On comparison of reversed hazard rates of two parallel systems comprising of independent gamma components

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Abstract

Let $X_1, \ldots, X_n$ ($Y_1, \ldots, Y_n$) be independent random variables such that X_i (Y_i) follows the gamma distribution with shape parameter α and mean $\frac{\alpha}{\lambda_i}$ ($\frac{\alpha}{\mu_i}$), $\alpha > 0, \lambda_i > 0$ ($\mu_i > 0$), $i = 1, \ldots, n$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and let $\tilde{r}_{n:n}(\boldsymbol{\lambda}; x)$ ($\tilde{r}_{n:n}(\boldsymbol{\mu}; x)$) denote the reversed hazard rate of $\max\{X_1, \ldots, X_n\}$ ($\max\{Y_1, \ldots, Y_n\}$). In this note we show that if $\boldsymbol{\lambda}$ weakly majorizes $\boldsymbol{\mu}$ then $\tilde{r}_{n:n}(\boldsymbol{\lambda}; x) \geq \tilde{r}_{n:n}(\boldsymbol{\mu}; x), \forall x > 0$, thereby strengthening and extending the results of Dykstra *et al.* [5], Khaledi *et al.* [11], and Lihong and Xinsheng [15].

Keywords: Gamma distribution, Hazard rate order, Majorization, Order statistics, Reversed hazard rate order, Usual stochastic order.

1. Introduction and a review of literature

Let $X_1, \ldots, X_n$ be independent and nonnegative random variables (i.e. corresponding distributions have the common support $\mathbb{R}_+ \equiv [0, \infty)$) representing the lifetimes of n components and let $Y_1, \ldots, Y_n$ be another set of independent and nonnegative random variables representing the lifetimes of another set of n components. For $k \in \{1, \ldots, n\}$, let $X_{k:n}$ and $Y_{k:n}$ respectively denote the kth order statistics based on random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Then $X_{k:n}$ and $Y_{k:n}$ are the lifetimes of $(n-k+1)$-out-of-n systems constructed from the two sets of components and thus a stochastic

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comparison of these two random variables may be of interest. A vast literature on stochastic comparisons of order statistics from two heterogeneous distributions is available. In order to provide a review of the literature on this topic we will require definitions of some stochastic orders and the concept of majorization, for which we refer the reader to Section 2 of the paper.

Suppose that the random variables X_i and Y_i have absolutely continuous distribution functions $F(x; \lambda_i)$ and $F(x; \mu_i)$, respectively, where $\lambda_i, \mu_i > 0$, $i = 1, \ldots, n$. Let $\bar{F}(x; \lambda_i) = 1 - F(x; \lambda_i)$ and $\bar{F}(x; \mu_i) = 1 - F(x; \mu_i)$ be the corresponding survival functions. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$.

First we will discuss results on stochastic comparisons of order statistics under the proportional hazard rates (PHR) model (i.e. $\bar{F}(x; \lambda) = [\bar{F}_0(x)]^{\lambda}, x \in \mathbb{R} \equiv (-\infty, \infty), \lambda > 0$, for some survival function $\bar{F}_0$). Under the PHR model, Pledger and Proschan [17] proved that

$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{k:n} \leq_{\text{st}} X_{k:n}, \quad k = 1, \ldots, n. \tag{1.1}$$

Proschan and Sethuraman [18] strengthened this result from componentwise stochastic ordering to multivariate stochastic ordering by proving that

$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \leq^{\text{st}} (X_{1:n}, \ldots, X_{n:n}). \tag{1.2}$$

For two-component parallel systems, Boland *et al.* [2] strengthened result (1.1) of Pledger and Proschan [17] by showing that

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{\text{hr}} X_{2:2}. \tag{1.3}$$

Using an example they also demonstrated that this result may not hold for $n \geq 3$ component parallel systems. However, for two-component parallel systems with exponentially distributed lifetimes, Dykstra *et al.* [5] showed that

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{\text{lr}} X_{2:2}. \tag{1.4}$$

