MTH-418a: Inference-I

2023-2024: II Semester
Quiz I

Time Allowed: 45 Minutes **Maximum Marks: 25**

1. Let $X_1, \ldots, X_n$ be a random sample from a population having the Lebesgue p.d.f.

$$f_\theta(x) = \begin{cases} \frac{xe^{-\frac{x}{\theta}}}{\theta^2}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in \Theta = (0, \infty)$ is unknown. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $T_1(\underline{X}) = \overline{X}^2 - 3\overline{X} + 2$, $T_2(\underline{X}) = e^{\overline{X}} - \overline{X} - 1$ and $T_3(\underline{X}) = (X_{(1)}, X_{(2)}, \sum_{i=2}^{n} X_{(i)})$. $\boxed{\text{2 x 5 =10 Marks}}$

   (a) Show that $\overline{X}$ is a complete and sufficient statistic;

   (b) Is $T_1(\underline{X})$ a minimal sufficient statistic?

   (c) Is $T_2(\underline{X})$ a minimal sufficient statistic?

   (d) Is $T_1(\underline{X})$ a complete statistic?

   (e) Is $T_3(\underline{X})$ a sufficient statistic?

2. Let X_1, X_2 be a random sample from a population having the Lebesgue p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1}{\theta}, & \text{if } \theta < x < 2\theta \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in \Theta = (0, \infty)$ is unknown. Let $\underline{T} = (X_{(1)}, \frac{X_{(2)}}{X_{(1)}})$. $\boxed{\text{4 +4 =8 Marks}}$

   (a) Show that $\underline{T}$ is a minimal sufficient statistic?

   (b) Is $\underline{T}$ a complete statistic?

3. Let

$$f_0(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \quad f_1(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

$$f_2(x) = \begin{cases} 1, & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}, \text{ and } f_3(x) = \begin{cases} e^{-(x-2)}, & \text{if } x > 2 \\ 0, & \text{otherwise} \end{cases}.$$

Let $\mathcal{P} = \{f_0, f_1, f_2, f_3\}$.

   (a) Find a minimal sufficient partition for $\mathcal{P}$;

   (b) Is

$$T(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 2, & \text{if } 1 \le x < 2 \\ e^x, & \text{otherwise} \end{cases},$$

   a sufficient statistic? Is it a minimal sufficient statistic? $\boxed{\text{4 +3 =7 Marks}}$

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Model Solutions

Problem No. 1

(a) The joint pdf of $\underline{X} = (X_1, \ldots, X_n)$ is

$$g_\theta(\underline{x}) = e^{\eta \bar{x} + B(\theta)} h(\underline{x}),$$

where $\eta(\theta) = -\frac{n}{\theta} \in (-\infty, 0) = \Theta_\eta$, $B(\theta) = -2n \ln\theta$ and

$$h(\underline{x}) = \prod_{i=1}^{n} \{ x_i I(x_i > 0) \}$$

Clearly $\Theta_\eta = (-\infty, 0)$ contains a one-dimensional rectangle (an interval). Consequently $\bar{X}$ is a complete and sufficient statistic. ... $\boxed{\text{2 MARKS}}$

(b) Since $\bar{X}$ is a complete and sufficient statistic (by (a)), it is also a minimal sufficient statistic. We have

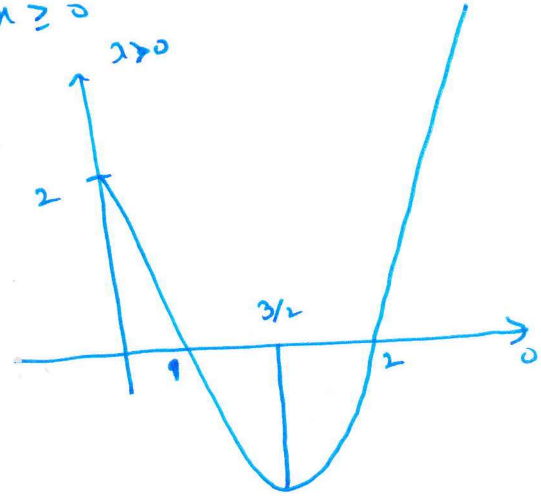
$$T_1(\underline{X}) = h_1(\bar{X}) = \bar{X}^2 - 3\bar{X} + 2,$$

where $h_1(x) = x^2 - 3x + 2 = (x-1)(x-2)$, $x \geq 0$

Since $h_1(\bar{X}) = \bar{X}^2 - 3\bar{X} + 2$ is not a 1-1 function of minimal sufficient statistic $\bar{X}$, it is not minimal suff.

$\Rightarrow T_1(\underline{X}) = \bar{X}^2 - 3\bar{X} + 2$ is not a minimal sufficient statistic

... $\boxed{\text{2 MARKS}}$



(c) We have

$$T_2(\underline{X}) = h_2(\underline{X}) = e^{\bar{X}} - \bar{X} - 1,$$

where $h_2(x) = e^{x} - x - 1$, $x \geq 0$.

