \sim \mathrm{Poisson}(\theta), \Theta = (0, \infty), \psi(\theta) = (P_\theta(X_1 = 2))^2$. Also find the UMVUE of $\psi(\theta) = \cos(\theta)$;

   (xi) $X_1 \sim \mathrm{Gamma}(\theta, \alpha_0), \Theta = (0, \infty), \psi(\theta) = \theta^r, r > -n\alpha_0, \alpha_0$ is a known positive constant;

   (xii) $X_1 \sim \mathrm{Gamma}(\sigma_0, \theta), \Theta = (0, \infty), \psi(\theta) = \theta^n, \sigma_0$ is a known positive constant;

   (xiii) $f_\theta(x) = \frac{\alpha(x)\theta^x}{c(\theta)}, x = 0, 1, 2, \ldots, \Theta = (0, \infty), \psi(\theta) = \theta^r$, r is a positive integer, $\alpha(x) \geq 0, x = 0, 1, 2, \ldots,$ $c(\theta)$ is a normalizing factor for the p.m.f.;

   (xiv) X_1 follows discrete uniform distribution on the set $\{1, 2, \ldots, \theta\}$, $\Theta = \{1, 2, \ldots\}$, $\psi(\theta)$ is an arbitrary real-valued function defined on Θ;

   (xv) X_1 follows discrete uniform distribution on the set $\{1, 2, \ldots, \theta\}$, $\Theta = \{1, 2, \ldots\} - \{\theta_0\}$, θ_0 is a fixed positive integer and $\psi(\theta)$ is an arbitrary real-valued function defined on Θ.

2. Let $X_1, \ldots, X_n$ be a random sample from a population with mean $\theta \in \mathbb{R}$ and finite variance. Let δ_U be the UMVUE of $\psi(\theta)$ and let δ_0 be any other unbiased estimator of $\psi(\theta)$. Show that $\mathrm{Cov}_\theta(\delta_U, \delta_0) = \mathrm{Var}_\theta(\delta_U)$.

3. (i) In an estimation problem, suppose that UMVUE of an estimand exists. Show that the UMVUE must be unique;

   (ii) Let δ_i be the UMVUE of $\psi_i(\theta), i = 1, \ldots, k, \theta \in \Theta$. Show that $\delta_P = \sum_{i=1}^{k} \delta_i$ is the UMVUE of $\psi(\theta) = \sum_{i=1}^{k} \psi_i(\theta)$.

4. Let $X_1, \ldots, X_n$ be a random sample from p.d.f.

$$f_\theta(x) = \begin{cases} \frac{\alpha x^{\alpha-1}}{\theta^\alpha}, & \text{if } 0 < x < \theta \\ 0, & \text{otherwsie} \end{cases}, \quad \theta \in \Theta = (0, \infty),$$

   where α is a known positive constant. Let $\psi : (0, \infty) \to \mathbb{R}$ be a given function.

   (i) Suppose that $\lim_{\theta \to 0}(\theta^{n\alpha}\psi(\theta)) = c$, for some non-zero constant c. Show that the UMVUE of $\psi(\theta)$ does not exist;

   (ii) Suppose that $\psi(\theta)$ is an u-estimable differentiable function. Find the UMVUE of $\psi(\theta)$. In particular, find the UMVUEs of $\psi_1(\theta) = e^{-\theta}$ and $\psi_2(\theta) = \theta \ln \theta$.

5. Let $X_1, \ldots, X_n$ $(n \geq 3)$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$, where $\underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty) = \Theta$, say.

   (i) Find the UMVUE of $\psi_r(\underline{\theta}) = \mu^r, r = 1, 2, 3, 4$;

   (ii) For estimating σ^r $(r < -n + 1)$ under the squared error loss function, find the minimum mean squared error (risk function under the SEL function) estimator of the form $\delta_\alpha(\underline{X}) = \alpha(\sum_{i=1}^{n}(X_i - \overline{X})^2)^{\frac{r}{2}}, \alpha > 0$. Find the risk functions of the minimum mean squared estimator and the UMVUE.

   (iii) Find the UMVUEs of $\psi_5(\underline{\theta}) = \frac{\mu}{\sigma}$;

   (iv) Find the UMVUE of $\xi_p(\underline{\theta})$, the p-the quantile of X_1;

   (v) Find the UMVUE of $\psi_6(\underline{\theta}) = \Phi(\frac{t_0 - \mu}{\sigma})$, where t_0 is a fixed real constant;

   (vi) Find the UMVUE of $\psi_7(\underline{\theta}) = \frac{1}{\sigma}\phi(\frac{t_0 - \mu}{\sigma})$, where t_0 is a fixed real constant.