For the PHR model, Da *et al.* [4] further studied this problem and proved that

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \text{ and } \lambda_2 - \lambda_1 \geq \mu_2 - \mu_1 \Rightarrow Y_{n:n} \leq_{\text{hr}} X_{n:n}. \tag{1.5}$$

Zhao and Balakrishnan [20] extended results (1.3) and (1.4) by establishing that:

$$\min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \leq \max(\mu_1, \mu_2) \leq \max(\lambda_1, \lambda_2) \text{ and}$$
$$(\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{\text{hr}} X_{2:2}; \tag{1.6}$$

and

$$\min(\lambda_1, \lambda_2) \le \min(\mu_1, \mu_2) \le \max(\mu_1, \mu_2) \le \max(\lambda_1, \lambda_2) \text{ and}$$
$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\mu_1, \mu_2) \Rightarrow Y_{2:2} \le_{\mathrm{lr}} X_{2:2}. \tag{1.7}$$

Earlier, Joo and Mi [7] had proved a weaker version of result (1.7) for the particular case of exponential distribution (i.e. $\bar{F}_0(x) = e^{-x}, x \in \mathbb{R}_+$). For parallel systems, Khaledi and Kochar [9] generalized result (1.1), due to Pledger and Proschan [17], in another direction by establishing that

$$\boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \le_{\mathrm{st}} X_{n:n}. \tag{1.8}$$

Using an example they demonstrated that this result may not hold for other order statistics. Khaledi and Kochar [9] also showed that:

$$\mu_i = \left(\prod_{j=1}^{n} \lambda_j\right)^{1/n}, \ i = 1, \ldots, n \Rightarrow Y_{n:n} \le_{\mathrm{hr}} X_{n:n}; \tag{1.9}$$

and

$$F_0 \text{ has DHR and } \mu_i = \left(\prod_{j=1}^{n} \lambda_j\right)^{1/n}, \ i = 1, \ldots, n \Rightarrow Y_{n:n} \le_{\mathrm{disp}} X_{n:n}. \tag{1.10}$$

Earlier, Khaledi and Kochar [8] had proved results (1.8)–(1.10) for the particular case of exponential distribution. Moreover, Dykstra *et al.* [5] had proved weaker forms of the results proved by Khaledi and Kochar [8]. For exponential distributions with $\mu_i = \frac{1}{n}\sum_{j=1}^{n} \lambda_j$, $i = 1, \ldots, n$, Dykstra *et al.* [5] established that $Y_{n:n} \le_{\mathrm{hr}} X_{n:n}$ and $Y_{n:n} \le_{\mathrm{disp}} X_{n:n}$. For exponential distributions, Dykstra *et al.* [5] also showed that

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \le_{\mathrm{rh}} X_{n:n}. \tag{1.11}$$

Kochar and Xu [13] demonstrated that result (1.9), due to Khaledi and Kochar [9], may not hold if we replace $\le_{\mathrm{hr}}$ ordering by $\le_{\mathrm{rh}}$ ordering or by $\le_{\mathrm{lr}}$ ordering. However, they extended result (1.9) by proving that

$$\mu_i = \frac{1}{n}\sum_{j=1}^{n} \lambda_j, \ i = 1, \ldots, n \Rightarrow Y_{n:n} \le_{\mathrm{lr}} X_{n:n}. \tag{1.12}$$

Recently, for two-component parallel systems, Zhao and Balakrishnan [21] proved that:

$$F_0 \text{ has DHR }, \min(\lambda_1,\lambda_2) \le \min(\mu_1,\mu_2) \le \max(\mu_1,\mu_2) \le \max(\lambda_1,\lambda_2) \text{ and}$$
$$(\lambda_1,\lambda_2) \overset{\text{rm}}{\succeq} (\mu_1,\mu_2) \Rightarrow Y_{2:2} \le_{\text{mrl}} X_{2:2}. \tag{1.13}$$

