New results on stochastic comparisons of two-component series and parallel systems

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Abstract

Let X_1, X_2, and X_3 be independent random variables with absolutely continuous distributions having the common support $[0, \infty)$. We show that if $X_1 \leq_{\text{hr[mrl, lr]}} X_3$ and $X_2 \leq_{\text{hr[mrl, lr]}} X_3$, then $\max\{X_1, X_2\} \leq_{\text{hr[mrl, lr]}} \max\{X_1, X_3\}$. We also show that if $X_2 \leq_{\text{rh[lr]}} X_1$ and $X_2 \leq_{\text{rh[lr]}} X_3$, then $\min\{X_1, X_2\} \leq_{\text{rh[lr]}} \min\{X_1, X_3\}$. These results generalize and extend some of the results given in Shaked and Shanthikumar (2007, Example 1.C.36, p. 56), Joo and Mi (2010), and Da *et al.* (2010).

Keywords: Hazard rate order, Likelihood ratio order, Mean residual life order, Reversed hazard rate order, Usual stochastic order.

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1. Introduction

A series (parallel) system functions if and only if each (at least one) of its components function. Consider n components $C_1, \ldots, C_n$ having random lifetimes $Y_1, \ldots, Y_n$, respectively. Then the lifetime of a series (parallel) system constructed from these components is given by $\min\{Y_1, \ldots, Y_n\}$ ($\max\{Y_1, \ldots, Y_n\}$). Performance of two systems constructed from different sets of components can be compared through stochastic comparisons of the corresponding system lifetimes. Normally a stochastic comparison on lifetimes of two different systems is made with respect to one of stochastic orders between lifetimes of systems. For an account on characterizations and properties of various stochastic orders one may refer to Shaked and Shanthikumar

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(2007) and Müller and Stoyan (2002). A vast literature on stochastic comparisons of lifetimes of series and parallel systems exist. See, for example, Boland *et al.* (1994), Dykstra *et al.* (1997), Khaledi and Kochar (2000), Da *et al.* (2010), Joo and Mi (2010), Zhao and Balakrishnan (2011), and references cited therein. In this paper we will derive some new results on stochastic comparisons of two-component series and parallel systems. These results generalize and extend some of the results known in the literature. First we recall definitions of various stochastic orders relevant to the context of this paper.

Let X and Y be random variables with the distribution functions F and G, the probability density functions f and g, the hazard functions r and μ, the reversed hazard functions $\tilde{r}$ and $\tilde{\mu}$, and the mean residual life functions m and l, respectively. Let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ denote the corresponding survival functions. Suppose that F and G have the common support $\mathbb{R}_+ \equiv [0, \infty)$ and $\{t : f(t) > 0\} = \{t : g(t) > 0\} = \mathbb{R}_+$. When we say that a function is increasing (decreasing) it means that the function is non-decreasing (non-increasing). Unless otherwise stated all the random variables considered in this study will be assumed to have absolutely continuous distributions.

Definition 1.1. *X is said to be smaller than Y in the*

- (i) *likelihood ratio order (written as $X \leq_{\mathrm{lr}} Y$) if $g(t)/f(t)$ is increasing in $t \in \mathbb{R}_+$;*
- (ii) *usual stochastic order (written as $X \leq_{\mathrm{st}} Y$) if $\bar{F}(t) \leq \bar{G}(t)$, $\forall t \in \mathbb{R}_+$;*
- (iii) *hazard rate order (written as $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ is increasing in $t \in \mathbb{R}_+$, or equivalently if $r(t) \geq \mu(t), \forall t \in \mathbb{R}_+$;*
- (iv) *reversed hazard rate order (written as $X \leq_{\mathrm{rh}} Y$) if $G(t)/F(t)$ is increasing in $t \in (0, \infty)$, or equivalently if $\tilde{r}(t) \leq \tilde{\mu}(t), \forall t \in (0, \infty)$;*
- (v) *mean residual order (written as $X \leq_{\mathrm{mrl}} Y$) if $\int_t^\infty \bar{G}(x)\,\mathrm{d}x / \int_t^\infty \bar{F}(x)\,\mathrm{d}x$ is increasing in $t \in \mathbb{R}_+$, or equivalently if $m(t) \leq l(t), \forall t \in \mathbb{R}_+$;*
- (vi) *increasing convex order (written as $X \leq_{\mathrm{icx}} Y$) if $\int_t^\infty \bar{G}(x)\,\mathrm{d}x \geq \int_t^\infty \bar{F}(x)\,\mathrm{d}x$ for all $t \in \mathbb{R}_+$.*

In this article we will mainly focus on two-component series and parallel systems. Let X_1, X_2, and X_3 be statistically independent and nonnegative random variables. Let $\mathcal{P}_i\,(\mathcal{S}_i)$ be the parallel (series) system consisting of two components having random lifetimes X_1 and X_i, $i = 2, 3$. Then the lifetime of $\mathcal{P}_i\,(\mathcal{S}_i)$ is given by $\max\{X_1, X_i\}$ $(\min\{X_1, X_i\})$, $i = 2, 3$. For the case when X_i has exponential distribution with hazard rate $\lambda_i\,(> 0)$, $i = 1, 2, 3$, Joo and Mi (2010, Theorem 2.2) proved that

$$\lambda_3 \le \lambda_2 \le \lambda_1 \Rightarrow \max\{X_1, X_2\} \le_{\text{hr}} \max\{X_1, X_3\}.$$

