On comparison of reversed hazard rates of two parallel systems comprising of independent gamma components

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Abstract

Let $X_1, \ldots, X_n$ ($Y_1, \ldots, Y_n$) be independent random variables such that $X_i$ ($Y_i$) follows the gamma distribution with shape parameter $\alpha$ and mean $\frac{\alpha}{\lambda_i}$, $\alpha > 0, \lambda_i > 0$ ($\mu_i > 0$), $i = 1, \ldots, n$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ and let $\tilde{r}_{n.n}(\lambda; x)$ ($\tilde{r}_{n.n}(\mu; x)$) denote the reversed hazard rate of $\max\{X_1, \ldots, X_n\}$ ($\max\{Y_1, \ldots, Y_n\}$). In this note we show that if $\lambda$ weakly majorizes $\mu$ then $\tilde{r}_{n.n}(\lambda; x) \geq \tilde{r}_{n.n}(\mu; x)$, $\forall x > 0$, thereby strengthening and extending the results of Dykstra \textit{et al.} [5], Khaledi \textit{et al.} [11], and Lihong and Xinsheng [15].

Keywords: Gamma distribution, Hazard rate order, Majorization, Order statistics, Reversed hazard rate order, Usual stochastic order.

1. Introduction and a review of literature

Let $X_1, \ldots, X_n$ be independent and nonnegative random variables (i.e. corresponding distributions have the common support $\mathbb{R}_+ \equiv [0, \infty)$) representing the lifetimes of $n$ components and let $Y_1, \ldots, Y_n$ be another set of independent and nonnegative random variables representing the lifetimes of another set of $n$ components. For $k \in \{1, \ldots, n\}$, let $X_{k:n}$ and $Y_{k:n}$ respectively denote the $k$th order statistics based on random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Then $X_{k:n}$ and $Y_{k:n}$ are the lifetimes of $(n - k + 1)$-out-of-$n$ systems constructed from the two sets of components and thus a stochastic

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comparison of these two random variables may be of interest. A vast literature on stochastic comparisons of order statistics from two heterogeneous distributions is available. In order to provide a review of the literature on this topic we will require definitions of some stochastic orders and the concept of majorization, for which we refer the reader to Section 2 of the paper.

Suppose that the random variables $X_i$ and $Y_i$ have absolutely continuous distribution functions $F(x; \lambda_i)$ and $F(x; \mu_i)$, respectively, where $\lambda_i, \mu_i > 0$, $i = 1, \ldots, n$. Let $\bar{F}(x; \lambda_i) = 1 - F(x; \lambda_i)$ and $\bar{F}(x; \mu_i) = 1 - F(x; \mu_i)$ be the corresponding survival functions. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$.

First we will discuss results on stochastic comparisons of order statistics under the proportional hazard rates (PHR) model (i.e. $\bar{F}(x; \lambda_i) = [\bar{F}_0(x)]^{\lambda_i}, x \in \mathbb{R} \equiv (-\infty, \infty), \lambda > 0$, for some survival function $\bar{F}_0$). Under the PHR model, Pledger and Proschan [17] proved that

$$\lambda \succeq \mu \Rightarrow Y_{k:n} \leq_{st} X_{k:n}, \quad k = 1, \ldots, n.$$  \hspace{1cm} (1.1)

Proschan and Sethuraman [18] strengthened this result from componentwise stochastic ordering to multivariate stochastic ordering by proving that

$$\lambda \succeq \mu \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \leq_{st} (X_{1:n}, \ldots, X_{n:n}).$$  \hspace{1cm} (1.2)

For two-component parallel systems, Boland et al. [2] strengthened result (1.1) of Pledger and Proschan [17] by showing that

$$(\lambda_1, \lambda_2) \succeq (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{hr} X_{2:2}.$$  \hspace{1cm} (1.3)

Using an example they also demonstrated that this result may not hold for $n \geq 3$ component parallel systems. However, for two-component parallel systems with exponentially distributed lifetimes, Dykstra et al. [5] showed that

$$(\lambda_1, \lambda_2) \succeq (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{hr} X_{2:2}.$$  \hspace{1cm} (1.4)

For the PHR model, Da et al. [4] further studied this problem and proved that

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \text{ and } \lambda_2 - \lambda_1 \geq \mu_2 - \mu_1 \Rightarrow Y_{n:n} \leq_{hr} X_{n:n}. \hspace{1cm} (1.5)$$

