MTH-418a: Inference-I

2023-2024: II Semester
Mid Semester Examination

Time Allowed: 120 Minutes **Maximum Marks: 50**

NOTE: (i) Start answer of every question on a new page. Moreover, attempt all the parts of a question at one place.
(ii) Answer each question legibly, clearly and concisely. Illegible answers will not be graded.
(ii) For each question, provide details of all the steps involved in arriving at the final answer(s)/conclusion(s) and box your final answer(s)/conclusion(s).

1. Identify TRUE and FALSE statements from the following ten statements. In support of your answers, provide appropriate arguments.

(i) If $|T|$ is a sufficient statistic for $\theta \in \Theta$, then T is also a sufficient statistic for $\theta \in \Theta$.

(ii) If T is a complete statistic, then $e^T - T - 1$ is also a complete statistic.

(iii) If T is a complete statistic and T^2 is a sufficient statistic, then T is a minimal sufficient statistic.

(iv) If T is a minimal sufficient statistic, then $T^3 - 2T^2 + 2T + 1$ is also a minimal sufficient statistic.

(v) Let $x_1 = 1.1, x_2 = 1.5$ and $x_3 = 0.4$ be the observed values of a random sample of size 3 from the pdf

$$f_\theta(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0,$$

where $\theta \in (0, \infty) = \Theta$. Then, the maximum likelihood estimate of θ is 0.5.

(vi) Let $x_1 = -1.5, x_2 = -0.2$ and $x_3 = 0.2$ be the observed values of a random sample of size 3 from the pdf

$$f_\theta(x) = \frac{e^{-|x-\theta|}}{2}, \quad -\infty < x < \infty,$$

where $\theta \in \mathbb{R} = \Theta$. Then, the method of moment estimate of θ is 0.2;

(vii) If δ_1 is the UMVUE of $\psi(\theta), \theta \in \Theta$, and δ_2 is an unbiased estimator of $\psi(\theta)$, then $\text{Var}_\theta(\delta_1) = 2\text{Cov}_\theta(\delta_1, \delta_2), \ \forall\ \theta \in \Theta$;

(viii) Let X_1, X_2 be a random sample from $\text{U}(0, \theta)$, where $\theta \in (0, \infty) = \Theta$. Let $X_{(1)} \leq X_{(2)}$ be the corresponding order statistics. Using Basu's theorem and the distribution of order statistics, $E_\theta(\frac{X_{(1)}}{X_{(2)}}) = 0.5, \ \forall\ \theta > 0$;

(ix) A minimal sufficient statistic is always complete;

(x) If δ_1 and δ_2 are two UMVUEs of $\psi(\theta)$, then $\text{Var}_\theta(\delta_1 - \delta_2) = 0, \ \forall\ \theta \in \Theta$, and $P_\theta(\delta_1 = \delta_2) = 1, \ \forall\ \theta \in \Theta$.

1.5 × 10=15 Marks

2. Let X_1, X_2 be a random sample from a population having the pdf

$$f_\theta(x) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - \theta)^2}, \quad x \in \mathbb{R},$$

where $\theta \in \mathbb{R} = \Theta$. Show that $\underline{T} = (X_{(1)}, X_{(2)})$ is a minimal sufficient statistic? Is $\underline{T}$ a boundedly complete statistic?

6+5=11 Marks

3. Let X_1, X_2, X_3 be a random sample of size 3 from the pdf

$$f_\theta(x) = \frac{2x}{\theta^2}, \quad 0 < x < \theta,$$

where $\theta \in (0, \infty) = \Theta$.

(i) Find the MLE of θ;

(ii) Find the MME of θ;

(iii) Using the Rao-Blackwell Theorem, find an estimator based on the complete sufficient statistic $X_{(3)} = \max\{X_1, X_2, X_3\}$ that is better than the MME under the absolute error loss function $L(\theta, a) = |a - \theta|, \ a \in \mathbb{R} = \mathcal{A}, \theta \in \Theta$.

4+4+4=12 Marks

4. Let X be a random variable having the pmf

$$f_\theta(x) = \begin{cases} \theta, & \text{if } x = -1 \\ (1-\theta)^2\theta^x, & \text{if } x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0, 1) = \Theta$. Show that the UMVUE of $\psi(\theta) = \theta(1 - \theta)^2$ does not exist.