Now we will provide a discussion on stochastic comparisons of order statistics under the scale model (i.e. $F(x;\lambda) = G_0(\lambda x), x \in \mathbb{R}, \lambda > 0$, for some distribution function G_0). Let r_{G_0} and $\tilde{r}_{G_0}$ respectively denote the hazard function and the reversed hazard function of G_0. Pledger and Proschan [17] proved the following results:

$$G_0 \text{ has DHR and } \boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{k:n} \le_{\text{st}} X_{k:n},\ k = 1,\ldots,n; \tag{1.14}$$

$$G_0 \text{ has DRHR and } \boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \le_{\text{st}} X_{n:n}; \tag{1.15}$$

and

$$G_0 \text{ has IHR and } \boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow X_{1:n} \le_{\text{st}} Y_{1:n}. \tag{1.16}$$

Hu [6] extended result (1.14) by showing that:

$$G_0 \text{ has DHR}, \psi_1(x) = x r_{G_0}(x), x \in \mathbb{R}_+, \text{ is an increasing function and}$$
$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow (Y_{1:n},\ldots,Y_{n:n}) \le^{\text{st}} (X_{1:n},\ldots,X_{n:n}). \tag{1.17}$$

It is worth mentioning here that the conditions of Hu [6] are satisfied by the gamma distribution with

$$G_0(x) = \frac{1}{\Gamma(q)} \int_0^x t^{q-1} e^{-t}\, \mathrm{d}t, \quad x > 0, 0 < q \le 1,$$

and by Weibull distribution with

$$G_0(x) = q \int_0^x t^{q-1} e^{-t^q}\, \mathrm{d}t, \quad x > 0, 0 < q \le 1.$$

For a function $\psi : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, let ψ' denote its derivative. Recently, Khaledi *et al.* [11] proved the following results:

$$\psi_2(x) = x^2 r'_{G_0}(x) \text{ is decreasing (increasing) on } \mathbb{R}_+, \text{ and}$$
$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{1:n} \le_{\text{hr}} (\ge_{\text{hr}}) X_{1:n}; \tag{1.18}$$

G_0 has DHR, $\psi_2(x) = x^2 r'_{G_0}(x)$ is decreasing (increasing) on $\mathbb{R}_+$, and

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{1:n} \leq_{\mathrm{disp}} \ (\geq_{\mathrm{disp}}) X_{1:n}; \tag{1.19}$$

$$\psi_3(x) = x\tilde{r}_{G_0}(x) \text{ is decreasing on } \mathbb{R}_+, \text{ and } \boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\mathrm{st}} X_{n:n}; \tag{1.20}$$

and

$$\psi_4(x) = x^2 \tilde{r}'_{G_0}(x) \text{ is increasing (decreasing) on } \mathbb{R}_+, \text{ and } \boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\mathrm{rh}} \ (\geq_{\mathrm{rh}}) X_{n:n}. \tag{1.21}$$

Khaledi *et al.* [11] defined the generalized gamma distribution $GG(p, q)$ as

$$G_0(x) = \frac{p}{\Gamma(q/p)} \int_0^x t^{q-1} e^{-t^p} \, \mathrm{d}t, \quad x > 0, p > 0, q > 0. \tag{1.22}$$

Note that for $p = 1$ the generalized gamma distribution reduces to gamma distribution and for $p = q$ it reduces to Weibull distribution. For $GG(p, q)$ distribution, given by (1.22), Khaledi *et al.* [11] established that:

$$p \leq 1 \text{ and } q \leq 1 \, (p \geq 1 \text{ and } q \geq 1) \Rightarrow G_0 \text{ has DHR (IHR)}; \tag{1.23}$$

$$\text{for every } p, q > 0, \psi_1(x) = x r_{G_0}(x), x \in \mathbb{R}_+, \text{ is an increasing function}; \tag{1.24}$$

$$p < 1 \text{ and } q < 1 \, (p > 1 \text{ and } q > 1) \Rightarrow \psi_2(x) = x^2 r'_{G_0}(x), x \in \mathbb{R}_+, \text{ is a decreasing (an increasing) function}; \tag{1.25}$$

$$\text{for every } p, q > 0, \psi_3(x) = x\tilde{r}_{G_0}(x) \text{ is a decreasing function}; \tag{1.26}$$

and

$$p < 1 \Rightarrow \psi_4(x) = x^2 \tilde{r}'_{G_0}(x), x \in \mathbb{R}_+, \text{ is an increasing function}. \tag{1.27}$$

