GENERIC CUSPIDAL REPRESENTATIONS OF $U(2,1)$

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Abstract. Let F be a non-Archimedean local field and let σ be a non-trivial Galois involution with fixed field F_0. When the residue characteristic of F_0 is odd, using the construction of cuspidal representations of classical groups by Stevens, we classify generic cuspidal representations of $U(2,1)(F/F_0)$.

1. Introduction

Let F be a non-Archimedean local field and let σ be a non-trivial Galois involution with fixed field F_0. Let V be a three dimensional F-vector space, and let $h : V \times V \to F$ be a non-degenerate hermitian form with

$$h(xv, yw) = x\sigma(y)\sigma(h(w,v)) \text{ for all } v, w \in V \text{ and } x, y \in F. \tag{1.1}$$

Let σ_h be the adjoint anti-involution induced by h on $\mathrm{End}_F(V)$. Let $\mathbf{G}$ be the unitary group associated to (V, h), and let G be the group of F_0-rational points of $\mathbf{G}$. In this article we obtain a classification of generic cuspidal representations of G using the underlying skew semisimple strata, in the construction of cuspidal representations of G.

Let $\mathbf{H}$ be a quasi-split reductive algebraic group defined over F_0. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{H}$, defined over F_0, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Let $\mathbf{T}$ be a maximal F_0-split torus of $\mathbf{H}$ contained in $\mathbf{B}$. For any algebraic group $\mathbf{R}$, defined over F_0, we denote by R the group of F_0-rational points of $\mathbf{R}$. A character ψ of U is said to be non-degenerate if ψ is non-trivial on each simple root group of T in U. A representation π of H is called generic if $\mathrm{Hom}_U(\pi, \Psi) \neq 0$, for some non-degenerate character Ψ of U.

A cuspidal representation (π, V) of H is a generic representation if and only if there exists a non-zero linear functional $l : V \to \mathbb{C}$ and a character Ψ of U such that

$$l(\pi(u)v) = \Psi(u)l(v), \text{ for all } u \in U, v \in V.$$

If such a linear functional exists, for (π, V), then the character Ψ necessarily satisfies the non-degenerate condition on the character Ψ (see Section 2.4). The functional l is called a Whittaker linear functional. Moreover, when H equals G, genericity of an irreducible smooth representation does not depend on the choice of the pair (U, Ψ). A Whittaker linear functional on an irreducible smooth representation, if it exists, is unique up to scalars (see [20] and [17]).

Whittaker functionals are first used by Jacquet and Langlands to define the local L, ϵ-factors for $\mathrm{GL}_2(F_0)$ (see [12, Theorem 2.18]). These methods and their generalisations have played a fundamental role in the Langlands program, and especially in the theory of automorphic L-functions (see [10] for a survey). The local Langlands correspondence gives a natural partition of irreducible smooth representations of H into finite sets called the L-packets. It is conjectured by Shahidi that there exists a unique generic representation inside an L-packet consisting of irreducible tempered representations (see [19, Conjecture 9.4]). When the characteristic of F is zero, the local Langlands correspondence for the group G is established by Rogawski in the book [18], and Shahidi's conjecture, for the group G, is proved in the paper [9]. We also refer to the work of Blasco [3] in the context of local Langlands correspondence for G. We hope that the results of this article are useful in understanding an explicit version of the local Langlands correspondence for G, relating the L-packets of cuspidal representations to their inducing data.

Every cuspidal representation of $\mathrm{GL}_n(F_0)$ is generic (see [1, Chapter 3, 5.18]), but this is no longer true for classical groups. The classification of generic positive depth cuspidal representations from the inducing data is expected to depend on some subtle arithmetic aspects of the inducing data. When the characteristic of F_0 is zero, F/F_0 is unramified, and the cardinality of the residue field of F_0 is odd, Murnaghan classified

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the generic cuspidal representations of G in the article [15, Theorem 7.13]. The methods used in [15] are based on character formulas for cuspidal representations–the Murnaghan–Kirillov formula–and using a local character expansion to relate with Shalika germs. DeBacker and Reeder also studied genericity of very cuspidal representations, arising from an unramified torus, of an unramified p-adic group (see [8]). Blondel and Stevens, using different techniques from Murnaghan, have classified generic cuspidal representations of $\mathrm{Sp}_4(F_0)$, for a non-Archimedean local field F_0 with odd residue characteristic (see [4]). The methods of this article are inspired by the work of Blondel and Stevens on $\mathrm{Sp}_4(F_0)$.

The explicit construction of cuspidal representations of G, when F/F_0 is an unramified extension of p-adic fields, goes back to the work of Moy and Jabon in the articles [14] and [11] respectively. Later, Blasco, in the article [2], gave an explicit construction of cuspidal representations in the line of Bushnell–Kutzko's work on the admissible dual of GL_n. In this article, we use the generalisation of Bushnell–Kutzko construction of cuspidal representations to classical groups by Stevens, culminating in the paper [22].

We now describe the results of this article using the language of strata from the theory of types (see Section 3 and references in *loc.cit*). Let $\mathfrak{r} = [\Lambda, n, 0, \beta]$ be any skew semisimple stratum in $\mathrm{End}_F(V)$, in particular, Λ is a lattice sequence, n is a non-negative integer, $\beta \in \mathrm{End}_F(V)$ with $\sigma_h(\beta) = -\beta$, and the G-stabilizer of β–for the adjoint action of G on its Lie algebra–is isomorphic to a product of unitary groups. Let $\Pi_{\mathfrak{r}}$ be the set of all cuspidal representations containing a type, in the sense of Bushnell–Kutzko, constructed from the stratum $\mathfrak{r}$. Let ψ be a fixed non-trivial additive character of F, and let ψ_β be the function sending $X \in \mathrm{End}_F(V)$ to $\psi(\mathrm{tr}(\beta(\mathrm{id}_V - X)))$. Let $\mathfrak{X}_\beta(F_0)$ be the set of all F_0-rational Borel subgroups $\mathbf{B}$ of $\mathbf{G}$ such that ψ_β defines a character on the group of F_0-rational points of the unipotent radical of $\mathbf{B}$. The main result of this article is the following theorem:

Theorem 1.0.1. *Assume that F has odd residue characteristic. Let $\mathfrak{r} = [\Lambda, n, 0, \beta]$ be any skew semisimple stratum with $n > 0$. The cuspidal representations in the set $\Pi_{\mathfrak{r}}$ are either all generic or all non-generic. If $\mathfrak{X}_\beta(F_0)$ is empty, then every cuspidal representation in the set $\Pi_{\mathfrak{r}}$ is non-generic. Except when β has a non-degenerate two dimensional eigenspace, a cuspidal representation in the set $\Pi_{\mathfrak{r}}$ is generic if and only if $\mathfrak{X}_\beta(F_0)$ is non-empty. If β has a non-degenerate two dimensional eigenspace, V_2, then the set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if (V_2, h) is isotropic. However, every cuspidal representation in the set $\Pi_{\mathfrak{r}}$ is non-generic.*

The set $\mathfrak{X}_\beta(F_0)$ is the set of F_0-rational points of a closed subvariety $\mathfrak{X}_\beta$ of the variety of Borel subgroups of $\mathbf{G}$. We determine necessary and sufficient conditions on β for the non-emptiness of $\mathfrak{X}_\beta(F_0)$. Hence, we obtain a more explicit form of Theorem 1.0.1, and for this, we refer to Theorem 9.0.1. The genericity of depth-zero cuspidal representations G is well understood (see [7, Section 6]). However, we recall these results for giving a complete analysis of genericity of cuspidal representations of G, especially, for those results not stated in the literature, for instance, when F/F_0 is ramified.

In general, the proofs use the explicit construction of cuspidal representations for classical groups by Stevens in the articles [21] and [22] and Mackey-decomposition to understand the restriction of a cuspidal representation to a maximal unipotent subgroup of G. Blondel and Stevens, in the article [4], related the set $\mathfrak{X}_\beta(F_0)$ with the problem of genericity of cuspidal representations of $\mathrm{Sp}_4(F_0)$. We use the approach in [4] to classify generic cuspidal representations of G; however, there are significant differences from the case of $\mathrm{Sp}_4(F_0)$. In the case of $\mathrm{Sp}_4(F_0)$, the variety $\mathfrak{X}_\beta$ is a $\mathbb{P}^1$-fibre space over a quadratic hypersurface–in a 3-dimensional projective space over F_0. Hence, the problem of finding rational points on $\mathfrak{X}_\beta$ is reduced to that of a quadratic hypersurface. For the unitary group in 3-variables, the variety $\mathfrak{X}_\beta$ is the intersection of two quadratic hypersurfaces in a 5-dimensional projective space over F_0. This poses some arithmetic difficulties in understanding the F_0-rational points on $\mathfrak{X}_\beta$. The other significant difference between $\mathrm{Sp}_4(F_0)$ and G is that the isomorphism class of a non-degenerate subspace of (V, h) is not determined by its dimension. For instance, a 2-dimensional non-degenerate subspace of (V, h) can be either isotropic or anisotropic. The problem of genericity depends on these differences.

We briefly sketch the contents of each section. The algebra $F[\beta]$ is a direct sum of fields, say

$$F[\beta] = F[\beta_1] \oplus F\beta_2] \oplus \cdots \oplus F[\beta_k]$$

with $\beta = \sum_{i=1}^{k} \beta_i$ and $\sigma_h(\beta_i) = -\beta_i$, for $1 \leq i \leq k$. This decomposition of $F[\beta]$ corresponds to a maximal orthogonal decomposition of $V = \perp_{i=1}^{k} V_i$, for the property that $F[\beta]$ acts on V_i via its projection onto

$F[\beta_i]$, for $1 \leq i \leq k$. The above decomposition is unique and is determined by β. In Section 2, we set up some preliminaries to prove non-genericity results. In Section 3, we review some useful results from the construction of cuspidal representations of G. In Section 4, we consider the case where $F[\beta]$ is a field. When the characteristic of F_0 is zero, the L-packet containing a cuspidal representation from the set $\Pi_{\mathfrak{r}}$ has cardinality 1. Hence, any representation in the set $\Pi_{\mathfrak{r}}$ is expected to be generic. We first prove that $\mathfrak{X}_\beta(F_0)$ is non-empty, and using this, we will show that every representation in the set $\Pi_{\mathfrak{r}}$ is generic.

In Section 5, we consider the case where $F[\beta]$ is a 3-dimensional algebra, $F[\beta] = F[\beta_2] \oplus F[\beta_1]$, and $\beta = \beta_2 + \beta_1$ such that $[F[\beta_2] : F] = 2$. We will completely determine when $\mathfrak{X}_\beta(F_0)$ is non-empty and this depends only on the valuation of β_i, in the field $F[\beta_i]$, and on the isomorphism class of the hermitian space (V_2, h). Then we will use these results to show that a representation in the set $\Pi_{\mathfrak{r}}$ is generic if and only if the set $\mathfrak{X}_\beta(F_0)$ is non-empty. In this case, we sometimes have to find a nice integral model of $\mathfrak{X}_\beta$ and lift points from its special fibre.

In Sections 6 and 7, we treat the cases where $F[\beta]$ is a direct sum of two copies of F and three copies of F, respectively. The strategy is similar to that of the previous sections. But, in Section 6, we will see examples when $\mathfrak{X}_\beta(F_0)$ is non-empty, and yet every representation in the set $\Pi_{\mathfrak{r}}$ is non-generic. We note that β is not a regular semisimple element in this case.

In Section 7, we have $\beta = \beta_1 + \beta_2 + \beta_2$, with $\beta_i \in F$ and $\sigma(\beta_i) = -\beta_i$. When F/F_0 is unramified, the non-emptiness of the set $\mathfrak{X}_\beta(F_0)$ depends only on the valuations of β_i. However, when F/F_0 is ramified the information on the valuations of β_i, for $1 \leq i \leq 3$, is not enough to determine whether the set $\mathfrak{X}_\beta(F_0)$ is empty or not. Although it is fairly easy to determine the conditions on β for the non-emptiness of $\mathfrak{X}_\beta(F_0)$, these conditions do not involve the natural invariants of the stratum $\mathfrak{r}$; hence, we did not make them explicit. Nonetheless, we will show that a representation in the set $\Pi_{\mathfrak{r}}$ is generic if and only if $\mathfrak{X}_\beta(F_0)$ is non-empty.