Zhao and Balakrishnan [20] extended results (1.3) and (1.4) by establishing that:

$$\min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \leq \max(\mu_1, \mu_2) \leq \max(\lambda_1, \lambda_2) \text{ and } \hspace{1cm} (1.6)$$

$$(\lambda_1, \lambda_2) \succeq (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{hr} X_{2:2};$$
and
\[
\min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \leq \max(\mu_1, \mu_2) \leq \max(\lambda_1, \lambda_2) \ 	ext{and} \ 
\begin{aligned}
(\lambda_1, \lambda_2) &\succeq (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{tr} X_{2:2}.
\end{aligned}
\tag{1.7}
\]

Earlier, Joo and Mi [7] had proved a weaker version of result (1.7) for the particular case of exponential distribution (i.e. $F_0(x) = e^{-x}, x \in \mathbb{R}^+\). For parallel systems, Khaledi and Kochar [9] generalized result (1.1), due to Pledger and Proschan [17], in another direction by establishing that
\[
\lambda^p \succeq \mu \Rightarrow Y_{n:n} \leq_{st} X_{n:n}. \tag{1.8}
\]

Using an example they demonstrated that this result may not hold for other order statistics. Khaledi and Kochar [9] also showed that:
\[
\mu_i = \left( \prod_{j=1}^{n} \lambda_j \right)^{1/n}, \ i = 1, \ldots, n \Rightarrow Y_{n:n} \leq_{hr} X_{n:n}; \tag{1.9}
\]

and
\[
F_0 \text{ has DHR and } \mu_i = \left( \prod_{j=1}^{n} \lambda_j \right)^{1/n}, \ i = 1, \ldots, n \Rightarrow Y_{n:n} \leq_{disp} X_{n:n}. \tag{1.10}
\]

Earlier, Khaledi and Kochar [8] had proved results (1.8)–(1.10) for the particular case of exponential distribution. Moreover, Dykstra et al. [5] had proved weaker forms of the results proved by Khaledi and Kochar [8]. For exponential distributions with $\mu_i = \frac{1}{n} \sum_{j=1}^{n} \lambda_j, \ i = 1, \ldots, n, \ Dykstra \ et \ al. \ [5]$ established that $Y_{n:n} \leq_{hr} X_{n:n}$ and $Y_{n:n} \leq_{disp} X_{n:n}$. For exponential distributions, Dykstra et al. [5] also showed that
\[
\lambda^m \succeq \mu \Rightarrow Y_{n:n} \leq_{rh} X_{n:n}. \tag{1.11}
\]

Kochar and Xu [13] demonstrated that result (1.9), due to Khaledi and Kochar [9], may not hold if we replace $\leq_{hr}$ ordering by $\leq_{rh}$ ordering or by $\leq_{lr}$ ordering. However, they extended result (1.9) by proving that
\[
\mu_i = \frac{1}{n} \sum_{j=1}^{n} \lambda_j, \ i = 1, \ldots, n \Rightarrow Y_{n:n} \leq_{lr} X_{n:n}. \tag{1.12}
\]
Recently, for two-component parallel systems, Zhao and Balakrishnan [21] proved that:

\( F_0 \) has DHR, \( \min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \leq \max(\mu_1, \mu_2) \leq \max(\lambda_1, \lambda_2) \) and

\[
(\lambda_1, \lambda_2)^{\text{rm}} \geq (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{\text{mrl}} X_{2:2}.
\] (1.13)

Now we will provide a discussion on stochastic comparisons of order statistics under the scale model (i.e. \( F(x; \lambda) = G_0(\lambda x), x \in \mathbb{R}, \lambda > 0, \) for some distribution function \( G_0 \)). Let \( r_{G_0} \) and \( \tilde{r}_{G_0} \) respectively denote the hazard function and the reversed hazard function of \( G_0 \). Pledger and Proschan [17] proved the following results:

\( G_0 \) has DHR and \( \lambda \geq \mu \Rightarrow Y_{k:n} \leq_{\text{st}} X_{k:n}, k = 1, \ldots, n; \) \hspace{1cm} (1.14)