Thus, using result (1.17) of Hu [6], it follows that

$$p \leq 1, q \leq 1, \text{ and } \boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \leq^{\mathrm{st}} (X_{1:n}, \ldots, X_{n:n}). \tag{1.28}$$

This is an extension of a similar result proved by Lihong and Xinsheng [15] for gamma distributions ($p = 1$). Using results (1.18)–(1.21) and (1.23)–(1.27), for $GG(p, q)$ distributions, it follows that:

$$p < 1, q < 1 \, (p > 1, q > 1) \text{ and } \boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{1:n} \leq_{\mathrm{hr}} \ (\geq_{\mathrm{hr}}) X_{1:n}; \tag{1.29}$$

$$p < 1, q < 1 \text{ and } \boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{1:n} \leq_{\text{disp}} X_{1:n}; \tag{1.30}$$

$$\text{for every } p, q > 0, \boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\text{st}} X_{n:n}; \tag{1.31}$$

$$p < 1 \text{ and } \boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\text{rh}} X_{n:n}. \tag{1.32}$$

Now we will discuss some of the results on stochastic comparisons of order statistics from heterogeneous gamma (i.e. $GG(1, \alpha), \alpha > 0$) distributions. The following results are obvious consequences of results (1.16), (1.28), and (1.31).

$$\alpha > 1 \text{ and } \boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow X_{1:n} \leq_{\text{st}} Y_{1:n}; \tag{1.33}$$

$$\alpha \leq 1 \text{ and } \boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \leq^{\text{st}} (X_{1:n}, \ldots, X_{n:n}); \tag{1.34}$$

$$\forall \alpha > 0, \boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\text{st}} X_{n:n}. \tag{1.35}$$

It is worth mentioning here that results (1.33) and (1.34) were independently proved by Lihong and Xinsheng [15]. They also proved result (1.35) under the stronger condition $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu}$. Zhao and Balakrishnan [22] proved that

$$\alpha \leq 1 \text{ and } \mu_i = \left(\prod_{j=1}^{n} \lambda_j \right)^{1/n}, i = 1, \ldots, n \Rightarrow Y_{n:n} \leq_{\text{hr}} X_{n:n}. \tag{1.36}$$

Note that result (1.36) is a generalization of a result due to Dykstra *et al.* [5] for the exponential case ($\alpha = 1$). Zhao and Balakrishnan [23] considered the case when $n = 2$ and proved the following results:

$$\mu_1 = \lambda_2, \lambda_2 \geq \max(\lambda_1, \mu_2) \text{ and } \lambda_1 \leq \mu_2 \Rightarrow Y_{2:2} \leq_{\text{disp}} X_{2:2}; \tag{1.37}$$

$$\min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \text{ and } \lambda_1 \lambda_2 = \mu_1 \mu_2 \Rightarrow Y_{2:2} \leq_{\text{disp}} X_{2:2}; \tag{1.38}$$

$$\begin{gathered} \min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \leq \max(\mu_1, \mu_2) \leq \max(\lambda_1, \lambda_2), \text{ and} \\ (\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{\text{disp}} X_{2:2} \text{ and } Y_{2:2} \leq_{*} X_{2:2}; \end{gathered} \tag{1.39}$$

and

$$\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right) \overset{\text{m}}{\succeq} \left(\frac{1}{\mu_1}, \frac{1}{\mu_2} \right) \Rightarrow Y_{2:2} \leq_{*} X_{2:2}. \tag{1.40}$$

Using a counter example, they demonstrated that result (1.39) on dispersive ordering cannot be extended to $n \geq 3$ case.

For a discussion on some recent results on stochastic comparisons of order statistics from heterogeneous PHR and scale probability models the reader may also refer to Khaledi and Kochar [10] and Kochar and Xu [12].