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2. Preliminaries

All representations in this article are defined over $\mathbb{C}$-vector spaces. Let G, H be two groups with $H \subset G$, and let ρ be a representation of H. We denote by ρ^g the representation of $g^{-1}Hg$ sending $h \in g^{-1}Hg$ to $\rho(ghg^{-1})$. The group $g^{-1}Hg$ is denoted by H^g.

For any real number x, we denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x. Let $\lceil x \rceil$ be the smallest integer greater than or equal to x. Let $x+$ be the smallest integer strictly bigger than x and $x-$ be the greatest integer strictly smaller than x.

Let (W, h) be a pair consisting of an F-vector space and a non-degenerate hermitian form h on W. Let (W', h) be a non-degenerate subspace of (W, h). Then let $\mathbf{1}_{W'}$ be the projection of W onto W' with kernel $(W')^{\perp}$.

2.1. For a non-Archimedean local field K, let $\mathfrak{o}_K$ be the ring of integers of K, let $\mathfrak{p}_K$ be the maximal ideal of $\mathfrak{o}_K$, let k_K be the residue field $\mathfrak{o}_K/\mathfrak{p}_K$, and q_K denotes the cardinality of the residue field k_K. Let ν_K be the normalised valuation of K. From now we assume that q_K is odd.

Let F be a non-Archimedean local field with a Galois involution σ. Let F_0 be the fixed field of σ. Let ϖ be a uniformizer of F such that $\sigma(\varpi) = (-1)^{e(F|F_0)-1}\varpi$. When F/F_0 is ramified, we set ϖ_0 to be the element $\mathrm{Nr}_{F/F_0}(\varpi)$, and when F/F_0 is unramified, we set $\varpi_0 = \varpi$. The element ϖ_0 is a uniformizer of F_0. Let ψ_0 be a fixed additive character of F_0 with conductor $\mathfrak{p}_{F_0}$. The character $\psi_0 \circ \mathrm{tr}_{F/F_0}$ is denoted by ψ_F. Let $F = F_0[\delta]$, where $\sigma(\delta) = -\delta$ and $\nu_F(\delta) = e(F|F_0) - 1$. Let ν_{F/F_0} be the valuation of F extending the normalised valuation of F_0.

For any F_0-scheme $\mathbf{X}$, we denote by X the set of F_0-rational points of $\mathbf{X}$. If $\mathbf{H}$ is any linear algebraic group over F_0, then the group H is considered as a topological group whose topology is induced from the non-Archimedean metric on F.

2.2. Let V be a three dimensional F-vector space and let h be a non-degenerate hermitian form on V, as defined in (1.1). Let σ_h be the adjoint anti-involution on $\mathrm{End}_F(V)$ induced by the hermitian form h. The hermitian space (V,h) is isotropic and the up to isometry (V,h) depends only on the determinant of (V,h). Let $\mathbf{G}$ be the unitary F_0-group scheme associated with the pair (V,h). The isomorphism class of $\mathbf{G}$ does not depend on the choice of the hermitian form h. To simplify some arguments, it is convenient to assume that the determinant of (V,h) is the trivial class in $F_0^{\times}/\mathrm{Nr}_{F/F_0}(F^{\times})$. We identify the Lie algebra, $\mathfrak{g}$, of $\mathbf{G}$ with

$$\{X \in \mathrm{End}_F(V) : \sigma_h(X) = -X\}.$$

From now the algebra $\mathrm{End}_F(V)$ is denoted by A.

2.3. A basis, (e_1, e_0, e_{-1}), of (V,h) is called a *Witt-basis* if $h(e_1,e_1) = h(e_{-1},e_{-1}) = 0$, $h(e_1,e_{-1}) = 1$, and $e_0 \in \langle e_1, e_{-1}\rangle^{\perp}$ with $h(e_0,e_0) = 1$. A basis, (e_1, e_{-1}), for a two dimensional hermitian space (V',h'), is called a *Witt-basis* if $h(e_1,e_1) = h(e_{-1},e_{-1}) = 0$, and $h(e_1,e_{-1}) = 1$. Let $\mathbf{B}$ be any F_0-rational Borel subgroup of $\mathbf{G}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Let $\mathbf{T}$ be a maximal F_0-split torus of $\mathbf{G}$ contained in $\mathbf{B}$. Let $\bar{\mathbf{U}}$ be the unipotent radical of the opposite Borel subgroup, $\bar{\mathbf{B}}$, of $\mathbf{B}$ with respect to $\mathbf{T}$. Let $\mathbf{Z}$ and $\mathbf{N}$ be the centraliser and the normaliser of $\mathbf{T}$ respectively. We denote by W_G the Weyl group $\mathbf{N}/\mathbf{Z}$.

There exists a Witt-basis (e_1, e_0, e_{-1}) of V–giving an embedding of G in $\mathrm{GL}_3(F)$–such that B stabilises the line $\langle e_1\rangle$. The groups T and Z are identified with the groups

$$\{\mathrm{diag}(t,1,t^{-1}) : t \in F_0^{\times}\} \text{ and } \{\mathrm{diag}(z,z',\sigma(z)^{-1}) : z,z' \in F^{\times},\ z'\sigma(z') = 1\}$$

respectively. The groups U and $\bar{U}$ are identified with the groups

$$\left\{ u(c,d) := \begin{pmatrix} 1 & c & d \\ 0 & 1 & -\sigma(c) \\ 0 & 0 & 1 \end{pmatrix} : c,d \in F,\ c\sigma(c) + d + \sigma(d) = 0 \right\},$$
$$\left\{ \bar{u}(c,d) := \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & -\sigma(c) & 1 \end{pmatrix} : c,d \in F,\ c\sigma(c) + d + \sigma(d) = 0 \right\}$$

respectively. The derived groups of U and $\bar{U}$, denoted by U_{der} and $\bar{U}_{\mathrm{der}}$ respectively, and they are identified with the groups $\{u(0,d) : d \in F, d + \sigma(d) = 0\}$ and $\{\bar{u}(0,d) : d \in F, d + \sigma(d) = 0\}$ respectively. Let $\{U_{\mathrm{der}}(r) : r \in \mathbb{Z}\}$ be a filtration of compact subgroups of U_{der} defined as follows:

$$U_{\mathrm{der}}(r) := \{u(0,y) : y \in \delta\mathfrak{p}_{F_0}^r\}. \tag{2.1}$$

Similarly, we set $\bar{U}_{\mathrm{der}}(r)$ to be the group $\{\bar{u}(0,y) : y \in \delta\mathfrak{p}_{F_0}^r\}$, for $r \in \mathbb{Z}$.

2.4. Let $\mathbf{U}$ be the unipotent radical of an F_0-rational Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. An irreducible smooth representation (π, W) of G is called a *generic representation* if and only if there exists a non-zero linear functional $l : W \to \mathbb{C}$ and a non-trivial character Ψ of U such that

$$l(\pi(u)w) = \Psi(u)l(w) \text{ for all } u \in U, w \in W. \tag{2.2}$$

The group Z acts transitively on the set of non-trivial characters of U, and hence the genericity of an irreducible smooth representation (π, W) of G does not depend on a choice of the pair (U, Ψ). The linear functional $l : W \to \mathbb{C}$ is called a *Whittaker linear functional*. We have

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi, \Psi) \leq 1,$$

for any irreducible smooth representation π of G, and a non-trivial character Ψ of U (see [20] and [17]). We point out that the non-degeneracy condition on the pair (Ψ, U) used in the definition of genericity of p-adic representations, for this particular group G, is equivalent to the condition that Ψ is non-trivial. If (π, W) is a supercuspidal representation, then (π, W) is generic if and only if the character Ψ is non-trivial–otherwise we get a non-trivial element in the Jacquet functor associated to the Borel subgroup $\mathbf{B}$.

2.5. Let β be an element in the algebra A. Let ψ_β be the function on A given by

$$\psi_\beta(X) = \psi_F(\mathrm{tr}(\beta(X - \mathrm{id}_V))) \text{ for all } X \in A.$$

Let $V_1 \subset V_2 \subset V$ be a complete flag of F-vector spaces, and let P be the Borel subgroup of $\mathrm{GL}_F(V)$ fixing this flag. Let Y be the unipotent radical of P. If $V_2 = V_1^{\perp}$, then $Y \cap G$ is the unipotent radical of the Borel subgroup $P \cap G$ of G. The function ψ_β is a character of Y if and only if

$$\beta V_1 \subset V_2. \tag{2.3}$$

Let $\mathfrak{B}$ be the variety of Borel subgroups of $\mathbf{G}$. Let $\mathfrak{X}_\beta(F_0)$ be the following subset of $\mathfrak{B}(F_0)$:

$$\mathfrak{X}_\beta(F_0) = \{\mathbf{B} \in \mathfrak{B}(F_0) : \psi_\beta \text{ is a character of } \mathrm{R}_u(\mathbf{B})(F_0)\}. \tag{2.4}$$

Here, $\mathrm{R}_u(\mathbf{B})$ is the unipotent radical of a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Note that the set $\mathfrak{X}_\beta(F_0)$ is the set of F_0-rational points of a closed sub-variety of $\mathfrak{B}$, to be denoted by $\mathfrak{X}_\beta$.

2.6. The following lemma is frequently used in proving certain cuspidal representations are non-generic. Let (e_1, e_0, e_{-1}) be a Witt-basis for (V, h) and let B be the Borel subgroup of G fixing the line $\langle e_1 \rangle$. Let U be the unipotent radical of B. Using the basis (e_1, e_0, e_{-1}), we identify G as a subgroup of $\mathrm{GL}_3(F)$.

Lemma 2.6.1. *Let $\beta \in A$ be an element such that $\sigma_h(\beta) = -\beta$. Let g be an element of G, and let r be an integer. The character ψ_β^g of $U_{\mathrm{der}}(r)$ is non-trivial if and only if*

$$\nu_{F_0}(\delta h(ge_1, \beta ge_1)) \leq -r.$$

Proof. We prove the lemma for $U_{\mathrm{der}}(r)$, and the other case is similar. Let X_{der} be the 3×3 matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have the following equality:

$$\psi_\beta(gu(0, \delta d)g^{-1}) = \psi_0(d(\mathrm{tr}_{F/F_0}(\delta \, \mathrm{tr}(\beta g X_{\mathrm{der}} g^{-1})))).$$

Note that $\mathrm{tr}(\beta g X_{\mathrm{der}} g^{-1})$ is equal to $\mathrm{tr}(g^{-1}\beta g X_{\mathrm{der}})$, and $\mathrm{tr}(g^{-1}\beta g X_{\mathrm{der}})$ is equal to $h(ge_1, \beta ge_1)$. Since $\sigma_h(\beta) = -\beta$, we get that

$$\mathrm{tr}_{F/F_0}(\delta h(ge_1, \beta ge_1)) = 2\delta h(ge_1, \beta ge_1).$$

Hence, the character ψ_β^g is trivial on $U_{\mathrm{der}}(r)$ if and only if the character $d' \mapsto \psi_0(d'\delta h(ge_1, \beta ge_1))$, for $d' \in \mathfrak{p}_{F_0}^r$, is trivial. Since the conductor of ψ_0 is equal to $\mathfrak{p}_{F_0}$, we get the required inequality. ☐

3. Strata and cuspidal representations

In this section, we recall the construction of cuspidal representations of G, via Bushnell–Kutzko's theory of types. We refer to the articles [21], [22] and [13] for more details.

3.1. An $\mathfrak{o}_F$-*lattice sequence*, Λ, on V is a function from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V satisfying the following conditions:

(1) $\Lambda(n+1) \subseteq \Lambda(n)$, for all $n \in \mathbb{Z}$,
(2) there exists a positive integer $e(\Lambda)$ such that $\Lambda(n + e(\Lambda)) = \mathfrak{p}_F \Lambda(n)$, for any $n \in \mathbb{Z}$.

