

MTH 515a: Inference-II
Assignment No. 4: Bayes and Minimax Estimation

1. Let X_1, \dots, X_n be a random sample from $\text{Bin}(1, \theta)$ distribution, where $\theta \in \Theta = (0, 1)$. Consider estimation of θ under the squared error loss function $L(\theta, a) = (a - \theta)^2$, $a, \theta \in \Theta = \mathcal{A}$. Consider the randomized decision rule δ_0 defined by:

$$\delta_0(a|\underline{x}) = \begin{cases} \frac{n}{n+1}, & \text{if } a = \bar{x} \\ \frac{1}{n+1}, & \text{if } a = \frac{1}{2} \end{cases} .$$

Compare the supremum of risks of \bar{X} and δ_0 and hence conclude that \bar{X} is not minimax.

2. What are the conjugate priors for:

- (a) $N_k(\underline{\theta}, I_k)$, $\underline{\theta} \in \mathbb{R}^k$;
 - (b) $\text{Bin}(n, \theta)$, $\theta \in (0, 1)$; (Binomial distribution with known number of trials $n \in \mathbb{N}$)
 - (c) $U(0, \theta)$, $\theta > 0$; (Uniform distribution)
 - (d) $E(0, \theta)$, $\theta > 0$; (Exponential Distribution)
 - (e) $\text{Bin}(n, \theta)$, $n \in \{1, 2, \dots\}$; (Binomial distribution with known success probability $\theta \in (0, 1)$)
3. Let X_1, \dots, X_n be i.i.d. $\text{Poisson}(\theta)$ random variables, where $\theta \in \Theta = (0, \infty)$. For a positive real number r , consider estimation of $g(\theta) = \theta^r$ under the SEL function and $\text{Gamma}(\alpha_0, \mu_0)$ prior ($\alpha_0, \mu_0 > 0$). Find the Bayes estimator. Also show that $\delta_0(\underline{X}) = \bar{X}$ can not be Bayes estimator of θ with respect to any proper prior distribution.
4. Let X be a random variable having a p.d.f. $f_\theta(x)$, $\theta \in \Theta$, $x \in \chi$, and let π be a prior distribution on $\Theta \subseteq \mathbb{R}$. For a real-valued function $g(\theta)$ and a non-negative function $w(\theta)$, such that $\int_\Theta w(\theta)g(\theta)d\pi(\theta) < \infty$, consider estimation of $g(\theta)$ under the loss function $L(\theta, a) = w(\theta)(a - g(\theta))^2$. Show that the Bayes action is

$$\delta_\pi(x) = \frac{\int_\Theta w(\theta)g(\theta)f_\theta(x)d\pi(\theta)}{\int_\Theta w(\theta)f_\theta(x)d\pi(\theta)}, \quad x \in \chi.$$

5. For $i = 1, \dots, p$, let δ_i be a Bayes estimator of θ_i under the SEL function. For real constants c_1, \dots, c_p , show that $\sum_{j=1}^p c_j \delta_j$ is a Bayes estimator of $\sum_{j=1}^p c_j \theta_j$ under the squared error loss.
6. Let X_1, \dots, X_n be a random sample from $N(\theta, 1)$ distribution, where $\theta \in \mathbb{R}$, and let $\pi \sim \text{DE}(0, 1)$ (Double Exponential distribution). Obtain the Bayes action under the squared error loss.

7. Let X_1, \dots, X_n be i.i.d. $N(\mu, \sigma^2)$ random variables, where $\underline{\theta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_{++}$ is unknown. Consider estimation of $g_1(\underline{\theta}) = \mu$ and $g_2(\underline{\theta}) = \sigma^2$ under SEL functions and the prior for $(\mu, \tau) = (\mu, \frac{1}{2\sigma^2})$ such that conditional prior distribution of μ given τ is $N(\mu_0, \frac{\sigma_0^2}{\tau})$ and the marginal prior distribution of τ is $\text{Gamma}(\alpha_0, \nu_0)$ ($\mu_0 \in \mathbb{R}, \sigma_0, \alpha_0, \nu_0 > 0$). Find Bayes estimators of $g_1(\underline{\theta})$ and $g_2(\underline{\theta})$.
8. Let $X \sim N(\mu, \sigma^2)$, with a known $\sigma > 0$ and unknown $\mu > 0$. Consider estimating μ under the squared error loss and non-informative prior $\pi =$ the Lebesgue measure on $(0, \infty)$. Show that the generalized Bayes action is

$$\delta(x) = x + \sigma \cdot \frac{\phi(\frac{x}{\sigma})}{\Phi(\frac{x}{\sigma})}, \quad x \in \mathbb{R}.$$