In this paper we continue the study on stochastic comparisons of orders statistics from heterogeneous gamma distributions further by generalizing result (1.11), due to Dykstra *et al.* [5], from the exponential case to the gamma case. Specifically, in Section 3 of the paper, we show that

$$\forall \alpha > 0, \boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\mathrm{rh}} X_{n:n}.$$

This result may also be viewed as an extension of results (1.33)–(1.35), and generalization of a result proved by Lihong and Xinsheng [15].

2. Notation and definitions

Let X and Y be random variables having the distribution functions F and G, the probability density functions f and g, the hazard functions r and μ, and the reversed hazard functions $\tilde{r}$ and $\tilde{\mu}$, respectively. Let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ be the corresponding survival functions. When we say that a function is increasing (decreasing) it means that the function is non-decreasing (non-increasing). Moreover all the distributions under study shall be assumed to be absolutely continuous with support $\mathbb{R}_+$. For any probability density funtion h, we will assume that $\{x \in \mathbb{R} : h(x) > 0\} = \mathbb{R}_+$.

Definition 2.1. *X is said to be smaller than Y in the*

(i) *usual stochastic order (written as $X \leq_{\mathrm{st}} Y$) if $\bar{F}(t) \leq \bar{G}(t)$, $\forall t \in \mathbb{R}_+$;*

(ii) *hazard rate order (written as $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ is increasing in $t \in \mathbb{R}_+$, or equivalently if $r(t) \geq \mu(t), \forall t \in \mathbb{R}_+$;*

(iii) *reversed hazard rate order (written as $X \leq_{\mathrm{rh}} Y$) if $G(t)/F(t)$ is increasing in $t \in (0, \infty)$, or equivalently if $\tilde{r}(t) \leq \tilde{\mu}(t), \forall t \in (0, \infty)$;*

(iv) *mean residual life order (written as $X \leq_{\mathrm{mrl}} Y$) if*

$$E(X_t) \leq E(Y_t), \text{or} \int_t^\infty \bar{F}(x)\,\mathrm{d}x/\bar{F}(t) \leq \int_t^\infty \bar{G}(x)\,\mathrm{d}x/\bar{G}(t), \forall t \geq 0,$$

where $X_t = [X - t | X > t]$ and $Y_t = [Y - t | Y > t]$ are residual lifetimes of random variables X and Y, respectively, at time $t > 0$;

(v) *dispersive ordering (written as* $X \leq_{\text{disp}} Y$*) if* $G^{-1}(\beta) - G^{-1}(\alpha) \geq F^{-1}(\beta) - F^{-1}(\alpha)$ *whenever* $0 < \alpha \leq \beta < 1$*; here* F^{-1} *and* G^{-1} *are the right continuous inverses of* F *and* G*, respectively;*

(vi) *star ordering (written as* $X \leq_* Y$*) if* $G^{-1}F(x)/x$ *is increasing in* $x \in \mathbb{R}_+$*.*

It is well known that

$$X \leq_{\text{hr[rh]}} Y \Rightarrow X \leq_{\text{st}} Y, \text{ and } X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{mrl}} Y.$$

Definition 2.2. (i) *X is said to be log-concave on* $\mathbb{R}_+$ *if*

$$f(\alpha x + (1-\alpha)y) \geq (f(x))^{\alpha}(f(y))^{1-\alpha} \quad \text{whenever } x, y \in \mathbb{R}_+ \text{ and } \alpha \in (0,1);$$

(ii) *X (or F) is said to have an increasing hazard rate (IHR)*

$$\bar{F}(\alpha x + (1-\alpha)y) \geq (\bar{F}(x))^{\alpha}(\bar{F}(y))^{1-\alpha} \quad \text{whenever } x, y \in \mathbb{R}_+ \text{ and } \alpha \in (0,1);$$

(iii) *X (or F) is said to have an increasing hazard rate in average (IHRA) if* $\bar{F}(\beta x) \geq \bar{F}^{\beta}(x)$*, for all* $0 < \beta < 1$ *and* $x \in \mathbb{R}_+$*;*

(iv) *X (or F) is said to have decreasing hazard rate (DHR) if $\bar{F}$ is log-convex on* $\mathbb{R}_+$*;*

(v) *X (or F) is said to have decreasing reversed hazard rate (DRHR) if F is log-concave on* $(0, \infty)$*.*

It is well known that X is log-concave $\Rightarrow$ X has IHR $\Rightarrow$ X has IHRA. Also X is log-concave $\Rightarrow$ X is DRHR.