\( G_0 \) has DRHR and \( \lambda \geq \mu \Rightarrow Y_{n:n} \leq_{\text{st}} X_{n:n}; \) \hspace{1cm} (1.15)

and

\( G_0 \) has IHR and \( \lambda \geq \mu \Rightarrow X_{1:n} \leq_{\text{st}} Y_{1:n}. \) \hspace{1cm} (1.16)

Hu [6] extended result (1.14) by showing that:

\( G_0 \) has DHR, \( \psi_1(x) = x r_{G_0}(x), x \in \mathbb{R}_+, \) is an increasing function and

\[ \lambda \geq \mu \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \leq_{\text{st}} (X_{1:n}, \ldots, X_{n:n}). \] (1.17)

It is worth mentioning here that the conditions of Hu [6] are satisfied by the gamma distribution with

\[
G_0(x) = \frac{1}{\Gamma(q)} \int_0^x t^{q-1} e^{-t} \, dt, \quad x > 0, 0 < q \leq 1,
\]

and by Weibull distribution with

\[
G_0(x) = q \int_0^x t^{q-1} e^{-t^q} \, dt, \quad x > 0, 0 < q \leq 1.
\]

For a function \( \psi : A \to \mathbb{R}, \) where \( A \subseteq \mathbb{R}, \) let \( \psi' \) denote its derivative. Recently, Khaledi et al. [11] proved the following results:

\( \psi_2(x) = x^2 r'_{G_0}(x) \) is decreasing (increasing) on \( \mathbb{R}_+, \) and

\[
\lambda \geq \mu \Rightarrow Y_{1:n} \leq_{\text{hr}} (\geq_{\text{hr}}) X_{1:n};
\] (1.18)
$G_0$ has DHR, $\psi_2(x) = x^2 r'_{G_0}(x)$ is decreasing (increasing) on $\mathbb{R}_+$, and
\[ \lambda \succeq \mu \Rightarrow Y_{1:n} \leq_{\text{disp}} (\geq_{\text{disp}}) X_{1:n}; \] (1.19)

$\psi_3(x) = x r_{G_0}(x)$ is decreasing on $\mathbb{R}_+$, and $\lambda \succeq \mu \Rightarrow Y_{n:n} \leq_{\text{st}} X_{n:n};$ (1.20)

and
\[
\psi_4(x) = x^2 r'_{G_0}(x) \text{ is increasing (decreasing) on } \mathbb{R}_+, \quad \lambda \succeq \mu \Rightarrow Y_{n:n} \leq_{\text{st}} (\geq_{\text{st}}) X_{n:n}.
\] (1.21)

Khaledi et al. [11] defined the generalized gamma distribution $GG(p, q)$ as
\[
G_0(x) = \frac{p}{\Gamma(q/p)} \int_0^x t^{q-1}e^{-t^{p}} \, dt, \quad x > 0, p > 0, q > 0.
\] (1.22)

Note that for $p = 1$ the generalized gamma distribution reduces to gamma distribution and for $p = q$ it reduces to Weibull distribution. For $GG(p, q)$ distribution, given by (1.22), Khaledi et al. [11] established that:

$p \leq 1 \text{ and } q \leq 1 \text{ (} p \geq 1 \text{ and } q \geq 1 \implies G_0 \text{ has DHR (IHR);} \]
\[ (1.23) \]

for every $p, q > 0, \psi_1(x) = x r_{G_0}(x), x \in \mathbb{R}_+$, is an increasing function; (1.24)

$p < 1 \text{ and } q < 1 \text{ (} p > 1 \text{ and } q > 1 \implies \psi_2(x) = x^2 r'_{G_0}(x), x \in \mathbb{R}_+, \text{ is a decreasing (an increasing) function;} \]
\[ (1.25) \]

for every $p, q > 0, \psi_3(x) = x r_{G_0}(x)$ is a decreasing function; (1.26)

and
\[ p < 1 \Rightarrow \psi_4(x) = x^2 r'_{G_0}(x), x \in \mathbb{R}_+, \text{ is an increasing function.} \] (1.27)

Thus, using result (1.17) of Hu [6], it follows that
\[ p \leq 1, q \leq 1, \text{ and } \lambda \succeq \mu \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \leq_{\text{st}} (X_{1:n}, \ldots, X_{n:n}). \] (1.28)