Given any lattice $\mathcal{L} \subset V$, let $\mathcal{L}^\#$ be the lattice $\{v \in V : \; h(v, \mathcal{L}) \subseteq \mathfrak{p}_F\}$. For any lattice sequence Λ, let $\Lambda^\#$ be the lattice sequence defined as:

$$\Lambda^\#(n) = \Lambda(-n)^\#, \text{ for all } n \in \mathbb{Z}.$$

A lattice sequence Λ is said to be *self-dual* if there exists an integer d such that $\Lambda^\#(n) = \Lambda(-n + d)$, for all $n \in \mathbb{Z}$. Since we only use $\mathfrak{o}_F$-lattice sequences, we call them directly as lattice sequences.

Let W be a subspace of the vector space V, and let Λ be a lattice sequence on V. We denote by $\Lambda \cap W$ the lattice sequence on W sending n to $\Lambda(n) \cap W$.

Given any lattice sequence Λ and integers $a, b \in \mathbb{Z}$, the lattice sequence $a\Lambda + b$ is defined by setting

$$(a\Lambda + b)(n) = \Lambda(\lceil (n-b)/a \rceil), \text{ for all } n \in \mathbb{Z}.$$

The set of lattice sequences $\{a\Lambda + b : \ a, b \in \mathbb{Z}\}$ is called the *affine class* of Λ. For any self-dual lattice-sequence Λ, we can find a lattice sequence Λ' in the affine class of Λ such that $e(\Lambda')$ is an even integer, and $(\Lambda')^{\#} = -\Lambda' + 1$. Henceforth, we assume that all self-dual lattice sequences satisfy these conditions.

3.2. Given any lattice sequence Λ, and an integer n, let $\tilde{\mathfrak{a}}_n(\Lambda)$ be the following sublattice of $\mathrm{End}_F(V)$:

$$\tilde{\mathfrak{a}}_n(\Lambda) = \{T \in \mathrm{End}_F(V) : T\Lambda(i) \subset \Lambda(i+n) \ \forall \ i \in \mathbb{Z}\}.$$

The decreasing sequence of lattices $\{\tilde{\mathfrak{a}}_n(\Lambda) : n \geq 0\}$ has trivial intersection. Given any element $T \in \mathrm{End}_F(V)$, we denote by $\nu_\Lambda(T)$ the unique integer k such that $T \in \tilde{\mathfrak{a}}_k(\Lambda)$ and $T \notin \tilde{\mathfrak{a}}_{k+1}(\Lambda)$. Let $\tilde{P}_0(\Lambda)$ be the units in the ring $\tilde{\mathfrak{a}}_0(\Lambda)$. For any positive integer n, let $\tilde{P}_n(\Lambda)$ be the compact open subgroup $\mathrm{id}_V + \tilde{\mathfrak{a}}_n(\Lambda)$ of $\mathrm{GL}_F(V)$. For any self-dual lattice sequence Λ, the lattices $\tilde{\mathfrak{a}}_n(\Lambda)$ are stable under σ_h. For any non-negative integer n, let $P_n(\Lambda)$ be the compact open subgroup $\tilde{P}(\Lambda) \cap G$ of G.

The group $P_0(\Lambda)/P_1(\Lambda)$ is the set of k_{F_0}-rational points of a (not necessarily connected) reductive algebraic group over k_{F_0}, and let $P^0(\Lambda)$ be the inverse image of the k_{F_0}-rational points of its connected component. The compact subgroup $P^0(\Lambda)$ is called the parahoric subgroup associated to Λ. If F/F_0 is unramified, then $P^0(\Lambda)$ is equal to $P_0(\Lambda)$ and has index 2 in $P_0(\Lambda)$ otherwise.

A *stratum* in $\mathrm{End}_F(V)$ is the data, $[\Lambda, n, r, \beta]$, consisting of a lattice sequence Λ on V, integers $n \geq r \geq 0$, and an element $\beta \in \mathrm{End}_F(V)$ such that $\beta \in \tilde{a}_{-n}(\Lambda)$. Two strata $[\Lambda, n, r, \beta_1]$ and $[\Lambda, n, r, \beta_2]$ are said to be equivalent if $\beta_2 - \beta_1 \in \tilde{a}_{-r}(\Lambda)$. A stratum $[\Lambda, n, r, \beta]$ is called a *zero stratum* if $n = r$ and $\beta = 0$. For $n \geq r \geq n/2 > 0$, the set of equivalence classes of strata are in bijection with the characters of the group $\tilde{P}_{r+1}(\Lambda)/\tilde{P}_{n+1}(\Lambda)$. The character corresponding to the equivalence class containing the stratum $[\Lambda, n, r, \beta]$ is given by

$$\psi_\beta : \mathrm{id}_V + X \mapsto \psi_F(\mathrm{tr}\beta X), \text{ for } X \in \tilde{a}_n(\Lambda).$$

A stratum is called *skew* if the lattice sequence Λ is self-dual and $\beta \in \mathfrak{g}$. We have the same notion of equivalence on skew strata. For $n \geq r \geq n/2 > 0$, an equivalence class of skew strata corresponds to a character on the group $P_{r+1}(\Lambda)/P_{n+1}(\Lambda)$, given by $\mathrm{res}_{P_{r+1}(\Lambda)} \psi_\beta$.

3.3. Recall that a stratum $[\Lambda, n, r, \beta]$ is called a *simple stratum* if it satisfies the following conditions:

(1) We have $n \geq r \geq 0$,
(2) The valuation of β with respect to Λ, denoted by $\nu_\Lambda(\beta)$, is equal to $-n$.
(3) The algebra $F[\beta]$ is a field and it normalises the lattice sequence Λ.
(4) We have $r < -k_0(\Lambda, \beta)$, where $k_0(\Lambda, \beta)$ is the *critical constant* defined in [21, Section 1.2.2].

Let $\mathfrak{r} = [\Lambda, n, 0, \beta]$ be a stratum in A and let $V = \bigoplus_{i=1}^r V_i$ be a decomposition of V as F-vector spaces. Let Λ_i be the lattice sequence defined by setting $\Lambda_i(k) = \Lambda(k) \cap V_i$, for $k \in \mathbb{Z}$. Let β_i be the element $\mathbf{1}_i \, \beta \, \mathbf{1}_i$, where $\mathbf{1}_i$ is the projection onto V_i with kernel $\bigoplus_{j \neq i} V_j$. The decomposition $V = \oplus_{i=1}^r V_i$ is called a *splitting* for the stratum $\mathfrak{r}$ if $\Lambda(k) = \oplus_{i=1}^n \Lambda(k)$, for $k \in \mathbb{Z}$, and $\beta = \sum_{i=1}^r \beta_i$.

A stratum $\mathfrak{r} = [\Lambda, n, 0, \beta]$ is called a *semisimple stratum* if it is either a zero stratum or $\nu_\Lambda(\beta) = -n$ and there exists a splitting $V = \bigoplus_{i=1}^r V_i$ for the stratum $\mathfrak{r}$ such that

(1) The stratum $[\Lambda_i, q_i, r, \beta_i]$–with $q_i = r$ if $\beta_i = 0$ and $q_i = -\nu_{\Lambda_i}(\beta_i)$ otherwise–is either a zero stratum or a simple stratum, and
(2) the stratum $[\Lambda_i + \Lambda_j, q, r, \beta_i + \beta_j]$, with $q = \max\{q_i, q_j\}$, is non-equivalent to a zero stratum or a simple stratum, for $1 \leq i, j \leq k$ and $i \neq j$.

The decomposition $V = \oplus_{i=1}^k V_i$ is uniquely determined upto ordering by the element β, called the *underlying splitting* of the semisimple stratum $[\Lambda, n, r, \beta]$. A semisimple stratum $[\Lambda, n, r, \beta]$ is called a *skew semisimple stratum* if the decomposition $V = \oplus_{i=1}^k V_i$ is an orthogonal decomposition with respect to the form h on V, and $\sigma_h(\beta_i) = -\beta_i$, for $1 \leq i \leq k$. Observe that the algebra $F[\beta]$ is isomorphic to the algebra

$$F[\beta_1] \oplus F[\beta_2] \oplus \cdots \oplus F[\beta_k].$$

We use the notation $\mathfrak{x}$ for a general skew semisimple stratum $[\Lambda, n, 0, \beta]$.

Let $C_\beta(A)$ be the centraliser of $F[\beta]$ in $\mathrm{End}_F(V)$. The group $G \cap C_\beta(A)$ is denoted by G_β. Let n be any integer and let $\tilde{\mathfrak{b}}_n(\Lambda)$ and $\mathfrak{b}_n(\Lambda)$ be the groups $\tilde{\mathfrak{a}}_n(\Lambda) \cap C_\beta(A)$ and $\mathfrak{a}_n(\Lambda) \cap C_\beta(A)$ respectively. For any non-negative integer n, let $\tilde{P}_n(\Lambda_\beta)$ and $P_n(\Lambda_\beta)$ be the groups $\tilde{P}_n(\Lambda) \cap C_\beta(A)^\times$ and $P_n(\Lambda) \cap G_\beta$ respectively.

3.4. We recall the construction of a cuspidal representation starting from a skew semisimple stratum $\mathfrak{x} = [\Lambda, n, 0, \beta]$. Stevens, generalising the Bushnell–Kutzko's construction, associates some special compact subgroups: $J^0(\Lambda, \beta)$ and $H^0(\Lambda, \beta)$ of G. Then a particular class of representations of $J^0(\Lambda, \beta)$ are compactly induced to the group G to obtain cuspidal representations. We will not go into the details of the construction of these compact subgroups here. It will suffice to briefly recall the general scheme of this construction. However, we describe these compact subgroups, in more detail, as required in the later part of the article.

Let $J^i(\Lambda, \beta)$ be the compact open subgroup $J^0(\Lambda, \beta) \cap P_i(\Lambda)$, for any non-negative integer i. For any skew-semisimple stratum $\mathfrak{x} = [\Lambda, n, 0, \beta]$, the construction of cuspidal representations of G begins with a specific set of characters of the group $H^1(\Lambda, \beta)$ called *skew semisimple characters*, denoted by $\mathcal{C}(\Lambda, 0, \beta)$, (see [21, Section 3.6] and the set $\mathcal{C}(\Lambda, 0, \beta)$ is denoted by $\mathcal{C}_-(\Lambda, 0, \beta)$ there). The group $P_{(n/2)+}(\Lambda)$ is contained in $H^1(\Lambda, \beta)$, and we have $\mathrm{res}_{P_{(n/2)+}(\Lambda)} \theta = \psi_\beta$, for any $\theta \in \mathcal{C}(\Lambda, 0, \beta)$. For any character $\theta \in \mathcal{C}(\Lambda, 0, \beta)$, the map sending $g_1, g_2 \in J^1(\Lambda, \beta)$ to $\theta([g_1, g_2])$ induces a perfect alternating pairing:

$$\kappa_\theta : \frac{J^1(\Lambda, \beta)}{H^1(\Lambda, \beta)} \times \frac{J^1(\Lambda, \beta)}{H^1(\Lambda, \beta)} \to \mathbb{C}^\times.$$

Using the theory of Heisenberg lifting, for any character $\theta \in \mathcal{C}(\Lambda, 0, \beta)$, there exists a unique representation η_θ of $J^1(\Lambda, \beta)$ such that $\mathrm{res}_{H^1(\Lambda,\beta)} \eta_\theta$ is equal to a multiple of θ. There are a particular set of extensions of the representation η_θ to the group $J^0(\Lambda, \beta)$ called β-extensions; these representations are denoted by κ (see [22, Section 4]).

The group $P_0(\Lambda_\beta)$ is contained in $J^0(\Lambda, \beta)$. The inclusion of $P_0(\Lambda_\beta)$ in $J^0(\Lambda, \beta)$ induces an isomorphism

$$P_0(\Lambda_\beta)/P_1(\Lambda_\beta) \simeq J^0(\Lambda, \beta)/J^1(\Lambda, \beta).$$

The group $P_0(\Lambda_\beta)/P_1(\Lambda_\beta)$ is the k_{F_0}-rational points of a (non-necessarily connected) reductive group over k_{F_0}. Let τ be a cuspidal representation of $P_0(\Lambda_\beta)/P_1(\Lambda_\beta)$. If $P^0(\Lambda_\beta)$ is a maximal parahoric subgroup of $C_\beta(A) \cap G$, then the induced representation

$$\mathrm{ind}_{J^0(\Lambda,\beta)}^G (\kappa \otimes \tau) \tag{3.1}$$

is irreducible, and this construction exhausts all cuspidal representations of G. The pair $(J^0(\Lambda, \beta), \kappa \otimes \tau)$ is a Bushnell–Kutzko type for the Bernstein component containing the representation (3.1). Let $\Pi_\mathfrak{x}$ be the set of cuspidal representations of G containing a Bushnell–Kutzko type of the form $(J^0(\Lambda, \beta), \kappa \otimes \tau)$, for some κ and τ as above.