Da *et al.* (2010) generalized this result to the parallel systems consisting of general (not necessarily exponentially distributed) independent components by proving that

$$X_1 \le_{\text{hr}} X_2 \le_{\text{hr}} X_3 \Rightarrow \max\{X_1, X_2\} \le_{\text{hr}} \max\{X_1, X_3\}. \tag{1.1}$$

For general independent components Da *et al.* (2010) also showed that

$$X_1 \le_{\text{mrl}} X_2 \le_{\text{mrl}} X_3 \Rightarrow \max\{X_1, X_2\} \le_{\text{mrl}} \max\{X_1, X_3\}. \tag{1.2}$$

Shaked and Shanthikumar (2007, Example 1.C.36, p. 56) provide a similar result for the likelihood ratio order. Specifically, for general independent components, this result says that:

$$X_2 \le_{\text{lr}} X_1 \le_{\text{lr}} X_3 \Rightarrow \max\{X_1, X_2\} \le_{\text{lr}} \max\{X_1, X_3\} \text{ and } \min\{X_1, X_2\} \le_{\text{lr}} \min\{X_1, X_3\}. \tag{1.3}$$

The purpose of this article is to strengthen results (1.1)–(1.3) by providing weaker conditions under which these results hold. We will also derive some additional results. The outline of the paper is as follows. Results on parallel systems are discussed in Section 2. In this section, we will prove that

$$X_1 \le_{\text{hr[mrl]}} X_3 \text{ and } X_2 \le_{\text{hr[mrl]}} X_3 \Rightarrow \max\{X_1, X_2\} \le_{\text{hr[mrl]}} \max\{X_1, X_3\},$$

thereby generalizing results (1.1) and (1.2) given by Da *et al.* (2010). We also prove a similar result for the likelihood ratio order, thereby generalizing the result (1.3) given in Shaked and Shanthikumar (2007). Section 3 deals with series systems where we show that

$$X_2 \le_{\text{rh[lr]}} X_1 \text{ and } X_2 \le_{\text{rh[lr]}} X_3 \Rightarrow \min\{X_1, X_2\} \le_{\text{rh[lr]}} \min\{X_1, X_3\},$$

thereby extending and generalizing the result (1.3) given in Shaked and Shanthikumar (2007).

2. Stochastic Comparisons of Parallel systems

Let X_1, X_2, and X_3 denote independent random variables with absolutely continuous distributions having the common support $\mathbb{R}_+$. Let $F_i, f_i, \bar{F}_i, r_i, \tilde{r}_i$, and m_i denote, respectively, the distribution function, the probability density function, the survival function, the hazard function, the reversed hazard function, and the mean residual life function of X_i, $i = 1, 2, 3$. In this section we will compare random variables $\max\{X_1, X_2\}$ and $\max\{X_1, X_3\}$ with respect to the hazard rate order, the mean residual life order, and the likelihood ratio order.

In the following theorem we generalize and extend the result (1.1) proved by Da *et al.* (2010).

Theorem 2.1. *Suppose that one (or both) of the following conditions holds:*

(a) $X_1 \leq_{\text{hr}} X_3$ *and* $X_2 \leq_{\text{hr}} X_3$*;*

(b) $X_2 \leq_{\text{hr}} X_3$ *and* $r_1(t)F_3(t) \geq r_3(t)F_1(t)$ *for all* $t \in \mathbb{R}_+$*.*

Then

$$\max\{X_1, X_2\} \leq_{\text{hr}} \max\{X_1, X_3\}.$$

Proof. For $t \in \mathbb{R}_+$, consider

$$\begin{aligned} h_1(t) &= \frac{P\left(\max\{X_1, X_3\} > t\right)}{P\left(\max\{X_1, X_2\} > t\right)} \\ &= \frac{\bar{F}_1(t) + \bar{F}_3(t) - \bar{F}_1(t)\bar{F}_3(t)}{\bar{F}_1(t) + \bar{F}_2(t) - \bar{F}_1(t)\bar{F}_2(t)} = \frac{N_1(t)}{D_1(t)}, \quad \text{say.} \end{aligned}$$

It can be easily verified that

$$\begin{aligned} D_1^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_1(t) &= r_1(t)\bar{F}_1(t)[\bar{F}_3(t) - \bar{F}_2(t)] \\ &\quad + r_2(t)\bar{F}_2(t)[1 - \bar{F}_1(t)][\bar{F}_1(t) + \bar{F}_3(t) - \bar{F}_1(t)\bar{F}_3(t)] \\ &\quad - r_3(t)\bar{F}_3(t)[1 - \bar{F}_1(t)][\bar{F}_1(t) + \bar{F}_2(t) - \bar{F}_1(t)\bar{F}_2(t)]. \end{aligned}$$

(a) Since $r_2(t) \geq r_3(t)$ for all $t \in \mathbb{R}_+$, it follows that

$$\begin{aligned} D_1^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_1(t) &\geq \bar{F}_1(t)[\bar{F}_3(t) - \bar{F}_2(t)][r_1(t) - r_3(t)F_1(t)] \qquad (2.1) \\ &\geq 0 \qquad (\text{since } r_1(t) \geq r_3(t), \text{ and } X_2 \leq_{\text{hr}} X_3 (\Rightarrow \bar{F}_2(t) \leq \bar{F}_3(t))), \end{aligned}$$

i.e. $h_1(t)$ is increasing in $t \in \mathbb{R}_+$.