This is an extension of a similar result proved by Lihong and Xinsheng [15] for gamma distributions ($p = 1$). Using results (1.18)–(1.21) and (1.23)–(1.27), for $GG(p, q)$ distributions, it follows that:

$p < 1, q < 1 \text{ (} p > 1, q > 1 \text{) and } \lambda \succeq \mu \Rightarrow Y_{1:n} \leq_{\text{hr}} (\geq_{\text{hr}}) X_{1:n}; \] (1.29)
\[ p < 1, q < 1 \text{ and } \lambda^m \geq \mu \Rightarrow Y_{1:n} \leq_{\text{disp}} X_{1:n}; \quad (1.30) \]

for every \( p, q > 0, \lambda \geq \mu \Rightarrow Y_{n:n} \leq_{\text{st}} X_{n:n}; \quad (1.31) \]
\[ p < 1 \text{ and } \lambda^m \geq \mu \Rightarrow Y_{n:n} \leq_{\text{st}} X_{n:n}. \quad (1.32) \]

Now we will discuss some of the results on stochastic comparisons of order statistics from heterogeneous gamma (i.e. \( GG(1, \alpha), \alpha > 0 \)) distributions. The following results are obvious consequences of results (1.16), (1.28), and (1.31).
\[ \alpha > 1 \text{ and } \lambda^m \geq \mu \Rightarrow X_{1:n} \leq_{\text{st}} Y_{1:n}; \quad (1.33) \]
\[ \alpha \leq 1 \text{ and } \lambda^m \geq \mu \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \leq_{\text{st}} (X_{1:n}, \ldots, X_{n:n}); \quad (1.34) \]
\[ \forall \alpha > 0, \lambda^p \geq \mu \Rightarrow Y_{n:n} \leq_{\text{st}} X_{n:n}. \quad (1.35) \]

It is worth mentioning here that results (1.33) and (1.34) were independently proved by Lihong and Xinsheng [15]. They also proved result (1.35) under the stronger condition \( \lambda^m \geq \mu \). Zhao and Balakrishnan [22] proved that
\[ \alpha \leq 1 \text{ and } \mu_i = \left( \prod_{j=1}^{n} \lambda_j \right)^{1/n}, \quad i = 1, \ldots, n \Rightarrow Y_{n:n} \leq_{\text{hr}} X_{n:n}. \quad (1.36) \]

Note that result (1.36) is a generalization of a result due to Dykstra et al. [5] for the exponential case (\( \alpha = 1 \)). Zhao and Balakrishnan [23] considered the case when \( n = 2 \) and proved the following results:
\[ \mu_1 = \lambda_2, \lambda_2 \geq \max(\lambda_1, \mu_2) \text{ and } \lambda_1 \leq \mu_2 \Rightarrow Y_{2:2} \leq_{\text{disp}} X_{2:2}; \quad (1.37) \]
\[ \min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \text{ and } \lambda_1 \lambda_2 = \mu_1 \mu_2 \Rightarrow Y_{2:2} \leq_{\text{disp}} X_{2:2}; \quad (1.38) \]
\[ \min(\lambda_1, \lambda_2) \leq \min(\mu_1, \mu_2) \leq \max(\mu_1, \mu_2) \leq \max(\lambda_1, \lambda_2), \text{ and} \]
\[ (\lambda_1, \lambda_2)^p \geq (\mu_1, \mu_2) \Rightarrow Y_{2:2} \leq_{\text{disp}} X_{2:2} \text{ and } Y_{2:2} \leq_{\ast} X_{2:2}; \quad (1.39) \]
and
\[ \left( \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right)^m \geq \left( \frac{1}{\mu_1}, \frac{1}{\mu_2} \right) \Rightarrow Y_{2:2} \leq_{\ast} X_{2:2}. \quad (1.40) \]

Using a counter example, they demonstrated that result (1.39) on dispersive ordering cannot be extended to \( n \geq 3 \) case.
For a discussion on some recent results on stochastic comparisons of order statistics from heterogeneous PHR and scale probability models the reader may also refer to Khaledi and Kochar [10] and Kochar and Xu [12].

In this paper we continue the study on stochastic comparisons of order statistics from heterogeneous gamma distributions further by generalizing result (1.11), due to Dykstra et al. [5], from the exponential case to the gamma case. Specifically, in Section 3 of the paper, we show that

\[ \forall \alpha > 0, \lambda \geq \mu \Rightarrow Y_{n:n} \leq_{rh} X_{n:n}. \]

This result may also be viewed as an extension of results (1.33)–(1.35), and generalization of a result proved by Lihong and Xinsheng [15].