Definition 2.3. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be random vectors. Then* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in multivariate stochastic ordering (written as* $\mathbf{X} \leq^{\text{st}} \mathbf{Y}$*) if*

$$E(\phi(\mathbf{X})) \leq E(\phi(\mathbf{Y}))$$

for all componentwise increasing functions ϕ*.*

It is well known that $\mathbf{X} \leq^{\text{st}} \mathbf{Y}$ implies $X_i \leq_{\text{st}} Y_i$, $i = 1, \ldots, n$. For definitions and properties of various stochastic orders and aging classes, one may refer to Shaked and Shanthikumar [19], and Barlow and Proschan [1].

Definition 2.4. *Let* $\mathbb{R}^n$ *and* $\mathbb{R}^n_+$ *denote, respectively, the n-dimensional Euclidean space and the product space* $\underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n-times}$*, and let* $\mathbb{A} \subseteq \mathbb{R}^n$*.*

(i) *A point* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{A}$ *is said to majorize another point* $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{A}$ *(written as* $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$*) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \quad j = 1, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)},$$

where $x_{(1)} \leq \cdots \leq x_{(n)}$ *and* $y_{(1)} \leq \cdots \leq y_{(n)}$ *denote the increasing arrangements of the components of* $\mathbf{x}$ *and* $\mathbf{y}$*, respectively.*

(ii) *A point* $\mathbf{x} \in \mathbb{A}$ *is said to weakly majorize another point* $\mathbf{y} \in \mathbb{A}$ *(written as* $\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}$*) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \quad j = 1, \ldots, n.$$

(iii) *A point* $\mathbf{x} \in \mathbb{R}_+^n$ *is said to be p-larger than another point* $\mathbf{y} \in \mathbb{R}_+^n$ *(written as* $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$*) if*

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, \quad j = 1, \ldots, n.$$

(iv) *A point* $\mathbf{x} \in \mathbb{R}_+^n$ *is said to reciprocal majorize another point* $\mathbf{y} \in \mathbb{R}_+^n$ *(written as* $\mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$*) if*

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \geq \sum_{i=1}^{j} \frac{1}{y_{(i)}}, \quad j = 1, \ldots, n.$$

(v) *A function* $\psi : \mathbb{A} \to \mathbb{R}$ *is said to be Schur-convex (Schur-concave) on* $\mathbb{A}$ *if*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \psi(\mathbf{x}) \geq [\leq] \, \psi(\mathbf{y}) \text{ for } \mathbf{x}, \mathbf{y} \in \mathbb{A}.$$

It is well known (see Khaledi and Kochar [8], Kochar and Xu [14] and Zhao and Balakrishnan [20]) that, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$,

$$\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$$

Readers may also refer to Marshall and Olkin [16] and Bon and Păltănea [3] for comprehensive details of majorization and p-larger order.

3. Comparison of reversed hazard rates

Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be independent gamma random variables with X_i (Y_i) having probability density function $f(x; \alpha, \lambda_i)$ $(f(x; \alpha, \mu_i))$, $\lambda_i > 0$ $(\mu_i > 0)$, $i = 1, \ldots, n$, where for $\lambda > 0$ and $\alpha > 0$,

$$f(x; \alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda x} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

To prove the main result we need the following lemmas.

Lemma 3.1. *Let W be a random variable having the probability density function*

$$h(u; \alpha, y) = \begin{cases} \dfrac{(1-u)^{\alpha-1} e^{yu}}{\int_0^1 (1-t)^{\alpha-1} e^{yt}\, \mathrm{d}t} & \textit{if } 0 < u < 1, \\ 0 & \textit{otherwise}, \end{cases} \tag{3.1}$$

where α and y are given positive constants. Then, W has IHR.