(b) From (2.1), we can write

$$D_1^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_1(t) \geq \bar{F}_1(t)[\bar{F}_3(t) - \bar{F}_2(t)][r_1(t)\bar{F}_3(t) + r_1(t)F_3(t) - r_3(t)F_1(t)]$$
$$\geq 0,\ \forall t \in \mathbb{R}_+,$$

where the last inequality follows from the assumption that $r_1(t)F_3(t) \geq r_3(t)F_1(t)$ for all $t \in \mathbb{R}_+$, and the fact that $X_2 \leq_{\text{hr}} X_3$ implies $\bar{F}_2(t) \leq \bar{F}_3(t)$ for all $t \in \mathbb{R}_+$. Thus, $h_1(t)$ is increasing in $t \in \mathbb{R}_+$. Hence the result follows. □

Remark 2.1. (i) From (2.1) it is clear that the conclusion of Theorem 2.1 remains true if $X_2 \leq_{\text{hr}} X_3$ and $r_1(t) \geq r_3(t)F_1(t)$ for all $t \in \mathbb{R}_+$.

(ii) The condition $r_1(t)F_3(t) \geq r_3(t)F_1(t), \forall t \in \mathbb{R}_+$, is equivalent to the condition $\tilde{r}_1(t)\bar{F}_3(t) \geq \tilde{r}_3(t)\bar{F}_1(t), \forall t \in (0, \infty)$. Therefore, the conclusion of Theorem 2.1 also holds if $X_2 \leq_{\text{hr}} X_3$ and $\tilde{r}_1(t)\bar{F}_3(t) \geq \tilde{r}_3(t)\bar{F}_1(t)$ for all $t \in (0, \infty)$.

As a consequence of Theorem 2.1, we have the following corollary.

Corollary 2.1. *Suppose that one (or more) of the following conditions holds:*

(a) $X_1 \leq_{\text{hr}} X_2 \leq_{\text{hr}} X_3$*;*
(b) $X_2 \leq_{\text{hr}} X_1 \leq_{\text{hr}} X_3$*;*
(c) $X_1 \leq_{\text{st}} X_3, X_2 \leq_{\text{hr}} X_3$*, and* $r_3(t)/r_1(t)$ *is decreasing in* $t \in \mathbb{R}_+$*;*
(d) $X_3 \leq_{\text{st}} X_1, X_2 \leq_{\text{hr}} X_3$*, and* $\tilde{r}_1(t)/\tilde{r}_3(t)$ *is decreasing in* $t \in (0, \infty)$*.*

Then

$$\max\{X_1, X_2\} \leq_{\text{hr}} \max\{X_1, X_3\}.$$

Proof. Parts (a) and (b) are immediate from Theorem 2.1(a).

(c) On following the proof of Corollary 2.3 of Misra *et al.* (2011), we can show that if $X_1 \leq_{\text{st}} X_3$ and $r_3(t)/r_1(t)$ is decreasing in $t \in \mathbb{R}_+$, then $r_1(t)F_3(t) \geq r_3(t)F_1(t)$ for all $t \in \mathbb{R}_+$. Now, the result follows from Theorem 2.1(b).

(d) Using the technique used in the proof of Corollary 3.2 of Misra *et al.* (2011), we can show that if $X_3 \leq_{\text{st}} X_1$ and $\tilde{r}_1(t)/\tilde{r}_3(t)$ is decreasing in $t \in (0, \infty)$, then $\tilde{r}_1(t)\bar{F}_3(t) \geq \tilde{r}_3(t)\bar{F}_1(t)$ for all $t \in (0, \infty)$. Now, the result follows from Remark 2.1(ii). □

Recall that the result in Corollary 2.1(a) was proved by Da *et al.* (2010). The following example illustrates a few situations on comparative usefulness of results given in Theorem 2.1 and Corollary 2.1.

Example 2.1. (i) Let $r_1(t) = 36t + 10, r_2(t) = 43t + 3$, and $r_3(t) = 35t + 3,\ t \in \mathbb{R}_+$. Obviously, $r_1(t) \geq r_3(t), r_2(t) \geq r_3(t)$ for all $t \in \mathbb{R}_+$, and $r_1(t) \geq (\leq)\, r_2(t)$ for all $0 \leq t \leq 1\, (t \geq 1)$. Moreover, it can be easily verified that $r_3(t)/r_1(t)$ is strictly increasing on $\mathbb{R}_+$. For $t \in \mathbb{R}_+$, consider

$$\begin{aligned}\phi_1(t) &= r_1(t)F_3(t) - r_3(t)F_1(t)\\ &= (36t + 10)(1 - e^{-\frac{35}{2}t^2 - 3t}) - (35t + 3)(1 - e^{-18t^2 - 10t}).\end{aligned}$$

Then, $\phi_1(0.01) = -0.000482848$ and $\phi_1(0.02) = 0.00227857$. Thus, the conditions of Theorem 2.1(a) are satisfied but the conditions of Theorem 2.1(b) and Corollary 2.1(a)–(d) are not satisfied.