We have $h_2'(x) = e^{x} - 1 > 0$, $\forall x > 0$, implying that $h_2(x)$ is a strictly increasing function of x on $[0, \infty)$.

Thus $T_2(\underline{X}) = e^{\bar{X}} - \bar{X} - 1$ is a 1-1 function of minimal sufficient statistic $\bar{X}$, further implying that $T_2(\underline{X}) = e^{\bar{X}} - \bar{X} - 1$ is a minimal sufficient statistic

... $\boxed{\text{2 MARKS}}$

(d) Since $T_1(\underline{X}) = \bar{X}^2 - 3\bar{X} + 2$ is a function of complete statistic $\bar{X}$, it is a complete statistic.

.. $\boxed{\text{2 MARKS}}$

(e) Let

$$T_3(\underline{X}) = \left(X_{(1)}, X_{(2)}, \sum_{i=2}^{n} X_{(i)}\right) = (T_{3\,1}, T_{3\,2}, T_{3\,3}) \text{ say.}$$

Then

$$\bar{X} = \frac{T_{3\,1} + T_{3\,3}}{n} = \psi(T_3(\underline{X})), \quad \text{where } \psi(t_1, t_2, t_3) = \frac{t_1 + t_3}{n},$$

$t_1 > 0$, $t_3 > 0$

Thus the minimal sufficient statistic $\bar{X}$ is a function of statistic $T_3(\underline{X})$, implying that $T_3(\underline{X})$ is a sufficient statistic

... $\boxed{\text{2 MARKS}}$

Problem No. 2

(a) The joint pdf of $\underset{\sim}{X} = (X_1, X_2)$ is

$$g_\theta(x_1, x_2) = \frac{1}{\theta^2} I\left(\frac{x_{(2)}}{2} < \theta < x_{(1)} \right) I\left(0 < x_{(2)} < 2x_{(1)}\right), \quad \theta > 0.$$

For sample points $\underset{\sim}{x} \in (0,\infty)^2$ and $\underset{\sim}{y} \in (0,\infty)^2$, $x_{(2)} < 2x_{(1)}$, $y_{(2)} < 2y_{(1)}$

$$\frac{g_\theta(x_1, x_2)}{g_\theta(y_1, y_2)} = \frac{I\left(\frac{x_{(2)}}{2} < \theta < x_{(1)} \right)}{I\left(\frac{y_{(2)}}{2} < \theta < y_{(1)} \right)}$$

is independent of θ iff (w.p.1) $\left(\frac{x_{(2)}}{2}, x_{(1)} \right) = \left(\frac{y_{(2)}}{2}, y_{(1)} \right)$

$\iff (x_{(1)}, x_{(2)}) = (y_{(1)}, y_{(2)})$

$\Rightarrow (X_{(1)}, X_{(2)})$ is a minimal sufficient statistic. .... $\boxed{2 \text{ MARKS}}$

As $T(\underset{\sim}{X}) = \left(X_{(1)}, \frac{X_{(2)}}{X_{(1)}} \right) = (T_1, T_2)$ is a 1-1 function of minimal sufficient statistic it is also minimal sufficient. ... $\boxed{2 \text{ MARKS}}$

(b) We have

$$g_\theta(x_1, x_2) = \frac{1}{\theta^2} g\left(\frac{x_1}{\theta}, \frac{x_2}{\theta} \right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where

$$g(y_1, y_2) = \begin{cases} 1, & \text{if } (y_1, y_2) \in (1, 2) \times (1, 2) \\ 0, & \text{o.w.} \end{cases}$$

$\longrightarrow$ Scale family.

Let

$$\psi(X_1, X_2) = \frac{X_{(2)}}{X_{(1)}}.$$

Then for any $c > 0$,

$$\psi(cx_1, cx_2) = \frac{\max\{cx_1, cx_2\}}{\min\{cx_1, cx_2\}} = \frac{c\max\{x_1, x_2\}}{c\min\{x_1, x_2\}} = \frac{x_{(2)}}{x_{(1)}} = \psi(x_1, x_2)$$

$\Rightarrow$ $\psi(X_1, X_2) = \dfrac{X_{(2)}}{X_{(1)}}$ is ancillary, so its expectation $E_\theta(\cdot)$ does not depend on θ. ... $\boxed{\text{2 MARKS}}$

but

$$E_\theta[\psi(X_1, X_2)] = c_0, \quad \forall\, \theta \in \Theta,$$

(Note that under $\theta = 1$ $\frac{X_{(2)}}{X_{(1)}}$ is bounded wp 1, so expectation exists)

where c_0 does not depend on θ

$$\Rightarrow \quad E_\theta\left(\frac{X_{(2)}}{X_{(1)}} - c_0\right) = 0, \quad \forall\, \theta \in \Theta$$

$$\text{but} \quad P_\theta\left(\frac{X_{(2)}}{X_{(1)}} - c_0 = 0\right) = 0, \quad \forall\, \theta \in \Theta$$