Proof. We consider the following two cases:
Case I: $0 < \alpha < 1$ and $y > 0$.
Let $\bar{H}$ denote the survival function of W. Then

$$K_1(u) = \frac{\mathrm{d}}{\mathrm{d}u} \ln \bar{H}(u) = -\frac{(1-u)^{\alpha-1} e^{yu}}{\int_u^1 (1-t)^{\alpha-1} e^{yt}\, \mathrm{d}t}, \quad 0 < u < 1.$$

Clearly, $K_1(u)$ is a decreasing function on $(0, 1)$. Therefore $\ln \bar{H}(\cdot)$ is concave on $(0, 1)$ and hence W has IHR.
Case II: $\alpha \geq 1$ and $y > 0$.
Using (3.1) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} \ln h(u) = -\frac{\alpha - 1}{(1-u)^2} \leq 0, \quad \forall u \in (0, 1),$$

i.e. $\ln h(\cdot)$ is concave on $(0, 1)$. Then it follows that W is log-concave, which in turn implies that W has IHR. □

Lemma 3.2. *(Barlow and Proschan [1, p. 118].) Let X be a nonnegative random variable with distribution function F and let $\mu_r = \int_0^\infty x^r\, \mathrm{d}F(x)$, $r = 1, 2$. If X has IHRA then $\mu_2 \leq 2\mu_1^2$.*

Theorem 3.1. *For any $\alpha > 0$ and $n \geq 2$*

$$\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\mathrm{rh}} X_{n:n}.$$

Proof. Fix $x > 0$. Then the reversed hazard rate of $X_{n:n}$ is

$$\tilde{r}_{n:n}(\boldsymbol{\lambda}, x) = \frac{1}{x} \sum_{i=1}^{n} \frac{(\lambda_i x)^{\alpha}\, e^{-\lambda_i x}}{\int_0^{\lambda_i x} u^{\alpha-1}\, e^{-u}\, \mathrm{d}u} = \frac{1}{x} \sum_{i=1}^{n} \varphi(\lambda_i x),$$

where

$$\varphi(y) = \frac{y^{\alpha}\, e^{-y}}{\int_0^{y} u^{\alpha-1}\, e^{-u}\, \mathrm{d}u} = \frac{1}{\int_0^1 (1-u)^{\alpha-1}\, e^{yu}\, \mathrm{d}u}, \quad y > 0.$$

In view of Theorem A.8 of Marshall and Olkin [16, p. 59] it suffices to show that, for each $x > 0$, $\tilde{r}_{n:n}(\boldsymbol{\lambda}, x)$ is decreasing in each λ_i, $i = 1, \ldots, n$, and is a Schur-convex function of $\boldsymbol{\lambda} \in \mathbb{R}_{+}^{n}$. Fix $x > 0$. Clearly $\tilde{r}_{n:n}(\boldsymbol{\lambda}, x)$ is decreasing in each λ_i, $i = 1, \ldots, n$. Then, in the light of Proposition C.1 of Marshall and Olkin [16, p. 64], it suffices to show that $\varphi(\cdot)$ is a convex function on $(0, \infty)$. Define

$$\psi(y) = \frac{1}{\varphi(y)} = \int_0^1 (1-u)^{\alpha-1}\, e^{yu}\, \mathrm{d}u, \quad y > 0.$$

Then, for $y > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}y} \varphi(y) = -\frac{\frac{\mathrm{d}}{\mathrm{d}y} \psi(y)}{\psi^2(y)},$$

and

$$\begin{aligned} \psi(y) \frac{\mathrm{d}^2}{\mathrm{d}y^2} \varphi(y) &= 2 \left(\frac{\frac{\mathrm{d}}{\mathrm{d}y} \psi(y)}{\psi(y)} \right)^2 - \frac{\frac{\mathrm{d}^2}{\mathrm{d}y^2} \psi(y)}{\psi(y)} \\ &= 2 \left(\frac{\int_0^1 u(1-u)^{\alpha-1}\, e^{yu}\, \mathrm{d}u}{\int_0^1 (1-u)^{\alpha-1}\, e^{yu}\, \mathrm{d}u} \right)^2 - \frac{\int_0^1 u^2 (1-u)^{\alpha-1}\, e^{yu}\, \mathrm{d}u}{\int_0^1 (1-u)^{\alpha-1}\, e^{yu}\, \mathrm{d}u} \\ &= 2(E[W])^2 - E[W^2], \end{aligned} \tag{3.2}$$