(ii) Let $r_1(t) = 5t{+}10, r_2(t) = 12t{+}3$, and $r_3(t) = 3t{+}3$, $t \in \mathbb{R}_+$. Obviously, $r_1(t) \geq r_3(t), r_2(t) \geq r_3(t)$ for all $t \in \mathbb{R}_+$, and $r_1(t) \geq (\leq)\, r_2(t)$ for all $0 \leq t \leq 1\, (t \geq 1)$. Moreover, $r_3(t)/r_1(t)$ is strictly increasing in $t \in \mathbb{R}_+$ and $\phi_1(t) \geq 0, \forall t \in \mathbb{R}_+$ (see Misra *et al.*, 2011, Example 2.2). Thus, the conditions of Theorem 2.1(a) and (b) are satisfied but the conditions of Corollary 2.1(a)–(d) are not satisfied.

(iii) Let $F_1(t) = (1 - e^{-t})^2, F_2(t) = 1 - e^{-2t}$, and $F_3(t) = 1 - e^{-t}, t \in \mathbb{R}_+$. Then we have $\tilde{r}_1(t) = 2/(e^t - 1)$ and $\tilde{r}_3(t) = 1/(e^t - 1), t \in (0, \infty)$. We can see that $X_3 \leq_{\text{st}} X_1, X_2 \leq_{\text{hr}} X_3$, and $\tilde{r}_1(t)/\tilde{r}_3(t)$ is decreasing in $t \in (0, \infty)$. Thus, the conditions of Corollary 2.1(d) are satisfied but the conditions of Theorem 2.1(a) and Corollary 2.1(a)–(c) are not satisfied. Moreover, on using Remark 2.1(ii), it is evident that the conditions of Theorem 2.1(b) are also satisfied.

The following lemma, which may be of independent interest, will be useful in proving the next theorem. The result given in this lemma strengthens Lemma 7.1(a) of Barlow and Proschan (1975, p. 120).

Lemma 2.1. *Let $A = \{(x, s) : 0 \leq s \leq x < \infty\}$ and let $K : A \to \mathbb{R}$ be a function such that $\int_t^\infty |K(x, s)|\, \mathrm{d}x < \infty$, and $\int_t^\infty K(x, s)\, \mathrm{d}x \geq 0$ whenever $0 \leq s \leq t < \infty$. Then, for any nonnegative and increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$,*

$$\int_s^\infty h(x)K(x, s)\, \mathrm{d}x \geq 0, \qquad \textit{for all } s \in \mathbb{R}_+.$$

Proof. Fix $s \in \mathbb{R}_+$. For a set $B \subseteq \mathbb{R}$, let $I_B(\cdot)$ denote its indicator function. Since h is a nonnegative and increasing function there exists a sequence $\{\psi_n\}_{n\geq 1}$ of simple functions such that $\lim_{n\to\infty} \psi_n(x) = h(x)$, $\forall x \in \mathbb{R}_+$, and

$$\psi_n(x) = \sum_{i=1}^{n} a_i I_{[t_i,\,\infty)}(x), \quad x \in \mathbb{R}_+,\, n = 1, 2, \ldots,$$

for some sequences $\{a_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ of positive real constants. Clearly, for each $x \in \mathbb{R}_+$, $\psi_n(x) \leq \psi_{n+1}(x)$, $n = 1, 2, \ldots$. Let $K^+(x, s) = \max\{K(x, s), 0\}$ and $K^-(x, s) = \max\{-K(x, s), 0\}$, $x \geq s$, so that $K^+(x, s) \geq 0$, $K^-(x, s) \geq 0$ and $K(x, s) = K^+(x, s) - K^-(x, s)$, $x \geq s$. Then

$$\int_s^\infty \psi_n(x) K(x, s)\, \mathrm{d}x = \sum_{i=1}^{n} a_i \int_{\max\,(t_i, s)}^\infty K(x, s)\, \mathrm{d}x \geq 0, \quad \forall n \geq 1,$$

and therefore

$$\lim_{n\to\infty} \int_s^\infty \psi_n(x) K(x, s)\, \mathrm{d}x \geq 0. \tag{2.2}$$

On using the monotone convergence theorem we obtain

$$\lim_{n\to\infty} \int_s^\infty \psi_n(x) K^+(x, s)\, \mathrm{d}x = \int_s^\infty h(x) K^+(x, s)\, \mathrm{d}x,$$

and

$$\lim_{n\to\infty} \int_s^\infty \psi_n(x) K^-(x, s)\, \mathrm{d}x = \int_s^\infty h(x) K^-(x, s)\, \mathrm{d}x.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \int_s^\infty \psi_n(x) K(x, s)\, \mathrm{d}x &= \lim_{n\to\infty} \int_s^\infty \psi_n(x) K^+(x, s)\, \mathrm{d}x \\ &\quad - \lim_{n\to\infty} \int_s^\infty \psi_n(x) K^-(x, s)\, \mathrm{d}x \\ &= \int_s^\infty h(x) K^+(x, s)\, \mathrm{d}x - \int_s^\infty h(x) K^-(x, s)\, \mathrm{d}x \\ &= \int_s^\infty h(x) K(x, s)\, \mathrm{d}x. \end{aligned}$$

Now the result follows on using (2.2). □

The following theorem generalizes the result (1.2) proved by Da *et al.* (2010).

