MTH 515a: Inference-II
Assignment No. 3: Location-Scale Invariant Estimation

1. Let $X_1, \ldots, X_n$ be i.i.d. $\text{N}(\mu, \sigma^2)$ RVs, where $\underline{\theta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_{++}$ is unknown. Consider estimation of σ^2 under the affine group of transformations and the scaled squared error loss function $L(\underline{\theta}, a) = (\frac{a}{\sigma^2} - 1)^2,\ a, \sigma \in \mathbb{R}_{++}$. Find the MRIE.

2. Let $X_1, \ldots, X_n$ be a random sample from a population with the Lebesgue p.d.f.

$$f_\mu(x) = \begin{cases} \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, & \text{if } x \geq \mu \\ 0, & \text{if } x < \mu \end{cases},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. Let $\underline{\theta} = (\mu, \sigma)$. Consider the affine group of transformations.

   (a) Find the MRIE of σ under the loss function $L(\underline{\theta}, a) = |\frac{a}{\sigma} - 1|^p$, for $p = 1$ or 2;

   (b) Under the loss function $L(\underline{\theta}, a) = \frac{(a-\mu)^2}{\sigma^2}$, find the MRIE of μ;

   (c) Under the loss function $L(\underline{\theta}, a) = \frac{(a-\eta)^2}{\sigma^2}$, find the MRIE of $\eta = \mu + \sigma$;

3. Let $X_1, \ldots, X_n$ be a random sample from $\text{U}(\mu - \sigma, \mu + \sigma)$ distribition, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. Let $\underline{\theta} = (\mu, \sigma)$.

   (a) Find the MRIE of σ under the loss function $L(\underline{\theta}, a) = |\frac{a}{\sigma} - 1|^p$, for $p = 1$ or 2;

   (b) Under the loss function $L(\underline{\theta}, a) = \frac{(a-\mu)^2}{\sigma^2}$, find the MRIE of μ. Can the findings be generalized to more general loss functions?;

   (c) Under the loss function $L(\underline{\theta}, a) = \frac{(a-\eta)^2}{\sigma^2}$, find the MRIE of $\eta = \mu + \sigma$;

4. Let $X_1, \ldots, X_n$ be i.i.d. $\text{N}(\mu, \sigma^2)$ RVs, where $\underline{\theta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_{++}$ is unknown. Consider estimation of $\psi(\underline{\theta}) = \mu$ under the affine group of transformations and the loss function $L(\underline{\theta}, a) = W(\frac{a-\mu}{\sigma}),\ a, \mu \in \mathbb{R}$, where $W : \mathbb{R} \to \mathbb{R}$ is a convex and even function. Find the MRIE.

5. Let $X_1, \ldots, X_n$ be i.i.d. $\text{N}(\mu, \sigma^2)$ RVs, where $\underline{\theta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_{++}$ is unknown. Consider estimation of $\eta = \mu + c\sigma$ under the affine group of transformations and the loss function $L(\underline{\theta}, a) = (\frac{a-\eta}{\sigma})^2,\ a, \mu \in \mathbb{R}$, where c is a given constant Find the MRIE. (**Note** for given $p \in (0, 1)$, if $c = \Phi^{-1}(p)$ then the estimation problem at hand reduces to estimation of the p-th quantile of $\text{N}(\mu, \sigma^2)$ distribution).

6. Let $\underline{X} = (X_1, \ldots, X_m)$ and $\underline{Y} = (Y_1, \ldots, Y_n)$ be two samples with joint p.d.f.

$$\frac{1}{\sigma_1^m \sigma_2^n} f\left(\frac{x_1 - \mu_1}{\sigma_1}, \cdots, \frac{x_m - \mu_1}{\sigma_1}, \frac{y_1 - \mu_2}{\sigma_2}, \cdots, \frac{y_n - \mu_2}{\sigma_2}\right).$$

(a) Suppose that $\mu_1 = \mu_2 = 0$ and consider estimation of $\eta = (\frac{\sigma_2}{\sigma_1})^b$, for a fixed $b \neq 0$, under the loss $L(\underline{\sigma}, a) = W(\frac{a}{\eta}), \underline{\sigma} = (\sigma_1, \sigma_2) \in (0, \infty)^2$. Show that the problem is invariant under the group of transformations $\mathcal{G} = \{g_{r,s} : r > 0, s > 0\}$, where $g_{r,s}(\underline{x}, \underline{y}) = (r\underline{x}, s\underline{y}), \underline{x} \in \mathbb{R}^m, \underline{y} \in \mathbb{R}^n, r > 0, s > 0$;

(b) Do the problem in (a) when μ_1 and μ_2 are unknown and the group of transformations is $\mathcal{G} = \{g_{r,s,c,d} : r > 0, s > 0, c \in \mathbb{R}, d \in \mathbb{R}\}$, where $g_{r,s,c,d}(\underline{x}, \underline{y}) = (r\underline{x} + c, s\underline{y} + d), \underline{x} \in \mathbb{R}^m, \underline{y} \in \mathbb{R}^n, r > 0, s > 0, c \in \mathbb{R}, d \in \mathbb{R}$;

(c) Under $\sigma_1 = \sigma_2 = \sigma > 0$ (σ is unknown), discuss invariant estimation of $\Delta = \mu_2 - \mu_1$ under a suitable group of transformations and loss function.

7. Consider Problem 6 (a) under the loss function $\frac{(a-\eta)^2}{\eta^2}$. Determine the MRIE of η in the following cases:

   (a) $m = n = 1$, $X \sim \text{Gamma}(\alpha_1, \sigma_1)$, $Y \sim \text{Gamma}(\alpha_2, \sigma_2)$ with known $\alpha_i > 0, i = 1, 2$;

   (b) $\underline{X} \sim N_m(0, \sigma_1^2 I_m)$ and $\underline{Y} \sim N_m(0, \sigma_2^2 I_n)$ are independent;

   (c) $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ are independent with $X_i \sim \text{U}(0, \sigma_1), i = 1, \ldots, m$ and $Y_i \sim \text{U}(0, \sigma_2), i = 1, \ldots, n$.

8. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two independent random samples, where $X_i s$ are i.i.d. having the common p.d.f. $\frac{1}{\sigma_1} f(\frac{x-\mu_1}{\sigma_1}), \mu_1 \in \mathbb{R}, \sigma_1 > 0$, and $Y_i s$ are i.i.d. having the common p.d.f. $\frac{1}{\sigma_2} f(\frac{x-\mu_2}{\sigma_2}), \mu_2 \in \mathbb{R}, \sigma_2 > 0$.

   (a) Under the loss function $L(\underline{\theta}, a) = \frac{(a-\eta)^2}{\eta^2}$ and the transformation given in Problem 6 (b), obtain the MRIE of $\eta = \frac{\sigma_2}{\sigma_1}$ when

      (i) f is the p.d.f. of $N(0, 1)$;

      (ii) f is the p.d.f. of $E(0, 1)$;

      (iii) f is the p.d.f. of $\text{U}(-\frac{1}{2}, \frac{1}{2})$;

   (b) In (i)-(iii) above find the MRIE of $\Delta = \mu_2 - \mu_1$ under the assumption $\sigma_1 = \sigma_2 = \sigma$ and under the loss $\frac{(a-\Delta)^2}{\sigma^2}$.