9. Let X_1, \dots, X_n be i.i.d. $N(\theta, \sigma_0^2)$ random variables, where $\theta \in \Theta = \mathbb{R}$ is unknown and $\sigma_0 (> 0)$ is known. Consider estimation of $g(\theta) = \theta$ under the SEL function and $N(\mu_0, \tau_0^2)$ prior for θ ($\mu_0, \tau_0 > 0$). Find the Bayes estimator. Also show that $\delta_0(\underline{X}) = \bar{X}$ can not be Bayes with respect to any proper prior distribution but it is an admissible and minimax estimator. Further show that $\delta_0(\underline{X}) = \bar{X}$ is the generalized Bayes estimator under the SEL function and non-informative prior $\pi =$ the Lebesgue measure on \mathbb{R} .
10. Let X_1, \dots, X_n be i.i.d. $N(\theta, \sigma_0^2)$ random variables, where $\theta \in \Theta = \mathbb{R}$ is unknown and $\sigma_0 (> 0)$ is known. Consider estimation of $g(\theta) = \theta$ under the SEL function. Let $\delta_{a,b}(\underline{X}) = a\bar{X} + b, a, b \in \mathbb{R}$. Show that $\delta_{a,b}$ is admissible whenever $0 < a < 1$, or, $a = 1$ but $b = 0$, and it is inadmissible whenever $a > 1$, or, $a < 0$, or, $a = 1$ but $b \neq 0$.
11. Let \underline{X} be a random vector having a distribution characterized by a parameter (possibly vector-valued) $\underline{\theta} \in \Theta \subseteq \mathbb{R}^m$. Let $\psi(\underline{\theta})$ be a real-valued estimand under the squared error loss function. Let $\delta_{1,0}(\underline{X})$ be an unbiased estimator of $\psi(\underline{\theta})$. Consider estimators $\delta_{a,b}(\underline{X}) = a\delta_{1,0}(\underline{X}) + b, a, b \in \mathbb{R}$. Show that $\delta_{a,b}$ is an inadmissible estimator under any of the following conditions: (a) $a > 1$; (b) $a < 0$; (c) $a = 1$ and $b \neq 0$.
12. Let \underline{X} be a random vector having a distribution characterized by a parameter (possibly vector-valued) $\underline{\theta} \in \Theta \subseteq \mathbb{R}^m$. Let $\psi(\underline{\theta}) \in (c, d)$ be a real-valued estimand under the squared error loss function. Let δ be an estimator of $\psi(\underline{\theta})$ such that $P_{\underline{\theta}_0}(\delta(\underline{X}) \notin (c, d)) > 0$, for some $\underline{\theta}_0 \in \Theta$. Show that δ is an inadmissible estimator.
13. Let X_1, \dots, X_n be i.i.d. $\text{Exp}(\theta)$ ($E_\theta(X_1) = \theta$) random variables, where $\theta \in \Theta = (0, \infty)$. Consider estimation of $g(\theta) = \theta$ under the SEL function and $\text{Gamma}(\alpha_0, \mu_0)$ prior for $\tau = \frac{1}{\theta}$ ($\alpha_0, \mu_0 > 0$). Let $\delta_{a,b}(\underline{X}) = a\bar{X} + b, a, b \in \mathbb{R}$.

(a) Find the Bayes estimator;