Theorem 2.2. *Suppose that* $X_1 \leq_{\mathrm{mrl}} X_3$ *and* $X_2 \leq_{\mathrm{mrl}} X_3$. *Then,*

$$\max\{X_1, X_2\} \leq_{\mathrm{mrl}} \max\{X_1, X_3\}.$$

Proof. For $t \in \mathbb{R}_+$, define

$$\begin{aligned} h_2(t) &= \frac{\int_t^\infty P\left(\max\{X_1, X_3\} > x\right)\,\mathrm{d}x}{\int_t^\infty P\left(\max\{X_1, X_2\} > x\right)\,\mathrm{d}x} \\ &= \frac{\int_t^\infty [\bar{F}_1(x) + \bar{F}_3(x) - \bar{F}_1(x)\bar{F}_3(x)]\,\mathrm{d}x}{\int_t^\infty [\bar{F}_1(x) + \bar{F}_2(x) - \bar{F}_1(x)\bar{F}_2(x)]\,\mathrm{d}x} \quad \left(= \frac{N_2(t)}{D_2(t)},\ \text{say}\right). \end{aligned}$$

We need to show that $h_2(t)$ is increasing in $t \in \mathbb{R}_+$. One can easily verify that

$$\begin{aligned} D_2^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_2(t) &= F_1(t)[\bar{F}_2(t) - \bar{F}_3(t)]\int_t^\infty \bar{F}_1(x)\,\mathrm{d}x + [\bar{F}_1(t) + F_1(t)\bar{F}_2(t)] \\ &\quad \times \int_t^\infty F_1(x)\bar{F}_3(x)\,\mathrm{d}x - [\bar{F}_1(t) + F_1(t)\bar{F}_3(t)]\int_t^\infty F_1(x)\bar{F}_2(x)\,\mathrm{d}x,\ t \in \mathbb{R}_+. \end{aligned} \tag{2.3}$$

Fix $t \in \mathbb{R}_+$. Then the following two cases arise.
Case I: $\bar{F}_2(t) \geq \bar{F}_3(t)$.
We have

$$D_2^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_2(t) \geq [\bar{F}_1(t) + F_1(t)\bar{F}_3(t)]\int_t^\infty F_1(x)[\bar{F}_3(x) - \bar{F}_2(x)]\,\mathrm{d}x.$$

Since $X_2 \leq_{\mathrm{mrl}} X_3$ implies that $X_2 \leq_{\mathrm{icx}} X_3$ (see Shaked and Shanthikumar, 2007, p. 195), it follows that $\int_y^\infty [\bar{F}_3(x) - \bar{F}_2(x)]\,\mathrm{d}x \geq 0$ for all $y \in \mathbb{R}_+$. Now, on using Lemma 2.1, we get $\int_y^\infty F_1(x)[\bar{F}_3(x) - \bar{F}_2(x)]\,\mathrm{d}x \geq 0$, for all $y \in \mathbb{R}_+$. In particular, for $y = t$, we have $\int_t^\infty F_1(x)[\bar{F}_3(x) - \bar{F}_2(x)]\,\mathrm{d}x \geq 0$. Hence $\frac{\mathrm{d}}{\mathrm{d}t}h_2(t) \geq 0$.
Case II: $\bar{F}_2(t) \leq \bar{F}_3(t)$.
Since $X_2 \leq_{\mathrm{mrl}} X_3$ implies $\bar{F}_2(s)\int_u^\infty \bar{F}_3(x)\,\mathrm{d}x \geq \bar{F}_3(s)\int_u^\infty \bar{F}_2(x)\,\mathrm{d}x$ for all $s \leq u$ (see Shaked and Shanthikumar, 2007, p. 82), on using Lemma 2.1, we obtain

$\bar{F}_2(s)\int_s^\infty F_1(x)\bar{F}_3(x) \geq \bar{F}_3(s)\int_s^\infty F_1(x)\bar{F}_2(x)$ for all $s \in \mathbb{R}_+$. In particular, for $s = t$, we have

$$\bar{F}_2(t)\int_t^\infty F_1(x)\bar{F}_3(x) \geq \bar{F}_3(t)\int_t^\infty F_1(x)\bar{F}_2(x).$$

Using this in (2.3), we obtain

$$\begin{aligned}
\bar{F}_3(t)D_2^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_2(t) &\geq F_1(t)\bar{F}_3(t)[\bar{F}_2(t)-\bar{F}_3(t)]\int_t^\infty \bar{F}_1(x)\,\mathrm{d}x \\
&\quad + \bar{F}_1(t)[\bar{F}_3(t)-\bar{F}_2(t)]\int_t^\infty F_1(x)\bar{F}_3(x)\,\mathrm{d}x \\
&\geq F_1(t)\bar{F}_3(t)[\bar{F}_2(t)-\bar{F}_3(t)]\int_t^\infty \bar{F}_1(x)\,\mathrm{d}x \\
&\quad + \bar{F}_1(t)[\bar{F}_3(t)-\bar{F}_2(t)]\int_t^\infty F_1(t)\bar{F}_3(x)\,\mathrm{d}x \\
&= F_1(t)[\bar{F}_3(t)-\bar{F}_2(t)]\int_t^\infty [\bar{F}_1(t)\bar{F}_3(x)-\bar{F}_3(t)\bar{F}_1(x)]\,\mathrm{d}x \\
&\geq 0 \qquad (\text{since } X_1 \leq_{\mathrm{mrl}} X_3).
\end{aligned}$$

Thus $h_2(t)$ is increasing in $t \in \mathbb{R}_+$. □

The following corollary is immediate from Theorem 2.2.