Let $\psi^*(T_1, T_2) = T_2 - c_0$. Then

$$E_\theta(\psi^*(T_1, T_2)) = 0, \quad \forall\, \theta \in \Theta$$

$$\text{but} \quad P_\theta(\psi^*(T_1, T_2) = 0) = 0, \quad \forall\, \theta \in \Theta$$

$\Rightarrow$ $T = (T_1, T_2) = \left(X_{(1)}, \dfrac{X_{(2)}}{X_{(1)}}\right)$ is not a complete statistic .. $\boxed{\text{2 MARKS}}$

Problem No. 3

$$f_0(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_1(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_2(x) = \begin{cases} 1, & \text{if } 1 < x < 2 \\ 0, & \text{o.w.} \end{cases}$$

$$f_3(x) = \begin{cases} e^{-(x-2)}, & \text{if } x > 2 \\ 0, & \text{o.w.} \end{cases}$$

$\mathcal{P} = \{f_0, f_1, f_2, f_3\}$; Let $x \equiv \frac{0}{0}$ (undefined) form

(a) clearly the support of dist^n of X is $[0, \infty)$

(i) For $x, y \in (0,1)$

$$\frac{f_\theta(x)}{f_\theta(y)} = \begin{cases} 1, & \theta = 0 \\ \frac{2x}{2y} = \frac{x}{y}, & \theta = 1 \\ x, & \theta = 2, 3 \end{cases} \longrightarrow \text{depends on } \theta$$

$\Rightarrow$ if $x, y \in (0,1)$ then they fall in different partition sets.

② For $x \in (0,1)$ & $y \in (1,2)$

$$\frac{f_\theta(x)}{f_\theta(y)} = \begin{cases} \infty, & \theta = 0,1 \\ 0, & \theta = 2 \\ x, & \theta = 3 \end{cases} \longrightarrow \text{depends on } \theta$$

$\Rightarrow$ if $x \in (0,1)$ & $y \in (1,2)$, then they fall in different partition sets

③ For $x \in (0,1)$ & $y \in (2,\infty)$

$$\frac{f_\theta(x)}{f_\theta(y)} = \begin{cases} \infty, & \theta = 0,1 \\ x, & \theta = 2 \\ 0, & \theta = 3 \end{cases} \longrightarrow \text{depends on } \theta$$

$\Rightarrow$ if $x \in (0,1)$ & $y \in (2,\infty)$, then they fall in different partition sets.

(4) For $x \in (1,2)$ & $y \in (1,2)$

$$\frac{f_\theta(x)}{f_\theta(y)} = \begin{cases} x, & \theta = 0, 1, 3 \\ 1, & \theta = 2 \end{cases} \longrightarrow \text{doesn't depend on } \theta$$

$\Rightarrow$ if $x \in (1,2)$ & $y \in (1,2)$, then they fall in the <u>same</u> partition sets.

(5) For $x \in (1,2)$ & $y \in (2,\infty)$

$$\frac{f_\theta(x)}{f_\theta(y)} = \begin{cases} x, & \theta = 0, 1 \\ \infty, & \theta = 2 \\ 0, & \theta = 3 \end{cases} \longrightarrow \text{depends on } \theta$$

$\Rightarrow$ if $x \in (1,2)$ & $y \in (2,\infty)$, then they fall in different partition sets.

(6) For $x \in (2,\infty)$ & $y \in (2,\infty)$

$$\frac{f_\theta(x)}{f_\theta(y)} = \begin{cases} x, & \theta = 0,1,2 \\ \dfrac{e^{-(x-2)}}{e^{-(y-2)}} = e^{-(x-y)}, & \theta = 3 \end{cases}$$

→ doesn't depend on θ

⇒ if $x \in (2,\infty)$ & $y \in (2,\infty)$, then they fall in the same partition sets.

--- $\boxed{\text{2 Marks}}$

Thus a minimal sufficient partition is

$$\mathcal{P}^* = \left\{ \{x\}_{x \in (0,1)},\ [1,2),\ [2,\infty) \right\}$$

--- $\boxed{\text{2 Marks}}$

(b) Using the minimal suff. partition defined in part (a), a minimal suff. statistic is

$$T_M(\underset{\sim}{X}) = \begin{cases} X, & 0 \le X < 1 \\ 2, & 1 \le X < 2 \\ 3, & X \ge 2 \end{cases}$$

We have $T(x) = h(X)$,

Where $h(x) = \begin{cases} 1, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \\ e^{x}, & \text{o.w.} \end{cases}$

Since $T_M(X)$ is not a function of $T(X) = h(X)$ as by knowing the value of $h(x)$, we can not get the value of $T_M(X)$ (e.g. if $h(X) = 1$ then we only know that $x \in [0,1)$ but from this we can not get x)

$\Rightarrow$ $T_M(X)$ is not a funⁿ of $T(X) = h(X)$

$\Rightarrow$ $T(X)$ is not sufficient stat.

$\Rightarrow$ $T(X)$ is not minimal suff. stat.

$\boxed{\text{3 Marks}}$