3.5. Let π be a cuspidal representation of $U(V, h)$ such that V is an F-vector space of prime dimension. Assume that $\pi \in \Pi_\mathfrak{r}$, where $\mathfrak{r} = [\Lambda, n, 0, \beta]$ is a skew simple strata. If $\mathfrak{r}$ is not a minimal strata then there exists a skew simple strata $\mathfrak{r}' = [\Lambda, n, -\kappa(\Lambda, \beta), \gamma]$ equivalent to $[\Lambda, n, -\kappa(\Lambda, \beta), \beta]$ (see [21, Proposition 3.4]). Using [6, Theorem 2.4.1], we get that $\gamma \in F$. This shows that we can twist the representation π with a character χ such that $\chi\pi \in \Pi_{\mathfrak{r}'}$, where $\mathfrak{r}'$ is a minimal skew simple strata.

3.6. For the convenience of the reader, we recall some frequently used results from [4]. Let us begin with the following lemma, which is useful in calculating the group $H^1(\Lambda, \beta) \cap U$.

Lemma 3.6.1 (Blondel–Stevens). *Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum in $\mathrm{End}_F(V)$ such that $C_\beta(A)$ does not contain any nilpotent elements. Let N be a maximal unipotent subgroup of G. For $k \geq m \geq 1$, we have*

$$P_m(\Lambda_\beta)P_k(\Lambda) \cap N = P_k(\Lambda) \cap N.$$

We refer to [4, Section 6.3, Lemma 6.5] for a proof of the above lemma. The following result is proved in greater generality by Blondel and Stevens (see [4, Section 4, Corollary 4.2, Theorem 4.3]); however, in the present context we will use the following simple version.

Proposition 3.6.2 (Blondel–Stevens). *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum such that $\mathfrak{X}_\beta(F_0)$ is non-empty. Assume that $J^0(\Lambda, \beta)/J^1(\Lambda, \beta)$ is anisotropic. Let $\mathbf{B} \in \mathfrak{X}_\beta(F_0)$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. If*

$$\mathrm{res}_{H^1(\Lambda,\beta)\cap U}\, \theta = \mathrm{res}_{H^1(\Lambda,\beta)\cap U}\, \psi_\beta,$$

for all $\theta \in \mathcal{C}(\Lambda, 0, \beta)$, then every cuspidal representation in the set $\Pi_\mathfrak{x}$ is generic.

Proof. Since we use this result in a crucial way, we briefly sketch the proof (see [4, Section 4, Corollary 4.2, Theorem 4.3]). Let π be a cuspidal representation in the set $\Pi_{\mathfrak{x}}$. Let $(J^0(\Lambda,\beta),\kappa)$ be a Bushnell–Kutzko type contained in the representation π. We have

$$\pi \simeq \operatorname{ind}_{J^0(\Lambda,\beta)}^{G} \kappa.$$

Let θ be a skew semisimple character contained in $\operatorname{res}_{H^1(\Lambda,\beta)} \kappa$.

Let $\tilde{H}^1$ be the group $(J^0(\Lambda,\beta)\cap U)H^1(\Lambda,\beta)$ and Θ_β be the character of $\tilde{H}^1$ defined by:

$$\Theta_\beta(jh) = \psi_\beta(j)\theta(h), \text{ for all } j \in J^0(\Lambda,\beta)\cap U,\ h \in H^1(\Lambda,\beta).$$

The group $\tilde{H}^1 \cap J^1(\Lambda,\beta)$ is equal to $(J^1(\Lambda,\beta)\cap U)H^1(\Lambda,\beta)$, and it is a totally isotropic subspace for the pairing κ_θ on $J^1(\Lambda,\beta)/H^1(\Lambda,\beta)$. The representation η_θ is the induced representation from an extension of the character Θ_β to the inverse image in $J^1(\Lambda,\beta)$ of a maximal isotropic subspace of $J^1(\Lambda,\beta)/H^1(\Lambda,\beta)$. Hence, $\operatorname{res}_{J^1\cap U}\eta_\theta$ contains the character $\operatorname{res}_{J^1(\Lambda,\beta)\cap U}\psi_\beta$. Since the group $J^0(\Lambda,\beta)\cap U$ is equal to $J^1(\Lambda,\beta)\cap U$, using Mackey-decomposition, we get that $\operatorname{Hom}_U(\pi,\psi_\beta)\neq 0$. □

It is convenient to partition the set of skew-semisimple strata in $\operatorname{End}_F(V)$ into four disjoint classes. A skew semisimple stratum $\mathfrak{x} = [\Lambda, n, 0, \beta]$ is of type **(A)** if $F[\beta]$ is a field. A skew semisimple stratum $\mathfrak{x}$ is of type **(B)** if $F[\beta]$ is a direct sum of two fields with one of them a quadratic extension of F. A skew semisimple stratum $\mathfrak{x}$ is of type **(C)** if the algebra $F[\beta]$ is a direct sum of two copies of F, and finally a skew semisimple stratum $\mathfrak{x}$ is of type **(D)** if the algebra $F[\beta]$ is a direct sum of three copies of F.

4. The simple case.

4.1. A skew semisimple stratum $[\Lambda, n, 0, \beta]$ is of type **(A)** if the algebra $F[\beta]$ is a field. When the characteristic of F_0 is zero, it was shown by Blasco that the cardinality of the L-packet, containing a cuspidal representation in the set $\Pi_{\mathfrak{x}}$, is equal to one (see [3]). When the characteristic of F_0 is zero, a tempered L-packet is known to contain a generic member (see [9]), and hence every cuspidal representation in the set $\Pi_{\mathfrak{x}}$ is generic. In this section, for any non-Archimedean local field F_0 with odd residue characteristic, we directly prove that every cuspidal representation in the set $\Pi_{\mathfrak{x}}$ is generic.

Lemma 4.1.1. *Let $[\Lambda, n, 0, \beta]$ be a skew simple stratum of type* **(A)***, then the set $\mathfrak{X}_\beta(F_0)$ is non-empty.*

Proof. The involution σ_h stabilises the field $F[\beta]$, and let D be the fixed field of σ_h in $F[\beta]$. We have the following diagram of field extensions:

$$\begin{tikzcd}[column sep=small] & F[\beta] \arrow[dl, dash] \arrow[dr, dash] & \\ F \arrow[dr, dash] & & D = F_0[\delta\beta] \arrow[dl, dash] \\ & F_0 & \end{tikzcd}$$

Let λ be a σ_h-equivariant non-zero F-linear form on $F[\beta]$. There exists a unique hermitian form $h_1 : V\times V \to F[\beta]$ with respect to σ_h such that

$$h(v,w) = \lambda((h_1(v,w)), \text{ for all } v,w \in V$$

(see [5, Lemma 5.2]). The set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only there exists a non-zero vector, $v \in V$, such that

$$h(v,v) = 0 \text{ and } h(v,\beta v) = 0.$$

Observe that $h(v,\beta v) = \lambda(h_1(v,\beta v)) = \lambda(\beta h_1(v,v))$. Since β does not stabilise a proper non-zero subspace of $F[\beta]$, the F-linear forms λ and $\lambda\circ\beta$ are linearly independent. Let W be the F-vector space $\ker(\lambda)\cap\ker(\lambda\circ\beta)$, we have $\dim_F W = 1$. Since $\sigma_h(\beta) = -\beta$ and λ is σ_h-equivariant, the F-vector space W is stable under the action of σ_h. Let W_0 be the F_0-vector space $W\cap D$, and note that $W_0 \otimes_{F_0} F = W$.

The form $h_1(x,y)$ is equal to $xa\sigma_h(y)$, for some $a \in D$. Let N_a be the set $\{xa\sigma_h(x) : x \in F[\beta]^\times\}$. Assume that the inclusion of $F_0^\times$ in $D^\times$ induces a surjection of $F_0^\times$ onto the quotient $D^\times/(\operatorname{Nr}_{F[\beta]/D}(F[\beta]^\times)$. Let w be a non-zero vector in W_0. If $w \notin N_a$, then there exists some element $x \in F_0^\times$ such that $xw_1 \in N_a$.

Hence, the set $N_a \cap W_0$ is non-empty. Since $h(w,w) = h(w, \beta w) = 0$, for all $w \in N_a \cap W_0$, the set $\mathfrak{X}_\beta(F_0)$ is non-empty.

Now, we will prove that the inclusion of $F_0^\times$ in $D^\times$ induces a surjection of $F_0^\times$ onto the quotient

$$D^\times/(\mathrm{Nr}_{F[\beta]/D}(F[\beta]^\times).$$

Recall the notations ϖ and ϖ_0 for a choice uniformizers of F and F_0, respectively, from Paragraph 2.1. Fix a valuation $\nu : F[\beta]^\times \to 1/e[F[\beta] : F]\mathbb{Z}$ such that $\nu(\varpi) = 1$. Assume that $\varpi_0 = \mathrm{Nr}_{F[\beta]/D}(x)$, for some $x \in F[\beta]$. Then, we have $\nu(\varpi) = 2\nu(x)$. If F/F_0 is unramified, then the equality $\nu(\varpi) = 2\nu(x)$ is impossible and therefore, we get that $\varpi_0 \notin \mathrm{Nr}_{F[\beta]/D}(F[\beta]^\times)$. Consider the case where F is a ramified extension of F_0. Let x be an element of $\mathfrak{o}_{F_0}^\times$ such that $\bar{x} \in k_{F_0}$ is not a square in k_{F_0}. The element x does not belong to $\mathrm{Nr}_{F[\beta]/D}(F[\beta]^\times)$; since the involution σ_h must act trivially on the residue field of $k_{F[\beta]}$ (note that $[k_{F[\beta]} : k_{F_0}]|3$). □

Proposition 4.1.2. *Let $\mathfrak{x} = [\Lambda, n, 0, \beta]$ be any skew simple stratum of type* **(A)**. *Then every cuspidal representation in the set $\Pi_\mathfrak{x}$ is generic.*

Proof. Let π be a cuspidal representation in the set $\Pi_\mathfrak{x}$. Since $\dim_F V = 3$, we may twist the representation π by a character, if necessary, and assume that $\mathfrak{x}$ is minimal. Let $(J^0(\Lambda, \beta), \kappa)$ be a Bushnell–Kutzko type contained in π. Then the representation π is isomorphic to $\mathrm{ind}_{J^0(\Lambda,\beta)}^G \kappa$. Here, the group $J^0(\Lambda, \beta)$ is equal to $\mathfrak{o}_{F[\beta]}^\times P_{(n/2)+}(\Lambda)$, and κ is a β-extension of the Heisenberg lift η_θ of a skew semisimple character θ of $H^1(\Lambda, \beta)$. Now, Lemma 4.1.1 implies that the set $\mathfrak{X}_\beta(F_0)$ is non-empty. Let $\mathbf{U}$ be the unipotent radical of a Borel subgroup in the set $\mathfrak{X}_\beta(F_0)$. Using Lemma 3.6.1, we get that the group $J^0(\Lambda, \beta) \cap U$ is equal to $P_{(n/2)+}(\Lambda) \cap U$. Hence, we have

$$\mathrm{res}_{H^1(\Lambda,\beta)\cap U}\, \theta = \mathrm{res}_{H^1(\Lambda,\beta)\cap U}\, \psi_\beta.$$

Now using Proposition 3.6.2, we get that the representation π is generic. □

5. The non simple type (B) strata

A skew semisimple stratum $\mathfrak{x} = [\Lambda, n, 0, \beta]$ is of type **(B)** if the underlying splitting of $\mathfrak{x}$ is of the form $V = V_1 \perp V_2$ with $\dim_F(V_2) = 2$ and the algebra $F[\beta_2]$ is a quadratic extension of F. Recall that β_i is equal to $\mathbf{1}_{V_i}\beta\mathbf{1}_{V_i}$, for $i \in \{1, 2\}$. From the definition of a skew semisimple stratum, we have $\beta = \beta_1 + \beta_2$. We recall the notation that $q_i = -\nu_{\Lambda_i}(\beta_i)$, for $i \in \{1, 2\}$. Genericity of a cuspidal representation, in the set $\Pi_\mathfrak{x}$, depends only on the isomorphism class of the hermitian space (V_2, h) and the integers q_1 and q_2. Since $\dim_F(V_2) = 2$, after twisting by a character, if necessary, we may assume that β_2 is minimal.