Corollary 2.2. *Suppose that any one of the following conditions holds:*

(a) $X_1 \leq_{\mathrm{mrl}} X_2 \leq_{\mathrm{mrl}} X_3$;
(b) $X_2 \leq_{\mathrm{mrl}} X_1 \leq_{\mathrm{mrl}} X_3$.

Then

$$\max\{X_1, X_2\} \leq_{\mathrm{mrl}} \max\{X_1, X_3\}.$$

Recall that the result in Corollary 2.2(a) was proved by Da *et al.* (2010). The following example gives a practical case in which the assumptions of Corollary 2.2 are not satisfied but the result of Theorem 2.2 is valid.

Example 2.2. Let $m_1(t) = \frac{5t+2}{4t+3}, m_2(t) = \frac{6t+5}{6(t+1)}$, and $m_3(t) = \frac{7t+5}{5t+4}, t \in \mathbb{R}_+$. Then, it is easy to verify that $m_1(t) \leq m_3(t)$ and $m_2(t) \leq m_3(t)$ for all $t \in \mathbb{R}_+$. Also, $m_2(0) - m_1(0) = 1/6 > 0$ and $m_2(1) - m_1(1) = -1/12 < 0$. Thus, $X_1 \leq_{\mathrm{mrl}} X_3$ and $X_2 \leq_{\mathrm{mrl}} X_3$, but there is no mean residual life ordering between the random variables X_1 and X_2.

Now, we derive sufficient conditions under which $\max\{X_1, X_2\} \leq_{\rm lr} \max\{X_1, X_3\}$ holds, thereby generalizes the result (1.3), given in Shaked and Shanthikumar (2007, Example 1.C.36, p. 56).

Theorem 2.3. *Suppose that* $X_2 \leq_{\rm lr} X_3$ *and* $F_1^2(t)f_3(t)/f_1(t)$ *(or* $F_1^2(t)f_2(t)/f_1(t)$*) is increasing in* $t \in \mathbb{R}_+$*. Then,*

$$\max\{X_1, X_2\} \leq_{\rm lr} \max\{X_1, X_3\}.$$

Proof. We will prove the result for the non-parenthetical part, the other part can be proved similarly. Let $\pi_2(t)$ and $\pi_3(t)$ denote the probability density functions of $\max\{X_1, X_2\}$ and $\max\{X_1, X_3\}$, respectively. Define

$$h_3(t) = \frac{\pi_3(t)}{\pi_2(t)} = \frac{f_1(t)F_3(t) + f_3(t)F_1(t)}{f_1(t)F_2(t) + f_2(t)F_1(t)}, \quad t \in \mathbb{R}_+.$$

It is easy to verify that for $t \in \mathbb{R}_+$

$$\begin{aligned}\pi_2^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_3(t) = {} & [2f_1^2(t) - f_1'(t)F_1(t)][f_3(t)F_2(t) - f_2(t)F_3(t)] + f_3'(t)F_1(t) \\ & \times [f_1(t)F_2(t) + f_2(t)F_1(t)] - f_2'(t)F_1(t)[f_1(t)F_3(t) + f_3(t)F_1(t)],\end{aligned}$$

where $f_i'(\cdot)$ denotes the derivative of $f_i(\cdot)$, $i = 1, 2, 3$. Since $X_2 \leq_{\rm lr} X_3$ implies $f_2(t)f_3'(t) - f_3(t)f_2'(t) \geq 0, \forall t \in \mathbb{R}_+$, it follows that

$$\begin{aligned}\pi_2^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_3(t) &\geq f_1(t)F_1(t)[f_3(t)F_2(t) - f_2(t)F_3(t)]\left[2\frac{f_1(t)}{F_1(t)} - \frac{f_1'(t)}{f_1(t)} + \frac{f_3'(t)}{f_3(t)}\right] \\ &= f_1(t)F_1(t)F_2(t)F_3(t)[\tilde{r}_3(t) - \tilde{r}_2(t)]\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(F_1^2(t)f_3(t)/f_1(t)\right) \\ &\geq 0,\end{aligned}$$

where the last inequality follows from the assumption that $F_1^2(t)f_3(t)/f_1(t)$ is increasing in $t \in \mathbb{R}_+$ and the fact that $X_2 \leq_{\rm lr} X_3$ implies $X_2 \leq_{\rm rh} X_3$. Hence $h_3(t)$ is increasing in $t \in \mathbb{R}_+$, i.e. $\max\{X_1, X_2\} \leq_{\rm lr} \max\{X_1, X_3\}$. ☐

The following corollary is a simple consequence of the above theorem.

Corollary 2.3. *Suppose that one (or more) of the following conditions holds:*

(a) $X_1 \leq_{\rm lr} X_3$ *and* $X_2 \leq_{\rm lr} X_3$*;*
(b) $X_1 \leq_{\rm lr} X_2 \leq_{\rm lr} X_3$*;*

(c) $X_2 \le_{\rm lr} X_1 \le_{\rm lr} X_3$;
(d) $X_2 \le_{\rm lr} X_3$ *and* $f_3(t)/\tilde{r}_1(t)$ *(or* $f_2(t)/\tilde{r}_1(t)$*) is increasing in* $t \in \mathbb{R}_+$;
(e) $X_2 \le_{\rm lr} X_3$ *and* $\tilde{r}_3(t)/\tilde{r}_1(t)$ *(or* $\tilde{r}_2(t)/\tilde{r}_1(t)$*) is increasing in* $t \in \mathbb{R}_+$.