2. Notation and definitions

Let \( X \) and \( Y \) be random variables having the distribution functions \( F \) and \( G \), the probability density functions \( f \) and \( g \), the hazard functions \( r \) and \( \mu \), and the reversed hazard functions \( \tilde{r} \) and \( \tilde{\mu} \), respectively. Let \( \bar{F} = 1 - F \) and \( \bar{G} = 1 - G \) be the corresponding survival functions. When we say that a function is increasing (decreasing) it means that the function is non-decreasing (non-increasing). Moreover all the distributions under study shall be assumed to be absolutely continuous with support \( \mathbb{R}^+ \). For any probability density function \( h \), we will assume that \( \{ x \in \mathbb{R} : h(x) > 0 \} = \mathbb{R}^+ \).

**Definition 2.1.** \( X \) is said to be smaller than \( Y \) in the

(i) usual stochastic order (written as \( X \leq_{st} Y \)) if \( \bar{F}(t) \leq \bar{G}(t) \), \( \forall t \in \mathbb{R}^+ \);

(ii) hazard rate order (written as \( X \leq_{hr} Y \)) if \( \bar{G}(t)/\bar{F}(t) \) is increasing in \( t \in \mathbb{R}^+ \), or equivalently if \( r(t) \geq \mu(t) \), \( \forall t \in \mathbb{R}^+ \);

(iii) reversed hazard rate order (written as \( X \leq_{rh} Y \)) if \( G(t)/F(t) \) is increasing in \( t \in (0, \infty) \), or equivalently if \( \tilde{r}(t) \leq \tilde{\mu}(t) \), \( \forall t \in (0, \infty) \);

(iv) mean residual life order (written as \( X \leq_{mrl} Y \)) if

\[ E(X_t) \leq E(Y_t), \text{or } \int_t^\infty \bar{F}(x)/\bar{F}(t) \, dx \leq \int_t^\infty \bar{G}(x)/\bar{G}(t) \, dx, \forall t \geq 0, \]

where \( X_t = [X - t | X > t] \) and \( Y_t = [Y - t | Y > t] \) are residual lifetimes of random variables \( X \) and \( Y \), respectively, at time \( t > 0 \);
(v) dispersive ordering (written as $X \leq_{\text{disp}} Y$) if $G^{-1}(\beta) - G^{-1}(\alpha) \geq F^{-1}(\beta) - F^{-1}(\alpha)$ whenever $0 < \alpha \leq \beta < 1$; here $F^{-1}$ and $G^{-1}$ are the right continuous inverses of $F$ and $G$, respectively;

(vi) star ordering (written as $X \leq_{\ast} Y$) if $G^{-1}F(x)/x$ is increasing in $x \in \mathbb{R}_+$.

It is well known that

$$X \leq_{\text{hr[rh]}} Y \Rightarrow X \leq_{\ast} Y, \text{ and } X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{mrl}} Y.$$

**Definition 2.2.**

(i) $X$ is said to be log-concave on $\mathbb{R}_+$ if

$$f(\alpha x + (1 - \alpha)y) \geq (f(x))^{\alpha}(f(y))^{1-\alpha} \text{ whenever } x, y \in \mathbb{R}_+ \text{ and } \alpha \in (0, 1);$$

(ii) $X$ (or $F$) is said to have an increasing hazard rate (IHR)

$$F(\alpha x + (1 - \alpha)y) \geq (F(x))^{\alpha}(F(y))^{1-\alpha} \text{ whenever } x, y \in \mathbb{R}_+ \text{ and } \alpha \in (0, 1);$$

(iii) $X$ (or $F$) is said to have an increasing hazard rate in average (IHRA) if $F(\beta x) \geq F^\beta(x)$, for all $0 < \beta < 1$ and $x \in \mathbb{R}_+$;

(iv) $X$ (or $F$) is said to have decreasing hazard rate (DHR) if $F$ is log-convex on $\mathbb{R}_+$;

(v) $X$ (or $F$) is said to have decreasing reversed hazard rate (DRHR) if $F$ is log-concave on $(0, \infty)$.