The ramification index of the extension $F[\beta_2]/F$ is determined by that of F/F_0. Note that the extension $F[\beta_2]/F_0$ is the composite of two disjoint fields F and $D = F[\delta\beta_2]$. Hence the Galois group of $F[\beta_2]/F_0$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If F/F_0 is unramified, then $F[\beta_2]/F$ is ramified, since all unramified extensions are Galois extensions with cyclic Galois group. Assume that F/F_0 is a ramified extension. If $F[\beta_2]/F$ is ramified then, then $q_2 = \nu_{F\beta_2/F}$ is an odd integer, and let ϖ be an uniformiser of $F[\beta_2]$ such that $\sigma_h(\varpi) = -\varpi$. Observe that the image of $\varpi^{-q_2}\beta_2$ is the residue field is zero singe $\sigma_h(\beta_2) = -\beta_2$. We therefore deduce that if F/F_0 is ramified, then $F[\beta_2]/F$ is unramified.

5.1. Lattice sequences.

Since G_β is anisotropic, the lattice sequence Λ is uniquely determined by β. First, we need to fix a Witt-basis of V which provides a splitting for Λ.

5.1.1. *The Unramified case.* Let F be an unramified quadratic extension of F_0, and consider the case where (V_2, h) is isotropic. Since F/F_0 is unramified, the extension $F[\beta_2]/F$ is ramified. In this case, we have $e(\Lambda) = 4$. There exists a Witt-basis, (e_1, e_{-1}), for the space V_2, and a unit vector $e_0 \in V_1$ (i.e., $h(e_0, e_0) = 1$) such that the lattice sequence Λ is given by:

$$\Lambda(-1) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1},\ \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1},$$
$$\Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1},\ \ \Lambda(2) = \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

The group $P(\Lambda)$ is an Iwahori subgroup of G. The filtration $\{P_m(\Lambda) : m > 0\}$ is sometimes called a non-standard filtration on the Iwahori subgroup $P(\Lambda)$ (see [15, Page 612]).

Now, consider the case where F/F_0 is unramified and (V_2, h) is anisotropic. There exists an orthogonal basis (v_2, v_3) of V_2 such that $h(v_2, v_2) = 1$ and $h(v_3, v_3) = \varpi_0$. Let v_1 be any vector in V_1 such that $h(v_1, v_1) = \varpi_0$. The period of the lattice sequence, $e(\Lambda)$, is equal to 2, and

$$\Lambda(0) = \mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_2 \oplus \mathfrak{o}_F v_3 \text{ and } \Lambda(1) = \mathfrak{o}_F v_1 \oplus \mathfrak{p}_F v_2 \oplus \mathfrak{o}_F v_3. \tag{5.1}$$

Lemma 5.1.1. *Let (V_2, h) be anisotropic, and let Λ be the lattice sequence in (5.1). There exists a Witt-basis (e_1, e_{-1}) for $(\langle v_1, v_3\rangle, h)$ such that*

$$\mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_3 = \mathfrak{p}_F e_1 \oplus \mathfrak{o}_F e_{-1}.$$

Proof. Let $\epsilon \in F$ be an element with $\epsilon\sigma(\epsilon) = -1$; such an element exists because F/F_0 is unramified. The vectors $e_1 = \epsilon v_1/2 + v_3/2$ and $\varpi_0 e_{-1} = -\epsilon v_1 + v_3$ are isotropic vectors, and $h(e_1, e_{-1}) = 1$. This implies that (e_1, e_{-1}) is a Witt-basis for $\langle v_1, v_3\rangle$, and since $\nu_F(\epsilon) = 0$, the tuple $(e_1, \varpi_0 e_{-1})$ is a $\mathfrak{o}_F$ basis for the lattice $\mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_3$. □

When (V_2, h) is anisotropic as above, we set e_0 to be the vector v_2. The Witt-basis (e_1, e_0, e_{-1}), where (e_1, e_{-1}) is a Witt-basis for $(\langle v_1, v_3\rangle, h)$ as in Lemma 5.1.1 provides a splitting for Λ. In the basis (e_1, e_0, e_{-1}), the lattice sequence Λ, with $e(\Lambda) = 2$, is given by

$$\Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

5.1.2. *Ramified case.* Let F/F_0 be a ramified quadratic extension. Then the extension $F[\beta_2]/F$ is an unramified quadratic extension. First consider the case where (V_2, h) is isotropic. The lattice sequence Λ has period 2. There exists a Witt-basis, (e_1, e_0, e_{-1}), for the hermitian space (V, h) with $e_1, e_{-1} \in V_2$ such that

$$\Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

Now, consider the case where (V_2, h) is anisotropic. There exists an orthogonal basis (v_2, v_3) of V_2 and a non-zero vector $v_1 \in V_1$ such that: $h(v_i, v_i) = \lambda_i \in \mathfrak{o}_F^\times$, for $1 \leq i \leq 3$, and the hermitian space $(\langle v_1, v_3\rangle, h)$ is isotropic. The period of the lattice sequence Λ is equal to 2, and we have:

$$\Lambda_2(-1) = \Lambda_2(0) = \mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_2 \oplus \mathfrak{o}_F v_3.$$

Lemma 5.1.2. *Let (V_2, h) be anisotropic. Then there exists a Witt-basis (e_1, e_{-1}) for the hermitian space $(\langle v_1, v_3\rangle, h)$ such that*

$$\mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_3 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_{-1}.$$

Proof. We fix an $\epsilon \in F$ such that $\epsilon\sigma(\epsilon) = -\lambda_3\lambda_1^{-1}$. The vectors $e_1 = \epsilon v_1/2 + v_3/2$ and $e_{-1} = (-\epsilon v_1 + v_3)\lambda_3^{-1}$ are isotropic and $h(e_1, e_{-1}) = 1$. Moreover, the vectors e_1 and e_{-1} are a basis for the $\mathfrak{o}_F$-lattice $\mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_3$. □

Let (e_1, e_{-1}) be a Witt-basis for the hermitian space $(\langle v_1, v_3\rangle, h)$ as in Lemma 5.1.2. Let e_0 be the vector v_2. In the basis (e_1, e_0, e_{-1}), the period 2 lattice sequence Λ is given by

$$\Lambda(-1) = \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}.$$

5.2. In each of the above cases, we fixed a Witt-basis (e_1, e_0, e_{-1}) which gives a splitting for the lattice sequence Λ. Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{G}$ such that B fixes the space $\langle e_1\rangle$. Let $\mathbf{T}$ be the maximal F_0-split torus of $\mathbf{G}$ such that T stabilises the decomposition $V = \langle e_1\rangle \oplus \langle e_0\rangle \oplus \langle e_{-1}\rangle$. Let $\bar{\mathbf{B}}$ be the opposite Borel subgroup of $\mathbf{B}$ with respect to $\mathbf{T}$. Let $\mathbf{U}$ (resp. $\bar{\mathbf{U}}$) be the unipotent radical of $\mathbf{B}$ (resp. $\bar{\mathbf{B}}$). We also recall the notations $u(c, d)$ and $\bar{u}(c, d)$ for elements in U and $\bar{U}$ respectively. In the basis (e_1, e_0, e_{-1}), let $\mathcal{I}$ be the Iwahori subgroup:

$$\mathcal{I} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F. \end{pmatrix} \cap G.$$

From the Iwasawa decomposition, we get that

$$G = \coprod_{w \in W_G} \mathcal{I} w B. \tag{5.2}$$

5.3. The integers q_1 and q_2 have the following constraints: If F/F_0 is unramified and (V_2, h) is isotropic, then $q_2 = 4m_2 + 2$ and $q_1 = 4m_1$ for some $m_1, m_2 \in \mathbb{Z}$. If F/F_0 is unramified and (V_2, h) is anisotropic, then $q_2 = 2m_2 + 1$ and $q_1 = 2m_1$, for some $m_1, m_2 \in \mathbb{Z}$. Finally consider the case where F/F_0 is ramified. Since the image of the element

$$y_{\beta_2} = \varpi^{q_2/g} \beta_2^{e(\Lambda_2)/g} = \varpi^{q_2/2} \beta_2$$

(here, $g = (q_2, e(\Lambda_2)) = 2$) in $k_{F[\beta_2]}$ must generate the degree 2 field extension $k_{F[\beta_2]}$ over k_F, we get that $q_2 = 4m_2$ and $q_1 = 4m_1 + 2$, for some $m_1, m_2 \in \mathbb{Z}$. Hence, in all the above cases $q_1 \neq q_2$.

5.4. We will need the structure of the compact subgroups $J^0(\Lambda, \beta)$ and $H^0(\Lambda, \beta)$ associated to $\mathfrak{r}$. First consider the case $q_2 < q_1$, and in this case the constant $k_0(\Lambda, \beta)$, defined in [21, equation 3.6], is equal to q_2. The stratum $[\Lambda, n, q_2, \beta_1]$ is a skew semisimple stratum equivalent to the stratum $[\Lambda, n, q_2, \beta]$. We then have:

$$\begin{aligned} J^0(\Lambda, \beta) =& P_0(\Lambda_\beta) P_{q_2/2}(\Lambda_{\beta_1}) P_{n/2}(\Lambda). \\ H^1(\Lambda, \beta) =& P_1(\Lambda_\beta) P_{(q_2/2)+}(\Lambda_{\beta_1}) P_{(n/2)+}(\Lambda). \end{aligned}$$

If $q_2 > q_1$, then we have

$$\begin{aligned} J^0(\Lambda, \beta) =& P_0(\Lambda_\beta) P_{n/2}(\Lambda). \\ H^1(\Lambda, \beta) =& P_1(\Lambda_\beta) P_{(n/2)+}(\Lambda). \end{aligned}$$

5.5. When $\mathfrak{r}$ is a skew semisimple stratum of type **(B)**, we prove a necessary and sufficient condition for the non-emptiness of the set $\mathfrak{X}_\beta(F_0)$. The field $F[\beta_2]$ is stable under the action of σ_h and let D be the fixed field of the automorphism σ_h. Let λ be the F-linear σ_h-equivariant linear functional $\lambda : F[\beta_2] \to F$ such that $\lambda(\beta_2) = \beta_1$ and $\lambda(1) = 1$. There exists a unique hermitian form $h' : V_2 \times V_2 \to F[\beta_2]$ such that

$$h'(xv, yw) = x\sigma_h(y)\sigma_h(h'(w, v)), \text{ for all } v, w \in V \text{ and } x, y \in F[\beta_2], \tag{5.3}$$

and

$$h(v, w) = \lambda(h'(v, w)), \text{ for all } v, w \in V_2.$$

Since $F_0[\delta]$ is a quadratic extension of F_0, the kernel of λ is equal to

$$(\beta_2 - \beta_1)F_0 \oplus \delta(\beta_2 - \beta_1)F_0.$$

The set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if there exists $v_1 \in V_1$, $v_2 \in V_2$ with $v_1 + v_2 \neq 0$ such that

$$h(v_1, v_1) + \lambda(h'(v_2, v_2)) = 0 \tag{5.4}$$

$$\beta_1 h(v_1, v_1) + \lambda(\beta_2 h'(v_2, v_2)) = 0. \tag{5.5}$$

For any two vectors v_1 and v_2 as above we must have $v_1 \neq 0$ and $v_2 \neq 0$. Note that $(\beta_1 - \beta_2)h'(v_2, v_2)$ is contained in the kernel of λ, and this implies that $h'(v_2, v_2) \in F_0^\times$. If d_1 and d_2 are the determinants of (V_2, h') and (V_1, h), then we have

$$- d_1 d_2^{-1} \in \mathrm{Nr}_{F[\beta_2]/D}(F[\beta_2]^\times). \tag{5.6}$$

Conversely, if the condition (5.6) holds, then we can always find a simultaneous solution to the equations (5.4) and (5.5), and hence, the set $\mathfrak{X}_\beta(F_0)$ is non-empty.