Then,

$$\max\{X_1, X_2\} \le_{\rm lr} \max\{X_1, X_3\}.$$

3. Stochastic Comparisons of Series systems

In this section, we deal with the stochastic comparisons of series systems having lifetimes $\min\{X_1, X_2\}$ and $\min\{X_1, X_3\}$. In the following theorems, we will compare random variables $\min\{X_1, X_2\}$ and $\min\{X_1, X_3\}$ with respect to the reversed hazard rate order and the likelihood ratio order.

Theorem 3.1. *Suppose that one (or both) of the following conditions holds:*

(a) $X_2 \le_{\rm rh} X_1$ *and* $X_2 \le_{\rm rh} X_3$;
(b) $X_2 \le_{\rm rh} X_3$ *and* $\tilde{r}_1(t)\bar{F}_2(t) \ge \tilde{r}_2(t)\bar{F}_1(t)$ *for all* $t \in (0, \infty)$.

Then

$$\min\{X_1, X_2\} \le_{\rm rh} \min\{X_1, X_3\}.$$

Proof. For $t \in (0, \infty)$, consider

$$\begin{aligned} h_4(t) &= \frac{P\left(\min\{X_1, X_3\} \le t\right)}{P\left(\min\{X_1, X_2\} \le t\right)} \\ &= \frac{F_1(t) + F_3(t) - F_1(t)F_3(t)}{F_1(t) + F_2(t) - F_1(t)F_2(t)} = \frac{N_3(t)}{D_3(t)}, \quad \text{say.} \end{aligned}$$

It can be easily verified that

$$\begin{aligned} D_3^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_4(t) = {} & \tilde{r}_1(t)F_1(t)[F_2(t) - F_3(t)] \\ & - \tilde{r}_2(t)\bar{F}_1(t)F_2(t)[F_1(t) + F_3(t) - F_1(t)F_3(t)] \\ & + \tilde{r}_3(t)\bar{F}_1(t)F_3(t)[F_1(t) + F_2(t) - F_1(t)F_2(t)]. \end{aligned}$$

(a) Since $\tilde{r}_3(t) \ge \tilde{r}_2(t)$ for all $t \in (0, \infty)$, it follows that

$$\begin{aligned} D_3^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_4(t) &\ge F_1(t)[F_2(t) - F_3(t)][\tilde{r}_1(t) - \tilde{r}_2(t)\bar{F}_1(t)] \\ &\ge 0, \quad \forall t \in (0, \infty), \end{aligned} \tag{3.1}$$

where the last inequality follows from the assumptions that $\tilde{r}_1(t) \geq \tilde{r}_2(t)$, $\forall t \in (0, \infty)$, and the fact that $X_2 \leq_{\mathrm{rh}} X_3$ implies $F_2(t) \geq F_3(t), \forall t \in \mathbb{R}_+$. It follows that $h_4(t)$ is increasing in $t \in \mathbb{R}_+$, i.e. $\min\{X_1, X_2\} \leq_{\mathrm{rh}} \min\{X_1, X_3\}$.

(b) From (3.1), we can write

$$\begin{aligned} D_3^2(t)\frac{\mathrm{d}}{\mathrm{d}t}h_4(t) &\geq F_1(t)[F_2(t) - F_3(t)][\tilde{r}_1(t)F_2(t) + \tilde{r}_1(t)\bar{F}_2(t) - \tilde{r}_2(t)\bar{F}_1(t)] \\ &\geq 0, \quad \forall t \in (0, \infty), \end{aligned}$$

where the last inequality follows from the assumptions that $\tilde{r}_1(t)\bar{F}_2(t) \geq \tilde{r}_2(t)\bar{F}_1(t)$, $\forall t \in (0, \infty)$, and the fact that $X_2 \leq_{\mathrm{rh}} X_3$ implies $F_2(t) \geq F_3(t), \forall t \in \mathbb{R}_+$. Thus, $h_4(t)$ is increasing in $t \in (0, \infty)$. Hence the result follows. □

Remark 3.1. From (3.1) it is clear that the conclusion of Theorem 3.1 remains true if $X_2 \leq_{\mathrm{rh}} X_3$ and $\tilde{r}_1(t) \geq \tilde{r}_2(t)\bar{F}_1(t)$ for all $t \in (0, \infty)$.

As a consequence of Theorem 3.1, we have the following corollary. The proof of the following corollary, being similar to that of Corollary 2.1, is omitted.

Corollary 3.1. *Suppose that one (or more) of the following conditions holds:*

(a) $X_2 \leq_{\mathrm{rh}} X_1 \leq_{\mathrm{rh}} X_3$*;*
(b) $X_2 \leq_{\mathrm{rh}} X_3 \leq_{\mathrm{rh}} X_1$*;*
(c) $X_2 \leq_{\mathrm{st}} X_1, X_2 \leq_{\mathrm{rh}} X_3$*, and* $\tilde{r}_1(t)/\tilde{r}_2(t)$ *is decreasing in* $t \in (0, \infty)$*;*
(d) $X_1 \leq_{\mathrm{st}} X_2, X_2 \leq_{\mathrm{rh}} X_3$*, and* $r_2(t)/r_1(t)$ *is decreasing in* $t \in \mathbb{R}_+$*.*

Then

$$\min\{X_1, X_2\} \leq_{\mathrm{rh}} \min\{X_1, X_3\}.$$

The following example illustrates a few situations on comparative usefulness of results given in Theorem 3.1 and Corollary 3.1.