12 Marks

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Model Solutions

Problem No. 1

(i) TRUE $\boxed{0.5 \text{ Marks}}$

Since the sufficient statistic $|T|$ is a function of statistic T, T is a sufficient statistic. $\boxed{1 \text{ MARK}}$

(ii) TRUE $\boxed{0.5 \text{ MARKS}}$

Any function of a complete statistic is a complete statistic $\boxed{1 \text{ MARK}}$

(iii) TRUE $\boxed{0.5 \text{ MARKS}}$

T^2 is sufficient $\Rightarrow$ T is sufficient (T^2 is a function of T)

$\Rightarrow$ T is complete-sufficient

$\Rightarrow$ T is minimal sufficient. $\boxed{1 \text{ MARK}}$

(iv) TRUE $\boxed{0.5 \text{ MARKS}}$

Consider

$$\psi(t) = t^3 - 2t^2 + 2t + 1 \qquad t \in \mathbb{R}$$

$$\psi'(t) = 3t^3 - 4t + 2$$

$$= 3\left(t^3 - \frac{4t}{3} + \frac{2}{3}\right)$$

$$= 3\left[\left(t - \frac{2}{3}\right)^2 + \frac{2}{9}\right] > 0, \quad \forall t \in \mathbb{R}$$

$\Rightarrow$ $\psi(T)$ is a 1-1 function function of minimal sufficient statistic

$\Rightarrow$ $T^3 - 2T^2 + 2T + 1$ is minimal sufficient $\boxed{1 \text{ MARK}}$

$\boxed{0.5 \text{ MARKS}}$

(V) FALSE

The log likelihood function is

$$l_{\underline{x}}(\theta) = -3\ln\theta - \frac{\sum x_i}{\theta}$$

$$\frac{\partial}{\partial\theta} l_{\underline{x}}(\theta) = -\frac{3}{\theta} + \frac{3\bar{x}}{\theta^2} = 0 \quad \Rightarrow \quad \theta = \bar{x}$$

$$\left[\frac{\partial^2}{\partial\theta^2} l_{\underline{x}}(\theta)\right]_{\theta=\bar{x}} = \left[\frac{3}{\theta^2} - \frac{6\bar{x}}{\theta^3}\right]_{\theta=\bar{x}} = -\frac{3}{\bar{x}^2} < 0$$

$\boxed{1 \text{ MARK}}$

$$\Rightarrow \quad \hat{\theta}_{MLE} = \bar{x} = 1$$

$\boxed{0.5 \text{ MARKS}}$

(VI) FALSE

$$E_\theta(X_i) = \theta$$

$$\Rightarrow \quad \hat{\theta}_{MME} = \bar{x} = -0.5$$

$\boxed{1 \text{ MARK}}$

$\boxed{0.5 \text{ MARKS}}$

(VII) FALSE

δ_1 is UMVUE and δ_2 is unbiased

$\Rightarrow$ δ_1 is UMVUE and $\delta_1 - \delta_2 \in \mathcal{U}$

$\Rightarrow$ $Cov_\theta(\delta_1, \delta_1 - \delta_2) = 0 \quad \forall\, \theta \in \Theta$

$\Rightarrow$ $Cov_\theta(\delta_1, \delta_2) = Var_\theta(\delta_1) \quad \forall\, \theta \in \Theta$

$\boxed{1 \text{ MARK}}$

$\boxed{0.5 \text{ MARKS}}$

(VIII) TRUE

$X_{(2)}$ is complete-sufficient and $\frac{X_{(1)}}{X_{(2)}}$ is ancillary

$\Rightarrow$ $X_{(2)}$ and $\frac{X_{(1)}}{X_{(2)}}$ are independent

$$\Rightarrow \quad E_\theta(X_{(1)}) = E_\theta\left(X_{(2)} \frac{X_{(1)}}{X_{(2)}}\right) = E_\theta(X_{(2)})\, E_\theta\left(\frac{X_{(1)}}{X_{(2)}}\right)$$

$$\Rightarrow E_\theta\left(\frac{X_{(1)}}{X_{(2)}}\right) = \frac{E_\theta(X_{(1)})}{E_\theta(X_{(2)})}$$

$$= \frac{2\int_0^1 x(1-x)\,dx}{2\int_0^1 x^2\,dx} = \frac{1}{2} = 0.5$$

1 MARK

0.5 MARKS

(IX) FALSE

Let X_1, X_2 be iid $U(\theta, \theta+1)$ $\theta \in \mathbb{R} = \Theta$. Then $(X_{(1)}, X_{(2)})$ is minimal sufficient, and $X_{(2)} - X_{(1)}$ is ancillary.

$$\Rightarrow E_\theta[X_{(2)} - X_{(1)}] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \quad \forall \theta \in \mathbb{R}$$

$$\Rightarrow E_\theta[X_{(2)} - X_{(1)} - \tfrac{1}{3}] = 0 \quad \forall \theta \in \mathbb{R}$$

But $P_\theta(X_{(2)} - X_{(1)} - \frac{1}{3} = 0) = 0 \quad \forall \theta \in \mathbb{R}$

1 MARK

$\Rightarrow (X_{(1)}, X_{(2)})$ is not complete.