5.6. **The case where (V_2, h) is isotropic.** In this part, we treat the case where (V_2, h) is isotropic, and let us begin with the case where $q_2 > q_1$.

Lemma 5.6.1. *Let F/F_0 be an unramified extension and let $\mathfrak{r}$ be a stratum of type* **(B)**. *If (V_2, h) is isotropic and $q_2 > q_1$, then every cuspidal representation in the set $\Pi_{\mathfrak{r}}$ is non-generic. Moreover, we have $\mathfrak{X}_\beta(F_0) = \emptyset$.*

Proof. Let π be a representation in the set $\Pi_{\mathfrak{r}}$. There exists a β-extension κ of $J^0(\Lambda, \beta)$ such that $\pi = \mathrm{ind}_{J^0(\Lambda,\beta)}^G \kappa$. If π is genetic, then there exists a non-trivial character Ψ of U, $p \in \mathcal{I}$, and $w \in W_G$ such that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^p \cap U^w}(\kappa^p, \Psi^w) \neq 0. \tag{5.7}$$

Let $\overline{U^w}$ be the unipotent radical of the opposite Borel subgroup, $\overline{B^w}$, of B^w containing T. Using the Iwahori decomposition of $\mathcal{I}$, with respect to $(\overline{B^w}, T)$, we have $p = p^+u^-$ with $p^+ \in B^w \cap \mathcal{I}$ and $u^- \in \overline{U^w} \cap \mathcal{I}$. Now the equation (5.7) implies that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^{u^-}\cap U^w}(\kappa^{u^-}, \Psi') \neq 0,$$

for some character Ψ' of U^w. The group $P_{(q_2/2)+}(\Lambda)$ is normalised by the element $u^- = \dot{u}(x, y)$, and hence, $P_{(q_2/2)+}(\Lambda)^{u^-} \cap U^w$ is equal to $P_{(q_2/2)+}(\Lambda) \cap U^w$.

We set $q_2 = 8k + 2r$, for some integer k and $r \in \{1, 3\}$. For the following calculations it is convenient to refer to the appendix A.3 for an explicit description of the filtration $\{a_m(\Lambda) : m \in \mathbb{Z}\}$. We have $(q_2/2)+ = 4k + r + 1$ and the intersection $P_{(q_2/2)+}(\Lambda) \cap U^w$ is given by:

$$P_{(q_2/2)+}(\Lambda) \cap U^w = \begin{cases} U(k + \lfloor r/2 \rfloor), & \text{if } w = \mathrm{id}, \\ U^w(k + 1 + \lfloor r/2 \rfloor), & \text{if } w \neq \mathrm{id}\,. \end{cases} \tag{5.8}$$

We define e_w and e_{-w} by setting $e_w = we_1$ and $e_{-w} = we_{-1}$, $w \in W_G$. Then, we have

$$h(u^-e_w, \beta u^-e_w) = \beta_1 x\bar{x} + h(e_w, \beta_2 e_w) + (y + \bar{y})h(e_w, \beta_2 e_{-w}) + y\bar{y}h(e_{-w}, \beta_2 e_{-w}). \tag{5.9}$$

Now, the valuation of the term $h(e_w, \beta_2 e_w)$ is strictly less than the valuation of all other terms in the right-hand side of (5.9). Hence, using Iwasawa decomposition, we get that $h(ge_1, \beta ge_1)) \neq 0$, for all $g \in G$. This shows that the set $\mathfrak{X}_\beta(F_0)$ is empty. We also get the valuation of $\delta h(u^-e_w, \beta u^-e_w)$ is equal to

$$\nu_F(\delta h(u^-e_w, \beta u^-e_w)) = \nu_F(h(e_w, \beta_2 e_w)) = \begin{cases} -(2k + \lfloor r/2 \rfloor) & \text{if } w = \mathrm{id}, \\ -(2k + \lfloor r/2 \rfloor + 1) & \text{if } w \neq \mathrm{id}\,. \end{cases} \tag{5.10}$$

From the equations (5.8) and (5.10), we get that

$$\nu(h(u^-e_w, u^-e_w)) \leq -s,$$

where s is given by the equality

$$U^w_{\mathrm{der}}(s) = P_{(q_2/2)+}(\Lambda) \cap U^w.$$

Now, Lemma 2.6.1 implies that the character $\psi_\beta^{u^-}$ is non-trivial on the group $P_{(q_2/2)+}(\Lambda) \cap U^w$ and we get a contradiction to (5.7). ☐

Lemma 5.6.2. *Let F/F_0 be a ramified extension and let $\mathfrak{x}$ be a stratum of type* **(B)** *such that (V_2, h) is isotropic. If $q_2 > q_1$, then the set $\mathfrak{X}_\beta(F_0)$ is empty, and every cuspidal representation in the set $\Pi_\mathfrak{x}$ is non-generic.*

Proof. First, we note that the group $P^0(\Lambda)$ is a special parahoric subgroup of G. Let $\pi = \mathrm{ind}_{J^0(\Lambda,\beta)}^G \kappa$ be a generic representation in the set $\Pi_\mathfrak{x}$. Using the Iwasawa decomposition $G = P(\Lambda)TU$, and the Mackey decomposition for the representation $\mathrm{res}_U \pi$, we get that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^g\cap U}(\kappa^g, \Psi) \neq 0, \tag{5.11}$$

for some $g \in P(\Lambda)$, and a non-trivial character Ψ of U. We refer to the appendix A.2 for an explicit description of the filtration $\{a_m(\Lambda) : m \in \mathbb{Z}\}$.

We set $q_2 = 4k_2$ and $q_1 = 4k_1 + 2$, for some integers k_1 and k_2. The group $P(\Lambda)$ normalises $P_{(q_2/2)+}(\Lambda)$ and hence, $P_{(q_2/2)+}(\Lambda)^g \cap U_{\mathrm{der}}$ is equal to $P_{(q_2/2)+}(\Lambda) \cap U_{\mathrm{der}}$. We have

$$P_{(q_2/2)+}(\Lambda) \cap U_{\mathrm{der}} = U_{\mathrm{der}}(\lceil (k_2 - 1)/2 \rceil). \tag{5.12}$$

Since, $\nu_\Lambda(e_1) = 1$, we have $ge_1 = ae_1 + \varpi be_0 \oplus \varpi ce_{-1}$, for some $a, b, c \in \mathfrak{o}_F$. We now try to estimate the valuation of $h(ge_1, \beta ge_1)$. Observe that $h(ge_1, \beta ge_1)$ is equal to

$$-\varpi^2\beta_1 b\sigma(b) + a\sigma(a)h(e_1, \beta_2 e_1) - \varpi a\sigma(c)h(e_1, \beta_2 e_{-1}) +$$
$$\varpi c\sigma(a)h(e_{-1}, \beta_2 e_1) - \varpi^2 c\sigma(c)h(e_{-1}, \beta_2 e_{-1}).$$

Recall that $F[\beta_2]$ is the unramified quadratic extension of F and the element $\tilde{\beta}_2 = \varpi^{q_2/2}\beta_2$ belongs to $a_0(\Lambda_2)$. Since $\nu_\Lambda(e_1) = 1$, the constants a and c both cannot belong to $\mathfrak{p}_F$. From the observation that $\nu_F(\mathrm{Nr}_{F[\beta_2]/F}(\beta_2)) = 0$, we have

$$a\sigma(a)h(e_1, 1/\varpi\tilde{\beta}_2 e_1) - a\sigma(c)h(e_1, \tilde{\beta}_2 e_{-1}) + c\sigma(a)h(e_{-1}, \tilde{\beta}_2 e_1) \\ + c\sigma(c)h(e_{-1}, \varpi\tilde{\beta}_2 e_{-1}) \not\equiv 0 \pmod{\mathfrak{p}_F}.$$

We note that

$$\nu_{F/F_0}(\varpi^2\beta_1 b\sigma(b)) \geq -k_1 + 1/2.$$

Since $q_2 > q_1$, we have $-k_2 < -k_1 - 1/2$, we deduce that $h(ge_1, \beta ge_1) \neq 0$ and $\nu_{F/F_0}(\delta h(ge_1, \beta ge_1))$ is equal to $-k_2 + 1$. From the equation (5.12), we get that

$$\nu_{F/F_0}(\delta h(ge_1, \beta ge_1)) \leq -s,$$

where $P_{(q_2/2)+}(\Lambda) \cap U_{\mathrm{der}} = U_{\mathrm{der}}(s)$. Now, using Lemma 2.6.1, we arrive at a contradiction to (5.11). Hence, any representation π in the set $\Pi_{\mathfrak{r}}$ is non-generic. Finally, using the Iwasawa decomposition $G = P(\Lambda)B$, we get that $h(ge_1, \beta ge_1) \neq 0$, for $g \in G$. Hence, the set $\mathfrak{X}_\beta(F_0)$ is empty. □

Lemma 5.6.3. *Let F/F_0 be an unramified extension and let $\mathfrak{r}$ be a stratum of type* **(B)** *such that (V_2, h) is isotropic. If $q_1 > q_2$, then the set $\mathfrak{X}_\beta(F_0)$ is non-empty.*

Proof. We continue using the notations introduced in 5.5. Since (V_2, h) is isotropic, we get that $\ker(\lambda)$ has non-trivial intersection with the set $\{h'(v,v) : v \in V_2, v \neq 0\}$. Since $F[\beta_2]$ is a ramified extension of F and F/F_0 is an unramified extension, we get that $F_0^\times$ is contained in $\mathrm{Nr}_{F[\beta_2]/D}(F[\beta_2]^\times)$. This implies that the determinant of the form (V_2, h') is the class in $D^\times/\mathrm{Nr}_{F[\beta_2]/D}(F[\beta_2]^\times)$ containing $\delta(\beta_2 - \beta_1)$. Since, $q_1 > q_2$ we have

$$\nu_{F[\beta_2]}(\delta(\beta_2 - \beta_1)) = \nu_{F[\beta_2]}(\beta_1) \in 2\mathbb{Z}.$$

From the observation that d_1, the determinant of (V_1, h), belongs to F_0, we see that

$$-d_1 d_2^{-1} \in \mathrm{Nr}_{F[\beta_2]/D}(F[\beta_2]^\times).$$

Now, using the criterion (5.6), we get that the set $\mathfrak{X}_\beta(F_0)$ is non-empty. □

Lemma 5.6.4. *Let F/F_0 be a ramified extension, and let $\mathfrak{r}$ be a stratum of type* **(B)** *such that (V_2, h) is isotropic. If $q_1 \geq q_2$, then the set $\mathfrak{X}_\beta(F_0)$ is non-empty.*

Proof. We continue using notations introduced in 5.5. Since the space (V_2, h) is isotropic, the set $\{h'(v,v) : v \in V_2, v \neq 0\}$ has a nontrivial intersection with $\ker(\lambda)$. Hence, the determinant class of (V, h') is equal to the coset in $D^\times/\mathrm{Nr}_{F[\beta_2]/D}(F[\beta_2]^\times)$ containing the element $\delta\beta_1(1 - \beta_2\beta_1^{-1})$. We note that $(1 - \beta_2\beta_1^{-1})$ belongs to $1 + \mathfrak{p}_{F[\beta_2]}$ and $\beta_1 \in \delta F_0$. Since k_D is a quadratic extension of k_{F_0}, we get that $F_0^\times$ is contained in $\mathrm{Nr}_{F[\beta_2]/D}(F[\beta_2]^\times)$. Hence, we have $\delta\beta_1 \in F_0$ and we get that $\delta(\beta_1 - \beta_2)$ belongs to $\mathrm{Nr}_{F[\beta_2]/D}(F[\beta_2]^\times)$. Now, using the criterion (5.6) we get that the set $\mathfrak{X}_\beta(F_0)$ is non-empty. □