It is well known that $X$ is log-concave $\Rightarrow$ $X$ has IHR $\Rightarrow$ $X$ has IHRA. Also $X$ is log-concave $\Rightarrow$ $X$ is DRHR.

**Definition 2.3.** Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be random vectors. Then $X$ is said to be smaller than $Y$ in multivariate stochastic ordering (written as $X \leq_{\ast} Y$) if

$$E(\phi(X)) \leq E(\phi(Y))$$

for all componentwise increasing functions $\phi$.

It is well known that $X \leq_{\ast} Y$ implies $X_i \leq_{\ast} Y_i$, $i = 1, \ldots, n$. For definitions and properties of various stochastic orders and aging classes, one may refer to Shaked and Shanthikumar [19], and Barlow and Proschan [1].

**Definition 2.4.** Let $\mathbb{R}^n$ and $\mathbb{R}_+^n$ denote, respectively, the $n$-dimensional Euclidean space and the product space $\mathbb{R}_+ \times \cdots \times \mathbb{R}_+$, and let $A \subseteq \mathbb{R}^n$. 

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(i) A point \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{A} \) is said to majorize another point \( \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{A} \) (written as \( \mathbf{x} \geq_m \mathbf{y} \)) if

\[
\sum_{i=1}^{j} x(i) \leq \sum_{i=1}^{j} y(i), \quad j = 1, \ldots, n - 1, \quad \text{and} \quad \sum_{i=1}^{n} x(i) = \sum_{i=1}^{n} y(i),
\]

where \( x(1) \leq \cdots \leq x(n) \) and \( y(1) \leq \cdots \leq y(n) \) denote the increasing arrangements of the components of \( \mathbf{x} \) and \( \mathbf{y} \), respectively.

(ii) A point \( \mathbf{x} \in \mathbb{A} \) is said to weakly majorize another point \( \mathbf{y} \in \mathbb{A} \) (written as \( \mathbf{x} \geq_w \mathbf{y} \)) if

\[
\sum_{i=1}^{j} x(i) \leq \sum_{i=1}^{j} y(i), \quad j = 1, \ldots, n.
\]

(iii) A point \( \mathbf{x} \in \mathbb{R}^n_+ \) is said to be \( p \)-larger than another point \( \mathbf{y} \in \mathbb{R}^n_+ \) (written as \( \mathbf{x} \geq_p \mathbf{y} \)) if

\[
\prod_{i=1}^{j} x(i) \leq \prod_{i=1}^{j} y(i), \quad j = 1, \ldots, n.
\]

(iv) A point \( \mathbf{x} \in \mathbb{R}^n_+ \) is said to reciprocal majorize another point \( \mathbf{y} \in \mathbb{R}^n_+ \) (written as \( \mathbf{x} \geq_{rm} \mathbf{y} \)) if

\[
\sum_{i=1}^{j} \frac{1}{x(i)} \geq \sum_{i=1}^{j} \frac{1}{y(i)}, \quad j = 1, \ldots, n.
\]

(v) A function \( \psi : \mathbb{A} \to \mathbb{R} \) is said to be Schur-convex (Schur-concave) on \( \mathbb{A} \) if

\[
\mathbf{x} \geq_m \mathbf{y} \implies \psi(\mathbf{x}) \geq [\leq] \psi(\mathbf{y}) \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{A}.
\]

It is well known (see Khaledi and Kochar [8], Kochar and Xu [14] and Zhao and Balakrishnan [20]) that, for \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+ \),

\[
\mathbf{x} \geq_m \mathbf{y} \Rightarrow \mathbf{x} \geq_w \mathbf{y} \Rightarrow \mathbf{x} \geq_p \mathbf{y} \Rightarrow \mathbf{x} \geq_{rm} \mathbf{y}
\]

Readers may also refer to Marshall and Olkin [16] and Bon and Páltánea [3] for comprehensive details of majorization and \( p \)-larger order.
3. Comparison of reversed hazard rates

Let $X_1, \ldots, X_n \ (Y_1, \ldots, Y_n)$ be independent gamma random variables with $X_i \ (Y_i)$ having probability density function $f(x; \alpha, \lambda_i) \ (f(x; \alpha, \mu_i))$, $\lambda_i > 0 \ (\mu_i > 0), \ i = 1, \ldots, n$, where for $\lambda > 0$ and $\alpha > 0$,

$$f(x; \alpha, \lambda) = \begin{cases} \frac{x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha) \lambda^{\alpha}} & x > 0, \\ 0 & \text{otherwise}. \end{cases}$$

To prove the main result we need the following lemmas.