6. Let $X_1, \ldots, X_n$ $(n \geq 4)$ be a random sample from $\mathrm{Exp}(\mu, \sigma)$, where $\underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty) = \Theta$, say.

   (i) Find the UMVUE of $\psi_r(\theta) = \mu^r, r = 1, 2$;

   (ii) For estimating σ^r $(r < -n + 1)$ under the squared error loss function, find the minimum mean squared error (risk function under the SEL function) estimator of the form $\delta_\alpha(\underline{X}) = \alpha(\sum_{i=1}^{n}(X_i - X_{(1)}))^r, \alpha > 0$. Find the risk functions of the minimum mean squared estimator and the UMVUE.

   (iii) Find the UMVUE of $\xi_p(\underline{\theta})$, the p-the quantile of X_1;

   (iv) Find the UMVUE of $\psi_6(\underline{\theta}) = P_{\underline{\theta}}(X_1 \leq t_0)$, where t_0 is a fixed real constant;

7. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from $\mathrm{N}(\mu_1, \sigma_1^2)$ and $\mathrm{N}(\mu_2, \sigma_2^2)$, respectively, where $\underline{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2) \in \mathbb{R}^2 \times (0, \infty)^2 = \Theta$, say.

   (i) Find the UMVUE of $\psi_r(\underline{\theta}) = (\mu_1 - \mu_2)^r, r = 1, 2, 3$;

(ii) Find the UMVUE of $\psi_{4,r}(\underline{\theta}) = \frac{\sigma_1^r}{\sigma_2^s}$, $(r > -n+1$ and $s < n-1)$;

(iii) Suppose that it is known apriori that $\sigma_1^2 = k\sigma_2^2$, where k is a known positive constant. Find the UMVUEs of $\psi_5(\underline{\theta}) = (\mu_1 - \mu_2)^3$ and $\psi_6(\underline{\theta}) = \sigma_1^3$.

8. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from $\text{Exp}(\mu_1, \sigma_1)$ and $\text{Exp}(\mu_2, \sigma_2)$, respectively, where $\underline{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2) \in \mathbb{R}^2 \times (0, \infty)^2 = \Theta$, say.

   (i) Find the UMVUE of $\psi_r(\underline{\theta}) = (\mu_1 - \mu_2)^r, r = 1, 2$;

   (ii) Find the UMVUE of $\psi_{3,r}(\underline{\theta}) = \frac{\sigma_1^r}{\sigma_2^s}$, $(r > -m+1$ and $s < n-1)$;

   (iii) Suppose that it is known apriori that $\sigma_1 = k\sigma_2$, where k is a known positive constant. Find the UMVUEs of $\psi_4(\underline{\theta}) = (\mu_1 - \mu_2)^2$ and $\psi_5(\underline{\theta}) = \sigma^2$.

9. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from $\text{Poisson}(\theta_1)$ and $\text{Poisson}(\theta_2)$, respectively, where $\underline{\theta} = (\theta_1, \theta_2) \in (0, \infty)^2$.

   (i) Using alone the sample $X_1, \ldots, X_m$, find the UMVUE of $\psi_1(\theta_1) = P_{\theta_1}(X_1 \in O)$, where O is the set of all odd integers;

   (ii) Find the UMVUE of $\psi_2(\theta) = (\theta_1 - \theta_2)^2$.

10. Let $\mathcal{P}$ be the collection of all Lebesgue p.d.f.s (p.d.f.s of all continuous distributions). Let $X_1, \ldots, X_n$ be a random sample from an unknown distribution in $\mathcal{P}$.

    (i) Show that $\underline{T} = (X_{(1)}, \ldots, X_{(n)})$ is a complete-sufficient statistic;

    (ii) Find the UMVUEs of $\theta = E_f(X_1), \sigma^2 = \text{Var}(X_1), \tau_1 = \theta^2$ and $\tau_2 = P_f(X_1 \leq t_0)$, where t_0 is a fixed real constant.

11. Let $\mathcal{P}$ be the collection of all symmetric Lebesgue p.d.f.s having finite mean. Let $X_1, \ldots, X_n$ be a random sample from an unknown distribution in $\mathcal{P}$.

    (i) Is $\underline{T} = (X_{(1)}, \ldots, X_{(n)})$ a complete statistic;

    (ii) Show that, for $n \geq 2$, no UMVUE of $\theta = E_f(X_1)$ exists;

    (iii) Show that, for $n = 1$, X_1 is complete for $\mathcal{P}$. In this case, show that $\delta_0(X_1) = X_1$ is the UMVUE of $\tau = E_f(X_1)$.