Lemma 5.6.5. *Let F/F_0 be a quadratic extension and let $\mathfrak{r}$ be a stratum of the type* **(B)** *such that (V_2, h) is isotropic. If $q_1 > q_2$, then any representation in the set $\Pi_{\mathfrak{r}}$ is generic.*

Proof. Let $\mathbf{U}$ be the unipotent radical of a Borel subgroup of in the set $\mathfrak{X}_\beta(F_0)$. Let θ be any skew semisimple character in the set $\mathcal{C}(\Lambda, 0, \beta)$. We will first check that

$$\mathrm{res}_{H^1(\Lambda,\beta)\cap U}\, \theta = \psi_\beta. \tag{5.13}$$

The group $H^1(\Lambda, \beta)$ is equal to

$$P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda_{\beta_1})P_{(q_1/2)+}(\Lambda).$$

We define H' to be the subgroup $P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda)$. Now, using Lemma 3.6.1, we get that $H' \cap U$ is equal to $P_{(q_2/2)+}(\Lambda) \cap U$. Since $H^1(\Lambda, \beta) \cap U$ is equal to $H^1(\Lambda, \beta) \cap (H' \cap U)$, we get that $H^1(\Lambda, \beta) \cap U$ is equal to $P_{(q_2/2)+}(\Lambda_{\beta_1})P_{(q_1/2)+}(\Lambda) \cap U$. Let $g_1 g_2 \in U$ for some $g_1 \in P_{(q_2/2)+}(\Lambda_{F[\beta_1]})$ and $g_2 \in P_{(q_1/2)+}(\Lambda)$. We recall the notations defined in 5.5. Let $v = v_1 + v_2$ be a non-trivial vector fixed by U. If $v_1 = 0$, then we have

$\lambda(h'(v_2, v_2)) = 0$ and $\lambda(\beta_2 h'(v_2, v_2)) = 0$. This implies that β_2 stabilises the kernel of λ and this absurd. Hence, we get that $v_1 \neq 0$. We have

$$g_2(v_1 + v_2) = g_1(v_1 + v_2) = xv_1 + g_1 v_2.$$

Now, comparing both sides we get that $x \in F^\times \cap P_{(q_1/2)+}(\Lambda)$. This implies that $\mathbf{1}_{V_1} g_1 \mathbf{1}_{V_1} \in F^\times \cap P_{(q_1/2)+}(\Lambda)$. Since, the determinant of $g_2 g_1$ is equal to 1, we get that determinant of $\mathbf{1}_{V_2} g_1 \mathbf{1}_{V_2}$ belongs to $F^\times \cap P_{(q_1/2)+}(\Lambda)$. From the definition of simple character (see [6, Definition 3.2.3(a)]), we get (5.13). The Lemma is now a consequence of Proposition 3.6.2. ☐

5.7. **The case where** (V_2, h) **is anisotropic.** As in the case where (V_2, h) is isotropic, the genericity of a cuspidal representation in the set $\Pi_{\mathfrak{x}}$ depends only on the integers q_1 and q_2. However, the condition for genericity becomes the opposite to the case where (V_2, h) is isotropic, i.e., the inequality $q_2 > q_1$ is necessary and sufficient condition for genericity of a cuspidal representation in $\Pi_{\mathfrak{x}}$.

Lemma 5.7.1. *Let F/F_0 be an unramified extension, and let $\mathfrak{x}$ be a stratum of type* (**B**) *such that (V_2, h) is anisotropic. If $q_1 > q_2$, then the set $\mathfrak{X}_\beta(F_0)$ is empty, and every cuspidal representation contained in the set $\Pi_{\mathfrak{x}}$ is non-generic.*

Proof. Let $\pi = \mathrm{ind}_{J^0(\Lambda,\beta)}^G \kappa$ be a generic representation in the set $\Pi_{\mathfrak{x}}$. The group $P(\Lambda)$ is a special parahoric subgroup of G. Using the Iwasawa decomposition $G = P(\Lambda)B$, we have

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^g \cap U}(\kappa^g, \Psi) \neq 0, \tag{5.14}$$

for some $g \in P(\Lambda)$ and a character Ψ of U.

Let $q_1 = 4m_1 + 2r$ and $q_2 = 2m_2 + 1$, for some integers m_1, m_2 and $r \in \{0, 1\}$. Note that $P_{(q_1/2)+}(\Lambda) \cap U_{\mathrm{der}}$ is equal to $U_{\mathrm{der}}(m_1)$ (see 5.1.1 and (A.2) for an explicit description of the lattice sequence Λ and the induced filtrations on $\mathrm{End}_F(V)$). We have $\nu_\Lambda(e_1) = 1$, and we get that $ge_1 = av_1 + \varpi_0 bv_2 + cv_3$, for some $a, b, c \in \mathfrak{o}_F$. Since e_1 is isotropic, we get that

$$a\sigma(a) + \varpi_0 b\sigma(b) + c\sigma(c) = 0.$$

Since $\nu_\Lambda(e_1) = 1$, the above equality implies that $a, c \in \mathfrak{o}_F^\times$. Note that $\nu_{F[\beta_2]}(\beta_2\beta_1^{-1}) > 0$, and in the basis (v_2, v_3) for V_2, the element $\beta_2\beta_1^{-1} \in \mathrm{End}_F(V)$ belongs to the following lattice of $\mathrm{End}_F(V_2)$:

$$\begin{pmatrix} \mathfrak{o}_F & \mathfrak{p}_F \\ \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix}.$$

Hence, we simultaneously get that $h(ge_1, \beta ge_1) \neq 0$ and

$$\nu_F(h(ge_1, \beta ge_1)) = \nu_F(\beta_1 a\sigma(a) + h(\varpi_0 bv_2 + cv_3, \beta_2(\varpi_0 bv_2 + cv_3))) = \nu_F(\beta_1) = -2m_1 - r \leq -m_1$$

Now, using Lemma 2.6.1, we get that the character ψ_β^g is non-trivial on the group $P_{(q_1/2)+}(\Lambda) \cap U_{\mathrm{der}}$. Hence, we get a contradiction for the assumption (5.14). This shows that the representation π in the set $\Pi_{\mathfrak{x}}$ is non-generic. Using the Iwasawa decomposition $G = P(\Lambda)B$, we get that $h(ge_1, \beta ge_1) \neq 0$, for all $g \in G$. Hence, the set $\mathfrak{X}_\beta(F_0)$ is empty. ☐

Lemma 5.7.2. *Let F/F_0 be a ramified extension, and let $\mathfrak{x}$ be a stratum of type* (**B**) *such that (V_2, h) is anisotropic. If $q_1 > q_2$, then the set $\mathfrak{X}_\beta(F_0)$ is empty, and every representation in the set $\Pi_{\mathfrak{x}}$ is non-generic.*

Proof. Let $\pi = \mathrm{ind}_{J^0(\Lambda,\beta)}^G \kappa$ be a generic representation in the set $\Pi_{\mathfrak{x}}$. The group $P^0(\Lambda)$ is a special parahoric subgroup of G, and using the Iwasawa decomposition $G = P(\Lambda)B$, we get that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^g \cap U}(\kappa^g, \Psi) \neq 0, \tag{5.15}$$

for some $g \in P(\Lambda)$, and a non-trivial character Ψ of U.

Let $q_1 = 4m_1 + 2$ and $q_2 = 4m_2$, for some integers m_1 and m_2. Observe that $P_{(q_1/2)+}(\Lambda)^g \cap U_{\mathrm{der}}$ is equal to $U_{\mathrm{der}}(\lceil m_1/2 \rceil)$. Since $\nu_\Lambda(e_1) = 0$, we get that $ge_1 = av_1 + bv_2 + cv_3$ for some $a, b, c \in \mathfrak{o}_F$. As the vector e_1 is isotropic we get that

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0.$$

Now, the space (V_2, h) is anisotropic and this implies that $a \in \mathfrak{o}_F^\times$. Now, we have

$$\begin{aligned} h(ge_1, \beta g e_1) =& \beta_1 a\sigma(a) + h(bv_2 + cv_2, \beta_2(bv_2 + cv_3)) \\ =& \beta_1(a\sigma(a) + \beta_1^{-1} h(bv_2 + cv_2, \beta_2(bv_2 + cv_3))). \end{aligned}$$

Since, $\nu_{F[\beta_2]}(\beta_2\beta_1^{-1}) > 0$, we simultaneously get that $h(ge_1, \beta g e_1) \neq 0$, and $\nu_F(\delta h(ge_1, \beta g e_1))$ is equal to $\nu_F(\delta\beta_1)$. From the inequality $\nu_F(\delta\beta_1) = -m_1 \leq -\lceil m_1/2 \rceil$, we get that ψ_β^g is a non-trivial character on $P_{(q_1/2)+}(\Lambda)^g \cap U_{\mathrm{der}}$. This is a contradiction to the assumption (5.15). Using the Iwasawa decomposition $G = P(\Lambda)B$, we get that $h(ge_1, \beta g e_1) \neq 0$, for all $g \in G$. This shows that the set $\mathfrak{X}_\beta(F_0)$ is empty. □

In the case where (V_2, h) is anisotropic, we could not use the criterion in 5.5. However, the following observation motivates the fact that $\mathfrak{X}_\beta(F_0)$ is non-empty in the case where $q_2 > q_1$. We suppose that F/F_0 is unramified. In the basis (v_1, v_2, v_3) consider a skew element β of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_2 \\ 0 & \beta_3 & 0 \end{pmatrix}. \tag{5.16}$$

Let e_1 be an isotropic vector in $\langle v_1, v_3 \rangle$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ such that B fixes the space $\langle e_1 \rangle$. The Borel subgroup $\mathbf{B}$ belongs to the set $\mathfrak{X}_\beta(F_0)$. Now, the class of β in the quotient $a_{-n}(\Lambda)/a_{1-n}(\Lambda)$, is represented by a matrix as in (5.16). This suggests that we may lift a point from the special fibre of an integral model of $\mathfrak{X}_\beta$, and we do this in the following lemma.

Lemma 5.7.3. *Let F/F_0 be a quadratic extension and let $\mathfrak{x}$ be a stratum of type* **(B)** *such that (V_2, h) is anisotropic. If $q_2 > q_1$, then the set $\mathfrak{X}_\beta(F_0)$ is non-empty.*

Proof. We will lift a point from the special fibre of a smooth model for $\mathfrak{X}_\beta$. Let (v_1, v_2, v_3) be a basis of V as defined in 5.1.1, when F/F_0 is unramified, and let (v_1, v_2, v_3) be a basis of V as defined in 5.1.2, when F/F_0 is ramified. Let $\beta = (\beta_{ij})$ be the matrix representation of β in the basis (v_1, v_2, v_3). We have $\beta_{11} = \beta_1$.

First consider the case where F/F_0 is unramified. A Borel subgroup $\mathbf{B}$, corresponding to the line $\langle xv_1 + yv_2 + zv_3 \rangle$, belongs to $\mathfrak{X}_\beta$ if and only if (x, y, z) satisfy the following equations:

$$\varpi x\sigma(x) + y\sigma(y) + \varpi z\sigma(z) = 0$$

and

$$\varpi\beta_1 x\sigma(x) + \beta_{22} y\sigma(y) + \beta_{33}\varpi z\sigma(z) + \beta_{23} y\sigma(z) + \varpi\beta_{32} z\sigma(y) = 0.$$

Changing y to $\varpi y'$ and rescaling the second equation by $\varpi^{\alpha-2}$ with $\alpha = -\nu_F(\beta_{32})$, we get the following set of equations:

$$x\sigma(x) + \varpi y'\sigma(y') + z\sigma(z) = 0 \tag{5.17}$$

and

$$\varpi^{\alpha-1}\beta_1 x\sigma(x) + \varpi^\alpha \beta_{22} y'\sigma(y') + \varpi^{\alpha-1}\beta_{33} z\sigma(z) + \varpi^{\alpha-1}\beta_{23} y'\sigma(z) + \varpi^\alpha \beta_{32} z\sigma(y') = 0. \tag{5.18}$$

Note that the coefficients of (5.18) are integral and the two equations (5.17) and (5.18) define a flat closed subscheme $\mathcal{X}_\beta$ of $\mathbb{P}^5_{\mathfrak{o}_{F_0}}$ such that the generic fibre is $\mathfrak{X}_\beta$. The special fibre is given by the set of equations

$$x\sigma(x) + z\sigma(z) = 0$$

and

$$C_1(y'\sigma(z) - z\sigma(y')) + C_2 x\sigma(x) + C_3 z\sigma(z) = 0,$$

where $C_1 = \overline{\varpi^{\alpha-1}\beta_{23}}$, $C_2 = \overline{\varpi^{\alpha-1}\beta_1}$, and $C_3 = \overline{\varpi^{\alpha-1}\beta_{33}}$. Note that $C_1 \neq 0$, and therefore, the special fibre $\overline{\mathcal{X}}_\beta$ is smooth. Since $\mathcal{X}_\beta$ is flat we get that $\mathcal{X}_\beta$ is smooth over $\mathfrak{o}_F$. The special fibre has a rational point and hence by Hensel's lemma the set $\mathfrak{X}_\beta(F_0)$ is non-empty.