0.5 MARKS

(X) TRUE

By (VII)

$$Var_\theta(\delta_1) = Var_\theta(\delta_2) = Cov_\theta(\delta_1, \delta_2) \quad \forall \theta \in \Theta$$

$$\Rightarrow E_\theta(\delta_1 - \delta_2) = 0 \quad \forall \theta \in \Theta$$

and $Var_\theta(\delta_1 - \delta_2) = 0 \quad \forall \theta \in \Theta$

1 MARK

$$\Rightarrow P_\theta(\delta_1 - \delta_2 = 0) = 1 \quad \forall \theta \in \Theta$$

Problem No. 2 For sample points $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$

$$\frac{g_\theta(\underline{x})}{g_\theta(\underline{y})} = \frac{[1+(\theta-x_{(1)})^2][1+(\theta-x_{(2)})^2]}{[1+(\theta-y_{(1)})^2][1+(\theta-y_{(2)})^2]}$$

$$= \frac{(\theta-x_{(1)}+i)(\theta-x_{(1)}-i)(\theta-x_{(2)}+i)(\theta-x_{(2)}-i)}{(\theta-y_{(1)}+i)(\theta-y_{(1)}-i)(\theta-y_{(2)}+i)(\theta-y_{(2)}-i)}$$

is independent of $\theta \in \mathbb{R}$ iff for some $k(\underline{x},\underline{y})$ (independent of θ)

$$(\theta-x_{(1)}+i)(\theta-x_{(1)}-i)(\theta-x_{(2)}+i)(\theta-x_{(2)}-i)$$
$$= k(\underline{x},\underline{y})(\theta-y_{(1)}+i)(\theta-y_{(1)}-i)(\theta-y_{(2)}+i)(\theta-y_{(2)}-i), \ \forall \theta \in \mathbb{R}$$

On the LHS and RHS we have two polynomials that match on whole $\mathbb{R}$ and thus

$$\Rightarrow k(\underline{x},\underline{y}) = 1$$

and polynomials $(\theta-x_{(1)}+i)(\theta-x_{(1)}-i)(\theta-x_{(2)}+i)(\theta-x_{(2)}-i)=0$ &

$(\theta-y_{(1)}+i)(\theta-y_{(1)}-i)(\theta-y_{(2)}+i)(\theta-y_{(2)}-i)=0$ $\boxed{\text{3 MARKS}}$

will have the same roots

$$\Rightarrow \{x_{(1)}-i,\ x_{(1)}+i,\ x_{(2)}-i,\ x_{(2)}+i\}$$
$$= \{y_{(1)}-i,\ y_{(1)}+i,\ y_{(2)}-i,\ y_{(2)}+i\}$$

$$\Rightarrow \{x_{(1)}, x_{(2)}\} = \{y_{(1)}, y_{(2)}\}$$

$$\Rightarrow (x_{(1)}, x_{(2)}) = (y_{(1)}, y_{(2)})$$

$\Rightarrow T = (X_{(1)}, X_{(2)})$ is minimal sufficient

$\boxed{\text{3 MARKS}}$

$\boxed{4/7}$

Note that $X_{(2)} - X_{(1)}$ is ancillary. ~~Consider~~ Then

$P_\theta(X_{(2)} - X_{(1)} \le 1) = c$ (c does not depend on θ).

Consider

$$\psi(X_{(1)}, X_{(2)}) = \begin{cases} 1-c, & \text{if } X_{(2)} - X_{(1)} \le c \\ -c & \text{if } X_{(2)} - X_{(1)} > c \end{cases}$$

The ψ is a bounded function

$E_\theta(\psi(X_{(1)}, X_{(2)})) = (1-c)c - c(1-c) = 0 \quad \forall \theta \in \Theta$

but

$P_\theta(\psi(X_{(1)}, X_{(2)}) = 0) = 0 \quad \forall \theta \in \Theta$

$\Rightarrow$ $T = (X_{(1)}, X_{(2)})$ is not boundedly complete. .. $\boxed{4\text{MARKS}}$

Problem No. 3

(i) The likelihood function

$$L_{\underline{x}}(\theta) = \frac{8\left(\prod_{i=1}^{3} x_i\right)}{\theta^6}, \quad \theta \ge x_{(3)}$$