- (b) Show that $\delta_0(\underline{X}) = \bar{X}$ can not be Bayes with respect to any proper prior distribution;
- (c) Show that $\delta_{a,b}$ is inadmissible under any of the following conditions: (i) $a > \frac{n}{n+1}$ and $b \geq 0$; (ii) $a < 0$; (iii) $b < 0$.
- (d) Show that the Bayes estimator is inadmissible if $\alpha_0 < 2$;
- (e) For $a \in [0, \frac{n}{n+1})$ and $b > 0$, show that $\delta_{a,b}(\underline{X})$ is an admissible estimator.
- (g) Can a minimax estimator be found?
- (h) Find the generalized Bayes estimator of θ under Jeffery's prior for θ . Also find the generalized Bayes estimator under the uniform prior $f_S(\theta) = 1$, $\theta > 0$ for θ .
14. Let X_1, \dots, X_n be i.i.d. $\text{Gamma}(\alpha, \theta)$, where $\theta \in (0, \infty) = \Theta$ is unknown and $\alpha > 0$ is known. Consider estimation of $g(\theta) = \theta$ under the loss function $L(\theta, a) = (\frac{a}{\theta} - 1)^2$, $a, \theta \in \Theta = \mathcal{A}$. Let $\delta_{a,b}(\underline{X}) = a\bar{X} + b$, $a, b \in \mathbb{R}$.
- (a) Show that $\delta_{a,b}$ is inadmissible under any of the following conditions: (i) $a > \frac{n}{n\alpha+1}$ and $b \geq 0$; (ii) $a < 0$; (iii) $b < 0$;
- (b) For $a \in [0, \frac{n}{n\alpha+1})$ and $b > 0$, show that $\delta_{a,b}(\underline{X})$ is an admissible estimator;
- (c) Find generalized Bayes estimator of θ with respect to Jeffery's prior for θ .
15. Let X_1, \dots, X_n be a random sample from $\text{Bin}(m, \theta)$ distribution, where $\theta \in \Theta = (0, 1)$ is unknown and m is a known positive integer.
- (a) Find the minimax estimator of θ under the SEL function and show that $\delta_0(\underline{X}) = \frac{\bar{X}}{m}$ is not a minimax estimator;
- (b) Show that $\delta_0(\underline{X}) = \frac{\bar{X}}{m}$ is an admissible and minimax estimator of θ under the loss function $L(\theta, a) = \frac{(a-\theta)^2}{\theta(1-\theta)}$, $a, \theta \in (0, 1)$;
- (c) Show that $\frac{\bar{X}}{m}$ is an admissible estimator of θ under the squared error loss function.
16. Let X_1, \dots, X_n be i.i.d. $N(\mu, \sigma^2)$ random variables, where $\underline{\theta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_{++}$ is unknown. Consider estimation of $g(\underline{\theta}) = \mu$ under SEL function. For any estimator δ show that $\sup_{\underline{\theta} \in \Theta} R_\delta(\underline{\theta}) = \infty$. If $\Theta = \mathbb{R} \times (0, c]$, for some positive constant c , show that $\delta_0(\underline{X}) = \bar{X}$ is minimax.
17. Let $X \sim N(\theta, 1)$, $\theta \in \mathbb{R}$. For estimating θ under the absolute error loss function $L(\theta, a) = |a - \theta|$, $a, \theta \in \mathbb{R}$ show that X is an admissible and minimax estimator.
18. Let $X \sim \text{Poisson}(\theta)$, where $\theta \in (0, \infty) = \Theta$ is unknown. Consider estimation of $g(\theta) = \theta$ under the SEL function. Show that, for any estimator δ , $\sup_{\theta \in \Theta} R_\delta(\theta) = \infty$.

19. Let $X \sim G(\theta)$ (Geometric distribution with support $\{1, 2, \dots\}$), where $\theta \in (0, 1)$ is unknown. Show that $I_{\{1\}}(X)$ is a minimax estimator of θ under the loss function $L(\theta, a) = \frac{(a-\theta)^2}{\theta(1-\theta)}$, $a, \theta \in (0, 1)$.
20. Let $X \sim \text{NB}(r, \theta)$, where $\theta \in \Theta = (0, 1)$ is unknown and r is a fixed positive integer. Consider estimation of $g(\theta) = \frac{1}{\theta}$ under the loss function $L(\theta, a) = \theta^2(a - \frac{1}{\theta})^2$, $a, \theta \in \Theta = \mathcal{A}$, and $\text{Beta}(\alpha, \beta)$ ($\alpha, \beta > 0$) prior. Find the Bayes estimator. Show that $\delta_0(X) = \frac{X+1}{r+1}$ is admissible under the SEL function.
21. Let X_1, \dots, X_n be a random sample from $N(\mu, \sigma^2)$ distribution, where $\underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty)$ is unknown. Show that \bar{X} is minimax estimator of μ under the loss function $L(\underline{\theta}, a) = \frac{(a-\mu)^2}{\sigma^2}$, $a \in \mathbb{R}$, $\underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty)$.
22. In a two-sample normal problem (independent random samples) with unknown means find a minimax estimator of $\Delta = \mu_2 - \mu_1$ under the squared error loss function when:
- the variances are known (possibly unequal);
 - the variances σ_1^2 and σ_2^2 are unknown but $\sigma_i^2 \in (0, c_i]$, $i = 1, 2$, for some known positive constants c_1 and c_2 .
23. Let $X \sim N(\theta, 1)$, $\theta \in \mathbb{R}$ and $d\pi(\theta) = e^\theta d\theta$, $\theta \in \mathbb{R}$. For estimating θ under the squared error loss function show that $\delta_{GB}(X) = X + 1$ is a generalized-Bayes estimator but neither minimax nor admissible.
24. Let X_1, \dots, X_n be a random sample from a distribution $F_\theta \sim: E(\theta, \sigma_0)$, $\theta \in \Theta = \mathbb{R}$ is unknown and $\sigma_0 > 0$ is known. Find a minimax estimator of θ under the squared error loss function.
25. Using heuristic (geometric) arguments show that:
- a Bayes rule may not be admissible;
 - Bayes rule may not be unique;
 - minimax rule may not be unique;
 - minimax rule may not be admissible;
 - Bayes rule may not be minimax.