where W is a random variable as defined in Lemma 3.1. Since the random variable W has IHR, it follows that W has IHRA. Now, from Lemma 3.2, it follows that $2\mu_1^2 = 2(E[W])^2 \geq E[W^2] = \mu_2$, i.e. $\frac{\mathrm{d}^2}{\mathrm{d}y^2} \varphi(y) \geq 0, \forall y > 0$. Hence $\varphi(y)$ is convex in $y \in (0, \infty)$. □

The following corollary is immediate from the above theorem and on using the fact that $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow \boldsymbol{\lambda} \overset{\text{w}}{\succeq} \boldsymbol{\mu}$.

Corollary 3.1. *For any $\alpha > 0$ and $n \geq 2$,*

$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{\text{rh}} X_{n:n},$$

i.e. the reversed hazard rate of $X_{n:n}$ is Schur-convex in $\boldsymbol{\lambda}$.

Recall that Theorem 3.1 for the particular case $\alpha = 1$ was proved by Dykstra *et al.* [5] under a stronger condition of $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu}$. Thus Theorem 3.1 is a generalization of the result proved by Dykstra *et al.* [5]. Lihong and Xinsheng [15] proved that, for any $\alpha > 0$, $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu}$ implies $Y_{n:n} \leq_{\text{st}} X_{n:n}$. Using the result of of Khaledi *et al.* [11] it follows that the result of Lihong and Xinsheng [15] holds under a weaker condition $\boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\mu}$. Thus, Corollary 3.1 also generalizes the result of Lihong and Xinsheng [15].

One may wonder whether the conclusion of Theorem 3.1 will remain true if we replace $\overset{\text{w}}{\succeq}$ by $\overset{\text{p}}{\succeq}$. The following example shows that the answer of this question is negative.

Example 3.1. For $n = 2$ and $\alpha = 1$, let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = (1, 55)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2) = (2, 44)$. It is easy to verify that $\boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \overset{\text{w}}{\not\succeq} \boldsymbol{\mu}$. For $x > 0$

$$\begin{aligned}\tilde{r}^*(x) &= \tilde{r}_{2:2}(\boldsymbol{\lambda}, x) - \tilde{r}_{2:2}(\boldsymbol{\mu}, x) \\ &= \frac{1}{x}\left(\varphi(x) + \varphi(55x) - \varphi(2x) - \varphi(44x)\right),\end{aligned}$$

where $\varphi(y) = y/(e^y - 1), y > 0$. It is easy to verify that $\tilde{r}^*(0.2) = 0.444451$ and $\tilde{r}^*(0.04) = -1.7994$. Thus $X_{2:2} \not\geq_{\text{rh}} Y_{2:2}$.

4. Conclusions and Comments

Consider a parallel system $\mathcal{P}$ consisting of $n\,(\geq 2)$ components having random lifetimes $X_1, \ldots, X_n$. Let $X_1, \ldots, X_n$ be independent gamma random variables with X_i having shape parameter α and mean $\frac{\alpha}{\lambda_i}, \alpha > 0, \lambda_i > 0$, $i = 1, \ldots, n$. In this paper we prove that for any $\alpha > 0$

$$\boldsymbol{\lambda} \overset{\text{w}}{\succeq} \boldsymbol{\mu} \Rightarrow X_{n:n} \geq_{\text{rh}} Y_{n:n}, \qquad n \geq 2.$$

A consequence of this result is that the reversed hazard rate of $\mathcal{P}$ is Schur-convex in $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, i.e. for any $\alpha > 0$

$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\mu} \Rightarrow X_{n:n} \geq_{\text{rh}} Y_{n:n}, \qquad n \geq 2.$$

With the help of an example we also show that $\boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{2:2} \leq_{\text{rh}} X_{2:2}$ may not hold even for exponential distribution (i.e. for $\alpha = 1$).

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