MTH 418a: Inference-I
Assignment No. 6a: Rao-Cramer Lower Bounds

1. Let $X_1, \ldots, X_n$ be a random sample from a p.d.f./p.m.f. f_θ, where $\theta \in \Theta$ is an unknown parameter. In each of the following cases find the Rao-Cramer lower bound on unbiased estimators of $g(\theta)$ and compare it with the variance of the UMVUE of $g(\theta)$:
(i) $X_1 \sim N(\mu, \theta^2)$, where μ is known, $\Theta = (0, \infty)$, $g(\theta) = \theta^2$;
(ii) $X_1 \sim N(\mu, \theta^2)$, where μ is known, $\Theta = (0, \infty)$, $g(\theta) = \theta$;
(iii) $X_1 \sim \text{Bin}(1, \theta)$, $\Theta = (0, 1)$, $g(\theta) = \theta$;
(iv) $X_1 \sim \text{Bin}(1, \theta)$, $\Theta = (0, 1)$, $g(\theta) = \theta(1 - \theta)$;
(v) $X_1 \sim \text{Poisson}(\theta)$, $\Theta = (0, \infty)$, $g(\theta) = \theta$;
(vi) $X_1 \sim \text{Poisson}(\theta)$, $\Theta = (0, \infty)$, $g(\theta) = e^{-\theta}(1 + \theta)$;
(vii) $X_1 \sim \text{Gamma}(\alpha, \theta)$, where $\alpha > 0$ is known, $\Theta = (0, \infty)$, $g(\theta) = \theta$;
(vii) $X_1 \sim \text{Gamma}(\alpha, \theta)$, where $\alpha > 0$ is known, $\Theta = (0, \infty)$, $g(\theta) = \theta^2$.

2. Suppose that $\underline{X}$ has a p.m.f./p.d.f. f_θ, where $\theta \in \Theta$ is an unknown real parameter. Let g be a real-valued function defined on Θ. For any $\theta \in \Theta$, define $A_\theta = \{x : f_\theta(x) > 0\}$ and $B_\theta = \{\Delta : \theta + \Delta \in \Theta, g(\theta) \neq g(\theta + \Delta),$ and $A_{\theta+\Delta} \subseteq A_\theta\}$, $\theta \in \Theta$. Let δ be an unbiased estimator of $g(\theta)$. Show that

$$V_\theta(\delta) \geq \sup_{\Delta \in B_\theta} \left[\frac{(g(\theta + \Delta) - g(\theta))^2}{E_\theta \left[\left(\frac{f_{\theta+\Delta}(\underline{X})}{f_\theta(\underline{X})} - 1 \right)^2 \right]} \right], \quad \theta \in \Theta.$$

3. Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta)$, where $\theta \in (0, \infty) = \Theta$. Using Problem 2, find a lower bound on unbiased estimators of θ. Is it attained by the UMVUE of θ?

4. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where $\underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty) = \Theta$. Find a lower bound on unbiased estimators of $\psi(\underline{\theta}) = \mu + \sigma$. Is this lower bound attained by the UMVUE of $\psi(\underline{\theta})$?

5. Let $X_1, \ldots, X_n$ be a random sample from $\text{Bin}(1, \theta)$, where $\theta = (0, \infty) = \Theta$. Let $\underline{\delta} = (\delta_1, \delta_2, \delta_3)$ be an unbiased estimator of $\underline{\psi}(\underline{\theta}) = (\theta, 1 - \theta, \theta(1 - \theta))$. Find a matrix B such that $V_\theta(\delta) - B$ is non-negative definite. Hence find lower bounds on unbiased estimators of $\psi_i(\underline{\theta}), i = 1, 2, 3$. Compare these lower bounds with those obtained in Problem 1 (iii) & (iv).

6. Let $X_1, \ldots, X_n$ be a random sample from $\text{Exp}(\theta)$, where $E_\theta(X_1) = \theta = (0, \infty) = \Theta$. Let $\underline{\delta} = (\delta_1, \delta_2, \delta_3)$ be an unbiased estimator of $\underline{\psi}(\underline{\theta}) = (\theta, \theta^2, \theta^3)$. Find a matrix B such that $V_\theta(\delta) - B$ is non-negative definite. Hence find lower bounds on unbiased estimators of $\psi_i(\underline{\theta}), i = 1, 2, 3$. Compare these lower bounds with the variances of the respective UMVUEs.