Lemma 3.1. Let $W$ be a random variable having the probability density function

$$h(u; \alpha, y) = \begin{cases} \frac{(1-u)^{\alpha-1} e^{yu}}{\int_0^u (1-t)^{\alpha-1} e^{yt} dt} & 0 < u < 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $\alpha$ and $y$ are given positive constants. Then, $W$ has IHR.

Proof. We consider the following two cases:

Case I: $0 < \alpha < 1$ and $y > 0$.

Let $\bar{H}$ denote the survival function of $W$. Then

$$K_1(u) = \frac{d}{du} \ln \bar{H}(u) = -\frac{(1-u)^{\alpha-1} e^{yu}}{\int_u^1 (1-t)^{\alpha-1} e^{yt} dt}, \quad 0 < u < 1.$$

Clearly, $K_1(u)$ is a decreasing function on $(0, 1)$. Therefore $\ln \bar{H}(\cdot)$ is concave on $(0, 1)$ and hence $W$ has IHR.

Case II: $\alpha \geq 1$ and $y > 0$.

Using (3.1) we have

$$\frac{d^2}{du^2} \ln h(u) = -\frac{\alpha - 1}{(1-u)^2} \leq 0, \quad \forall u \in (0, 1),$$

i.e. $\ln h(\cdot)$ is concave on $(0, 1)$. Then it follows that $W$ is log-concave, which in turn implies that $W$ has IHR.

Lemma 3.2. (Barlow and Proschan [1, p. 118].) Let $X$ be a nonnegative random variable with distribution function $F$ and let $\mu_r = \int_0^\infty x^r dF(x), \ r = 1, 2$. If $X$ has IHRA then $\mu_2 \leq 2\mu_1^2$.  

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Theorem 3.1. For any $\alpha > 0$ and $n \geq 2$

$$\lambda^w \succeq \mu \Rightarrow Y_{n:n} \leq_{rh} X_{n:n}.$$  

Proof. Fix $x > 0$. Then the reversed hazard rate of $X_{n:n}$ is

$$\tilde{r}_{n:n}(\lambda, x) = \frac{1}{x} \sum_{i=1}^{n} (\lambda_i x)^{\alpha} e^{-\lambda_i x} \frac{1}{\int_0^{\lambda_i x} u^{\alpha-1} e^{-u} du} = \frac{1}{x} \sum_{i=1}^{n} \varphi(\lambda_i x),$$

where

$$\varphi(y) = \frac{y^{\alpha} e^{-y}}{\int_0^{y} u^{\alpha-1} e^{-u} du} = \frac{1}{\int_0^{1} (1-u)^{\alpha-1} e^{yu} du}, \quad y > 0.$$  

In view of Theorem A.8 of Marshall and Olkin [16, p. 59] it suffices to show that, for each $x > 0$, $\tilde{r}_{n:n}(\lambda, x)$ is decreasing in each $\lambda_i$, $i = 1, \ldots, n$, and is a Schur-convex function of $\lambda \in \mathbb{R}_+^n$. Fix $x > 0$. Clearly $\tilde{r}_{n:n}(\lambda, x)$ is decreasing in each $\lambda_i$, $i = 1, \ldots, n$. Then, in the light of Proposition C.1 of Marshall and Olkin [16, p. 64], it suffices to show that $\varphi(\cdot)$ is a convex function on $(0, \infty)$. Define

$$\psi(y) = \frac{1}{\varphi(y)} = \int_0^{1} (1-u)^{\alpha-1} e^{yu} du, \quad y > 0.$$  