is maximized at $\theta = x_{(3)}$

... $\boxed{4\text{MARKS}}$

Thus $\hat{\theta}_{MLE} = X_{(3)}$

(ii) $E_\theta(X_1) = \frac{2}{3}\theta$

$\Rightarrow \frac{2}{3}\hat{\theta}_{MME} = \bar{X}$

$\Rightarrow \hat{\theta}_{MME} = \frac{3}{2}\bar{X}$ $\boxed{4\text{MARKS}}$

(iii) The dominating estimator is

$$\hat{\theta}^* = E_\theta\left[\tfrac{3}{2}\bar{X} \mid X_{(3)}\right]$$

$$= \frac{3}{2}\sum_{i=1}^{3} E_\theta[X_i \mid X_{(3)}]$$

$$= \frac{3}{2} E_\theta[X_1 \mid X_{(3)}] = \frac{3}{2} E_\theta\left[\frac{X_1}{X_{(3)}} X_{(3)} \mid X_{(3)}\right]$$

$$= \frac{3}{2} E_\theta\left[\frac{X_1}{X_{(3)}} \mid X_{(3)}\right] X_{(3)}$$

$X_{(3)}$ is complete sufficient and $\frac{X_1}{X_{(3)}}$ is ancillary

$\Rightarrow$ $X_{(3)}$ and $\frac{X_1}{X_{(3)}}$ are independent

$\Rightarrow$

$$\Rightarrow \quad \hat{\theta}^* = \frac{3}{2} E_\theta\left(\frac{X_1}{X_{(3)}}\right) X_{(3)}$$ .. $\boxed{2 \text{ MARKS}}$

Again using Basu's Theorem we get

$$E_\theta(X_1) = E_\theta\left(\frac{X_1}{X_{(3)}} X_{(3)}\right) = E_\theta\left(\frac{X_1}{X_{(3)}}\right) E_\theta(X_{(3)})$$

$$\Rightarrow \quad E_\theta\left(\frac{X_1}{X_{(3)}}\right) = \frac{E_\theta(X_1)}{E_\theta(X_{(3)})} = \frac{2/3}{3\int_0^1 x\, x^4 \times 2x\, dx}$$

$$= \frac{7}{9}$$

$$\hat{\theta}^* = \frac{3}{2} \times \frac{7}{9} X_{(3)} = \frac{7}{6} X_{(3)}$$ .. $\boxed{2 \text{ MARKS}}$

Problem No. 4

An unbiased estimator of $\psi(\theta) = (1-\theta)^2 \theta$ is

$$\delta_0(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

At $\theta = \theta_0$, the LMVUE of $\psi(\theta)$ is

$$\delta_0(x) = \delta_0(x) - U^*(x)$$

6/7

Where $U^* \in \mathcal{U}$ is such that

$$E_{\theta_0}\left[\left(\delta_0(X) - U^*(X)\right)^2\right] = \inf_{U \in \mathcal{U}} E_{\theta_0}\left[\left(\delta_0(X) - U(X)\right)^2\right] \quad .. \quad \boxed{3 \text{ MARKS}}$$

We have

$$U \in \mathcal{U} \iff E_\theta[U(X)] = 0 \quad \forall\, \theta \in (0,1)$$

$$\iff U(-1)\theta + \sum_{k=0}^{\infty} U(k)(1-\theta)^2\theta^k = 0 \quad \forall\, \theta \in (0,1)$$

$$\iff U(-1)\theta(1-\theta)^{-2} + \sum_{k=0}^{\infty} U(k)\theta^k = 0 \quad \forall\, \theta \in (0,1)$$

$$\iff U(-1)\theta(1+2\theta+3\theta^2+\dots) + \sum_{k=0}^{\infty} U(k)\theta^k = 0 \quad \forall\, \theta \in (0,1)$$

$$\iff U(0) = 0,\quad U(-1) + U(1) = 0,\quad kU(-1) + U(k) = 0,\quad k = 2, \dots$$

$$\iff U(x) = ax \text{ for some } a \in \mathbb{R} \quad \dots \quad \boxed{5 \text{ MARKS}}$$

The unique LMVUE of $\psi(\theta)$ at $\theta = \theta_0$ is

$$\delta_U(X) = \delta_0(X) - a^* X,$$

where $a^* \equiv a^*(\theta_0)$ minimizes

$$E_{\theta_0}\left[(\delta_0(X) - aX)^2\right] = a^2\theta_0 + (1-a)^2\theta_0(1-\theta_0)^2 + a^2\sum_{k=2}^{\infty} k^2(1-\theta_0)^2\theta_0^k$$

We have

$$a^*(\theta_0) = \frac{\theta_0(1-\theta_0)^2}{\theta_0 + \theta_0(1-\theta_0)^2 + \sum_{k=2}^{\infty} k^2(1-\theta_0)^2\theta_0^k}$$

$$= \frac{\theta_0(1-\theta_0)^2}{1 - P_{\theta_0}(X=0)} = \frac{\theta_0(1-\theta_0)^2}{1-(1-\theta_0)^2}$$

$\longrightarrow$ depends on θ_0 $\quad .. \quad \boxed{4 \text{ MARKS}}$

Thus the UMVUE of $\psi(\theta)$ does not exist.