Example 3.1. (i) Let $F_1(t) = (1 - e^{-5t})^4, F_2(t) = (1 - e^{-35t})^2, F_3(t) = (1 - e^{-30t})^5, t \in \mathbb{R}_+$, and let $\phi_2(t) = F_1(t)/F_3(t), t \in (0, \infty)$. It can be easily verified that $X_2 \leq_{\mathrm{rh}} X_1$ and $X_2 \leq_{\mathrm{rh}} X_3$. Also, we have $\phi_2(0.01) = 0.00483737, \phi_2(0.02) = 0.00438607$, and $\phi_2(0.03) = 0.00511513$, which implies that there is no reversed hazard rate ordering between the random variables X_1 and X_3. For $t \in (0, \infty)$, consider

$$\phi_3(t) = \frac{\tilde{r}_1(t)}{\tilde{r}_2(t)} = \frac{2e^{30t}(1 - e^{-35t})}{7(1 - e^{-5t})},$$

and

$$\begin{aligned}\phi_4(t) &= \tilde{r}_1(t)\bar{F}_2(t) - \tilde{r}_2(t)\bar{F}_1(t) \\ &= \frac{20e^{-5t}[1 - (1 - e^{-35t})^2]}{1 - e^{-5t}} - \frac{70e^{-35t}[1 - (1 - e^{-5t})^4]}{1 - e^{-35t}}.\end{aligned}$$

Then, it can be easily verified that

$$[7(1 - e^{-5t})]^2 \frac{\mathrm{d}}{\mathrm{d}t}\phi_3(t) = 7e^{-5t}(60e^{35t} - 70e^{30t} + 10) \geq 0, \ \forall t \in (0, \infty),$$

i.e. $\phi_3(t)$ is increasing in $t \in (0, \infty)$. Moreover, we have $\phi_4(0.08) = 0.316523$ and $\phi_4(0.10) = -0.293543$. Thus, the conditions of Theorem 3.1(a) are satisfied but the conditions of Theorem 3.1(b) and Corollary 3.1(a)–(d) are not satisfied.

(ii) Let $\bar{F}_i(t) = e^{-\lambda_i t}, t \in \mathbb{R}_+$, $i = 1, 2, 3$, with $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Clearly, $X_1 \leq_{\rm st} X_2, X_2 \leq_{\rm rh} X_3$, and $r_2(t)/r_1(t)$ is decreasing in $t \in \mathbb{R}_+$. Thus, the conditions of Corollary 3.1(d) are satisfied but the conditions of Theorem 3.1(a) and Corollary 3.1(a)–(c) are not satisfied. Since the conditions that $r_2(t)/r_1(t)$ is decreasing in $t \in \mathbb{R}_+$ and $X_1 \leq_{\rm st} X_2$ implies that $r_1(t)F_2(t) \geq r_2(t)F_1(t), \forall t \in \mathbb{R}_+$, which is equivalent to the condition $\tilde{r}_1(t)\bar{F}_2(t) \geq \tilde{r}_2(t)\bar{F}_1(t), \forall t \in (0, \infty)$, it follows that the result of Theorem 3.1(b) is also valid.

The following theorem generalizes the result (1.3), given in Shaked and Shanthikumar (2007, Example 1.C.36, p. 56). The proof of the following theorem, being similar to that of Theorem 2.3, is omitted.

Theorem 3.2. *Suppose that* $X_2 \leq_{\rm lr} X_3$ *and* $f_1(t)/(f_2(t)\bar{F}_1^2(t))$ *(or* $f_1(t)/(f_3(t)\bar{F}_1^2(t))$*) is increasing in* $t \in \mathbb{R}_+$*. Then,*

$$\min\{X_1, X_2\} \leq_{\rm lr} \min\{X_1, X_3\}.$$

The following corollary immediately follows from above theorem.

Corollary 3.2. *Suppose that one (or more) of the following conditions holds:*

(a) $X_2 \leq_{\mathrm{lr}} X_3$ *and* $X_2 \leq_{\mathrm{lr}} X_1$*;*
(b) $X_2 \leq_{\mathrm{lr}} X_1 \leq_{\mathrm{lr}} X_3$*;*
(c) $X_2 \leq_{\mathrm{lr}} X_3 \leq_{\mathrm{lr}} X_1$*;*
(d) $X_2 \leq_{\mathrm{lr}} X_3$ *and* $r_1(t)/f_2(t)$ *(or* $r_1(t)/f_3(t)$*) is increasing in* $t \in \mathbb{R}_+$*;*
(e) $X_2 \leq_{\mathrm{lr}} X_3$ *and* $r_1(t)/r_2(t)$ *(or* $r_1(t)/r_3(t)$*) is increasing in* $t \in \mathbb{R}_+$*.*

Then,

$$\min\{X_1, X_2\} \leq_{\mathrm{lr}} \min\{X_1, X_3\}.$$

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