MTH 515a: Inference-II
Assignment No. 5: Asymptotically Efficient Estimators

1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with finite mean μ and finite variance $\sigma^2 > 0$. Let $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$. Show that $\sqrt{n}(S^2 - \sigma^2) \xrightarrow{d} Z \sim \mathrm{N}(0, (\gamma-1)\sigma^4)$, as $n \to \infty$, where γ is the kurtosis of the parent distribution. Hence show that S^2 is a consistent estimator of σ^2.

2. Consider the set-up of Problem 1. Find the asymptotic distributions of $\sqrt{n}\begin{pmatrix} \overline{X} - \mu \\ S^2 - \sigma^2 \end{pmatrix}$ and $\sqrt{n}(\overline{X} + S - \mu - \sigma)$.

3. Let $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}, i = 1, 2, \ldots$ be a sequence of i.i.d. random vectors with $E(X_1) = \mu_1 \in \mathbb{R}, E(Y_1) = \mu_2 \in \mathbb{R}, \mathrm{Var}(X_1) = \sigma_1^2 > 0, \mathrm{Var}(Y_1) = \sigma_2^2 > 0, \mathrm{Corr}(X_1, Y_1) = \rho \in (-1, 1), \mathrm{Var}(X_1^4) < \infty$ and $\mathrm{Var}(Y_1^4) < \infty$. Let ${S_1}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2, {S_2}^2 = \frac{1}{n}\sum_{i=1}^n (Y_i - \overline{Y})^2, S_{1,2} = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y})$ and $r = \frac{S_{1,2}}{S_1 S_2}$. Show that, as $n \to \infty$,
$$\sqrt{n}\left(\begin{pmatrix} S_1^2 \\ S_2^2 \\ S_{1,2} \end{pmatrix} - \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \rho\sigma_1\sigma_2 \end{pmatrix}\right) \xrightarrow{d} N_3(\underline{0}, \Sigma^*),$$
for some p.d. matrix Σ^*. What are elements of Σ^*?. When $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}, i = 1, 2, \ldots$ follow bivariate normal distribution, show that, as $n \to \infty$,
$$\sqrt{n}(r - \rho) \xrightarrow{d} N(0, (1-\rho^2)^2)$$
$$\text{and} \quad \sqrt{n}\left(\frac{1}{2}\ln\frac{1+r}{1-r} - \frac{1}{2}\ln\frac{1+\rho}{1-\rho}\right) \xrightarrow{d} N(0, 1)$$

4. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $\mathrm{U}(0, \theta)$ random variables, where $\theta \in \Theta = (0, \infty)$. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}, n = 1, 2, \ldots$. Show that $n(\theta - X_{(n)}) \xrightarrow{d} Z \sim \mathrm{Exp}(\theta)$, as $n \to \infty$. Hence show that $\delta_n(\underline{X}) = \frac{n}{n+1}X_{(n)}$ is a consistent estimator of θ.

5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common p.m.f./p.d.f.
$$g_\theta(x) = e^{\theta T(x) - \psi(\theta)}, \; x \in \chi \subseteq \mathbb{R}, \theta \in \Theta \subseteq \mathbb{R},$$
where Θ is an open set. Show that $\sqrt{n}(\frac{1}{n}\sum_{i=1}^n T(X_i) - \psi'(\theta)) \xrightarrow{d} N(0, \psi''(\theta))$, as $n \to \infty$. Hence show that $\frac{1}{n}\sum_{i=1}^n T(X_i)$ is a consistent estimator of $\psi'(\theta)$.

6. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{N}(\theta, 1)$ random variables. Find a consistent estimator of
$$g(\theta) = \begin{cases} 0, & \text{if } \theta \neq 0 \\ 1, & \text{if } \theta = 0 \end{cases}.$$

7. Let the random variables $X_1, \ldots, X_n$ have the common mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0, \infty)$, and that $\text{Cov}(X_i, X_j) = \rho_{j-i},\ j \neq i$. For estimating μ, show that:

   (a) the sample mean $\overline{X}$ may not be consistent if $\rho_{j-i} = \rho \neq 0,\ \forall\ i \neq j$;

   (b) the sample mean $\overline{X}$ is consistent if $|\rho_{j-i}| \leq M r^{j-i}$, with $|r| < 1$.

8. Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean μ and finite variance $\sigma^2 > 0$. Let

$$\delta_n(\underline{X}) = \begin{cases} \overline{X}, & \text{w.p. } 1 - \epsilon_n \\ A_n, & \text{w.p. } \epsilon_n \end{cases},$$

   where ϵ_n and A_n $(n = 1, 2, \ldots)$ are constants satisfying $\epsilon_n \to 0$ and $A_n \epsilon_n \to \infty$, as $n \to \infty$. Show that δ_n is consistent for estimating μ but $E_{\underline{\theta}}((\delta_n - \mu)^2) \nrightarrow 0$, as $n \to \infty$.

9. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \theta), \theta > 0$, and let $T = \max\{X_1, \ldots, X_n\}$. Let h be a four times differentiable function on $(0, \infty)$ with bounded fourth derivative on $(0, \infty)$. Show that

$$\begin{aligned} E_\theta(h(T)) &= h(\theta) - \frac{\theta}{n} h'(\theta) + \frac{1}{n^2}[\theta h'(\theta) + \theta^2 h''(\theta)] + O(\frac{1}{n^3}) \\ \text{and } \text{Var}_\theta(h(T)) &= \frac{\theta^2}{n^2}[h'(\theta)]^2 + O(\frac{1}{n^3}). \end{aligned}$$

10. Let $X_1, X_2, \ldots$ be i.i.d. $\text{Bin}(1, \theta)$, where $\theta \in \Theta = (0, 1)$. For estimating $g(\theta) = \text{Var}_\theta(X_1)$, let δ_1 be the UMVUE. Find the limiting bias, limiting variance, asymptotic bias and asymptotic variance of δ_1.