Then, for $y > 0$,

$$\frac{d}{dy} \varphi(y) = -\frac{d}{dy} \psi(y) \frac{\psi^2(y)}{\psi'(y)},$$

and

$$\psi(y) \frac{d^2}{dy^2} \varphi(y) = 2 \left( \frac{d}{dy} \psi(y) \right)^2 - \frac{d^2}{dy^2} \psi(y) \frac{\psi^2(y)}{\psi'(y)}$$

$$= 2 \left( \frac{\int_0^{1} u(1-u)^{\alpha-1} e^{yu} du}{\int_0^{1} (1-u)^{\alpha-1} e^{yu} du} \right)^2 - \frac{\int_0^{1} u^2(1-u)^{\alpha-1} e^{yu} du}{\int_0^{1} (1-u)^{\alpha-1} e^{yu} du}$$

$$= 2E[W^2] - E[W^2], \quad (3.2)$$

where $W$ is a random variable as defined in Lemma 3.1. Since the random variable $W$ has IHR, it follows that $W$ has IHRA. Now, from Lemma 3.2, it follows that $2\mu^2_1 = 2(E[W]^2) \geq E[W^2] = \mu_2$, i.e. $\frac{d^2}{dy^2} \varphi(y) \geq 0, \forall y > 0$. Hence $\varphi(y)$ is convex in $y \in (0, \infty)$. $\square$
The following corollary is immediate from the above theorem and on using
the fact that $\lambda^m \geq \mu \Rightarrow \lambda^w \geq \mu$.

**Corollary 3.1.** For any $\alpha > 0$ and $n \geq 2$,

$$\lambda \geq \mu \Rightarrow Y_{n:n} \leq_{\text{rh}} X_{n:n},$$

i.e. the reversed hazard rate of $X_{n:n}$ is Schur-convex in $\lambda$.

Recall that Theorem 3.1 for the particular case $\alpha = 1$ was proved by
Dykstra et al. [5] under a stronger condition of $\lambda \geq^m \mu$. Thus Theorem 3.1
is a generalization of the result proved by Dykstra et al. [5]. Lihong and
Xinsheng [15] proved that, for any $\alpha > 0$, $\lambda \geq^m \mu$ implies $Y_{n:n} \leq_{\text{st}} X_{n:n}$. Using the result of of Khaledi et al. [11] it follows that the result of Lihong
and Xinsheng [15] holds under a weaker condition $\lambda \geq^p \mu$. Thus, Corollary
3.1 also generalizes the result of Lihong and Xinsheng [15].

One may wonder whether the conclusion of Theorem 3.1 will remain tru e
if we replace $\geq^w$ by $\geq^p$. The following example shows that the answer of this
question is negative.

**Example 3.1.** For $n = 2$ and $\alpha = 1$, let $\lambda = (\lambda_1, \lambda_2) = (1, 55)$ and
$\mu = (\mu_1, \mu_2) = (2, 44)$. It is easy to verify that $\lambda \geq^p \mu$ and $\lambda \not\geq^w \mu$. For $x > 0$

$$\tilde{r}^*(x) = \tilde{r}_{2:2}(\lambda, x) - \tilde{r}_{2:2}(\mu, x)$$

$$= \frac{1}{x} \left( \varphi(x) + \varphi(55x) - \varphi(2x) - \varphi(44x) \right),$$

where $\varphi(y) = y/(e^y - 1), y > 0$. It is easy to verify that $\tilde{r}^*(0.2) = 0.444451$
and $\tilde{r}^*(0.04) = -1.7994$. Thus $X_{2:2} \not\geq_{\text{rh}} Y_{2:2}$.

4. Conclusions and Comments

Consider a parallel system $\mathcal{P}$ consisting of $n \geq 2$ components having
random lifetimes $X_1, \ldots, X_n$. Let $X_1, \ldots, X_n$ be independent gamma random
variables with $X_i$ having shape parameter $\alpha$ and mean $\frac{\alpha}{\lambda_i}, \alpha > 0, \lambda_i > 0, i = 1, \ldots, n$. In this paper we prove that for any $\alpha > 0$

$$\lambda \geq \mu \Rightarrow X_{n:n} \geq_{\text{rh}} Y_{n:n}, \quad n \geq 2.$$
A consequence of this result is that the reversed hazard rate of $\mathcal{P}$ is Schur-convex in $\lambda = (\lambda_1, \ldots, \lambda_n)$, i.e. for any $\alpha > 0$

$$\lambda \succeq^m \mu \Rightarrow X_{n:n} \succeq_{rh} Y_{n:n}, \quad n \geq 2.$$  

With the help of an example we also show that $\lambda \succeq^p \mu \Rightarrow Y_{2:2} \leq_{rh} X_{2:2}$ may not hold even for exponential distribution (i.e. for $\alpha = 1$).

References