11. Let $X_1, X_2, \ldots$ be i.i.d. $\text{Bin}(1, \theta)$, where $\theta \in \Theta = (0, 1)$. For estimating $g(\theta) = \theta$, let $\delta_n^{(1)}$ be the UMVUE and let $\delta_n^{(2)}$ be the minimax estimator under the SEL function. Find the LRE and ARE of $\delta_n^{(1)}$ relative to $\delta_n^{(2)}$. Are these estimators asymptotically efficient?

12. Let $X_1, \ldots, X_n$ be i.i.d. gaussian random variables with finite mean θ and variance 1. Let u_0 be a given real constant. For estimating $g(\theta) = P_\theta(X_1 \leq u_0)$, let δ_1 be the UMVUE and let $\delta_2 = \frac{1}{n}\sum_{i=1}^n I(X_i \leq u_0)$. Find the LRE l_{δ_1, δ_2} and the ARE e_{δ_1, δ_2}. Are these estimators asymptotically efficient? Which estimator will you prefer?

13. Let $X_1, \ldots, X_n$ be i.i.d. normal random variables having unknown mean μ and finite known variance $\sigma^2 > 0$. Let $h_r(\mu) = \mu^r,\ r = 2, 3, 4$. Determine up to $O(\frac{1}{n})$,

   (a) the variance of the UMVUE of $h_r(\mu),\ r = 2, 3, 4$;

   (b) the bias of the MLE of $h_r(\mu),\ r = 2, 3, 4$.

   Repeat (a) and (b) when μ is known and σ is unknown and $h_r(\sigma) = \sigma^r,\ r = 2, 4$.

14. Let $X_1, \ldots, X_n$ be i.i.d. Poisson random variables having mean $\theta > 0$.

(a) Find the variance of the UMVUE of $P_\theta(X_1 = 0)$ up to $O(\frac{1}{n})$. Is the UMVUE CAN?

(b) For estimating $h(\theta) = e^{-\theta}$, find the LRE and ARE of $\delta_{2,n} = [\#X_i = 0]/n$ relative to $\delta_{1,n} = e^{-\overline{X}}$.

15. Let $X \sim \text{Bin}(n, \theta)$ and let $h(\theta) = \theta(1-\theta)$. Find the UMVUE of $h(\theta)$. Is it CAN?

16. Let $X_1, \ldots, X_n$ be i.i.d. $\text{N}(0, \sigma^2)$, where $\sigma > 0$.

    (a) Show that $\delta_{1,n} = \frac{k}{n}\sum_{i=1}^n |X_i|$ is a consistent estimator of σ iff $k = \sqrt{\frac{\pi}{2}}$;

    (b) Determine the LRE and ARE of $\delta_{1,n}$, with $k = \sqrt{\frac{\pi}{2}}$, relative to MLE $\delta_{2,n} = \sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2}$.

17. Let $X_1, \ldots, X_n$ be i.i.d. with $E_{\underline{\theta}}(X_1) = \mu$, $\text{Var}_{\underline{\theta}}(X_1) = 1$ and $E_{\underline{\theta}}((X_1 - \mu)^4) = \mu_4 \in \mathbb{R}$. Consider the unbiased estimators $\delta_{1,n} = \frac{1}{n}\sum_{i=1}^n X_i^2 - 1$ and $\delta_{2,n} = \overline{X}^2 - \frac{1}{n}$ of the estimand $h(\underline{\theta}) = \mu^2$.

    (a) Determine the ARE $e_{2,1}$;

    (b) Show that $e_{2,1} \geq 1$ if the distribution of X_1 is symmetric;

    (c) Find a distribution for which $e_{2,1} < 1$.

    (d) Is $\delta_{2,n}$ CAN?

18. Let $X_1, \ldots, X_n$ be i.i.d. Pareto random variables with p.d.f.

$$f_\theta(x) = \begin{cases} \frac{\theta}{x^{\theta+1}}, & \text{if } x > 1 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0, \infty) = \Omega$ is unknown. Show that the MME estimator based on $\ln X_i,\ i = 1, \ldots, n$, has asymptotic variance equal to CRLB. Compare the variance of UMVUE of θ with CRLB.

19. Let $X_1, \ldots, X_n$ be i.i.d. double exponential random variables with p.d.f.

$$f_\theta(x) = \frac{1}{2\theta}e^{-\frac{|x|}{\theta}},\ -\infty < x < \infty,$$

where $\theta \in (0, \infty) = \Omega$ is unknown. Suppose that the estimand is $h(\theta) = \theta$.

    (a) Show that the MME and the UMVUE are the same and their variances attain CRLB;

    (b) Show that $\overline{X}$ is asymptotically normal but not consistent;

    (c) Find a MME (say, $\delta_n^{(1)}$) based on $X_i^2,\ i = 1, \ldots, n$ and show that it is CAN. Compare its asymptotic variance with that of $\delta_n^{(2)} = \frac{1}{n}\sum_{i=1}^n |X_i|$. Find the LRE and ARE of $\delta_n^{(1)}$ relative to $\delta_n^{(2)}$. Are $\delta_n^{(1)}$ and $\delta_n^{(2)}$ asymptotically efficient?