Now, consider the case where F/F_0 is a ramified extension. A Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ fixing the isotropic subspace $\langle av_1 + bv_2 + cv_3 \rangle$, belongs to $\mathfrak{X}_\beta$ if and only if:

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0,$$

and

$$\lambda_1\beta_1 a\sigma(a) + \lambda_2\beta_{22} b\sigma(b) + \lambda_3\beta_{33} c\sigma(c) + b\sigma(c)\beta_{23}\lambda_2 + \lambda_3\beta_{32} c\sigma(b) = 0.$$

In the present case $F[\beta_2]$ is an unramified extension of F. After rescaling by a power of ϖ, if necessary, we may assume that

$$\beta_2 = \begin{pmatrix} \beta_{22} & \beta_{23} \\ \beta_{32} & \beta_{33} \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix},$$

and $\beta_1 \in \mathfrak{o}_F$. Because β_1 is skew, we get that $\beta_1 \in \mathfrak{p}_F$. By a change of variable b to $\varpi b'$, the above set of equations become:

$$\lambda_1 a\sigma(a) + \lambda_2 \varpi_0 b'\sigma(b') + \lambda_3 c\sigma(c) = 0$$

and

$$\varpi^{-1}\lambda_1\beta_1 a\sigma(a) - \varpi\lambda_2\beta_{22}b\sigma(b) + \varpi^{-1}\lambda_3\beta_{33}c\sigma(c) + \lambda_2\beta_{23}b\sigma(c) - \lambda_3\beta_{32}c\sigma(b) = 0.$$

Since β is skew, we get that $\sigma(\beta_{22}) + \beta_{22} = 0$, $\sigma(\beta_{33}) + \beta_{33} = 0$, and $\lambda_1\beta_{23} = -\lambda_3\sigma(\beta_{32})$. Hence, the above two equations have integral coefficients. The above two equations define a flat projective sub-variety $\mathcal{X}_\beta$ in $\mathbb{P}^5_{\mathfrak{o}_{F_0}}$ with generic fibre $\mathfrak{X}_\beta$. The special fibre is given by

$$\overline{\lambda_1}a^2 + \overline{\lambda_3}c^2 = 0,$$

and

$$C_1a^2 + C_2c^2 + C_3bc = 0,$$

where $C_1 = \overline{\varpi^{-1}\lambda_1\beta_1}$, $C_2 = \overline{\varpi^{-1}\lambda_3\beta_{33}}$, and $C_3 = \overline{\lambda_2\beta_{23}}$. Clearly $C_3 \neq 0$ as the element β_2 is minimal. The special fibre is smooth and hence, $\mathcal{X}_\beta$ is a smooth model for $\mathfrak{X}_\beta$. Note that the special fibre has a rational point. Using Hensel's lemma, we get that the set $\mathfrak{X}_\beta(F_0)$ is non-empty. □

Lemma 5.7.4. *Let F/F_0 be a quadratic extension and let $\mathfrak{r}$ be a stratum of type* **(B)** *such that (V_2, h) is anisotropic. If $q_2 > q_1$, then every representation in the set $\Pi_\mathfrak{r}$ is generic.*

Proof. Using Lemma 5.7.3, we get that the set $\mathfrak{X}_\beta(F_0)$ is non-empty. Let $\mathbf{U}$ be the unipotent radical of a Borel subgroup in the set $\mathfrak{X}_\beta(F_0)$. Note that the group $H^1(\Lambda, \beta)$ is equal to $P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda)$ and it follows from Lemma 3.6.1 that $H^1(\Lambda, \beta) \cap U$ is equal to $P_{(q_2/2)+}(\Lambda) \cap U$. Hence, we get that

$$\mathrm{res}_{H^1(\Lambda,\beta)\cap U}\,\theta = \psi_\beta,$$

where θ is any skew semisimple character of $H^1(\Lambda, \beta)$. Now, genericity is a consequence of Proposition 3.6.2. □

6. Non simple type (C) strata

We say that a skew semisimple stratum $[\Lambda, n, 0, \beta]$, denoted by $\mathfrak{r}$, is of type **(C)** if the underlying splitting of $\mathfrak{r}$ is given by $V = V_1 \perp V_2$ with $\dim_F V_i = i$, and $[F[\beta_i] : F] = 1$. Here, β_i is equal to $\mathbf{1}_{V_i}\beta\mathbf{1}_{V_i}$, for $i \in \{1, 2\}$. From the definition of skew semisimple stratum, we have $\sigma(\beta_i) = -\beta_i$, for $i \in \{1, 2\}$, and $\beta = \beta_1 + \beta_2$. We will show that every representation contained in the set $\Pi_\mathfrak{r}$ is non-generic.

6.1. **Lattice sequences.** We will describe the lattice sequences up to G_β-conjugacy. Note that Λ is a lattice sequence on V such that $P^0(\Lambda_\beta)$ is a maximal parahoric subgroup of G_β. We will fix a Witt-basis of (V, h) which gives a splitting for these lattice sequences.

6.1.1. *The unramified case.* Consider the case where F/F_0 is unramified and (V_2, h) is isotropic. We fix a Witt-basis (e_1, e_0, e_{-1}) for (V, h) such that $e_1, e_{-1} \in V_2$. The lattice sequence Λ, upto G_β-conjugation, is given by one of the following two lattice sequences:

$$e(\Lambda) = 2,\ \Lambda(-1) = \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}$$

or

$$e(\Lambda) = 2,\ \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

If F/F_0 is an unramified extension and (V_2, h) is anisotropic, we fix vectors $v_1 \in V_1$, $v_2, v_3 \in V_2$ such that (v_1, v_2, v_3) is an orthogonal basis for V and

$$h(v_1, v_1) = h(v_3, v_3) = \varpi \text{ and } h(v_2, v_2) = 1.$$

The lattice sequence Λ, up to G_β-conjugation, is given by the following lattice sequence with

$$e(\Lambda) = 2,\ \Lambda(0) = \mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_2 \oplus \mathfrak{o}_F v_3 \text{ and } \Lambda(1) = \mathfrak{o}_F v_1 \oplus \mathfrak{p}_F v_2 \oplus \mathfrak{o}_F v_3.$$

Using Lemma 5.1.1, there exists a Witt-basis (e_1, e_0, e_{-1}) of V with $e_1, e_{-1} \in \langle v_1, v_3\rangle$ such that (e_1, e_0, e_{-1}) provides a splitting for the lattice sequence Λ, and we have

$$\Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1}, \text{ and } \Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

6.1.2. *The ramified case.* In this part, we assume that the extension F/F_0 is ramified. If (V_2, h) is isotropic, we fix a Witt-basis (e_1, e_{-1}) for (V_2, h) and an unit vector $e_0 \in V_1$ such that (e_1, e_0, e_{-1}) is a Witt-basis for (V, h). The lattice sequence Λ, up to G_β-conjugation, is given by the following lattice sequence:

$$e(\Lambda) = 2, \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

Now assume that (V_2, h) is anisotropic, and fix an orthogonal basis (v_1, v_2, v_3) of (V, h) such that $v_1 \in V_1$ and $v_2, v_3 \in V_2$, $h(v_i, v_i) \in \mathfrak{o}_F^\times$, and the hermitian space $(\langle v_1, v_3\rangle, h)$ is isotropic. We denote by λ_i the constant $h(v_i, v_i)$, for $1 \le i \le 3$. Up to G_β-conjugacy, the lattice sequence Λ is given by the following lattice sequence:

$$e(\Lambda) = 2,\ \Lambda(-1) = \Lambda(0) = \mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_2 \oplus \mathfrak{o}_F v_3.$$

There exists a Witt-basis (e_1, e_0, e_{-1}) for the space (V, h) with $e_1, e_{-1} \in \langle v_1, v_3\rangle$ such that

$$\Lambda(-1) = \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}$$

The groups $J^0(\Lambda, \beta)$ and $H^1(\Lambda, \beta)$ are given by

$$P_0(\Lambda_\beta) P_{(n/2)}(\Lambda)$$

and

$$P_1(\Lambda_\beta) P_{(n/2)+}(\Lambda)$$

respectively.

6.2. **The case where** (V_2, h) **is anisotropic.** In this subsection, we assume that the hermitian space (V_2, h) is anisotropic. To show that any representation $\pi \in \Pi_{\mathfrak{r}}$ is non-generic, we will prove that the character ψ_β^g is non-trivial on $U_{\mathrm{der}} \cap P_{(n/2)+}(\Lambda)$, for all $g \in P(\Lambda)$.

Lemma 6.2.1. *Let F/F_0 be any quadratic extension and let $\mathfrak{r}$ be a strata of type* **(C)** *such that (V_2, h) is anisotropic. Every cuspidal representation in the set $\Pi_{\mathfrak{r}}$ is non-generic.*

Proof. We recall, from Subsection 6.1 the two F-basis (v_1, v_2, v_3) and (e_1, e_0, e_{-1}) for the vector space V, when (V_2, h) is anisotropic. Note that Λ is uniquely determined by β, and recall the description from Subsection 6.1. Let $\mathbf{U}$ be the unipotent radical of a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that U fixes the vector e_1. Since (e_1, e_0, e_{-1}) provides a splitting for the lattice Λ and $P^0(\Lambda)$ is a special parahoric subgroup, we get that $G = P(\Lambda)B$. Let π be a cuspidal representation in the set $\Pi_{\mathfrak{r}}$. Then we have $\pi \simeq \mathrm{ind}_{J^0(\Lambda,\beta)}^G \kappa$, where $(J^0(\Lambda, \beta), \kappa)$ is a Bushnell–Kutzko type contained in π. Now, assume that π is generic. Then there exists a $g \in P(\Lambda)$ and a non-trivial character Ψ of U such that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^g \cap U}(\kappa^g, \Psi) \neq 0. \tag{6.1}$$

First consider the case where F/F_0 is a ramified extension. Let $ge_1 = av_1 + bv_2 + cv_3$, for some $a, b, c \in \mathfrak{o}_F$. Since e_1 is isotropic, we get that

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0.$$

Since $-\lambda_2\lambda_3^{-1} \notin \mathrm{Nr}_{F/F_0}(F^\times)$, the above equality implies that $a \in \mathfrak{o}_F^\times$. Let $\nu_\Lambda(\beta) = -n = -(4m + 2r)$, for some integer m and $r \in \{0, 1\}$. Since $\sigma(\beta) = -\beta$, we get that $r = 1$. Now, $(n/2)+ = 2m + 2$ and we have $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$ is equal to $U_{\mathrm{der}}([(m + 1)/2])$. We also have

$$\nu_{F/F_0}(\delta h(ge_1, \beta ge_1)) = \nu_{F/F_0}(\delta(\beta_1 - \beta_2)\lambda_1 a\sigma(a)) = -m.$$

As $\nu_{F/F_0}(h(ge_1, ge_1) \le -[(m + 1)/2]$, for any $m \ge 1$, we get that ψ_β^g is a non-trivial character on the group $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$. This is a contradiction to the equation (6.1). Hence, any representation in the set $\Pi_{\mathfrak{r}}$ is non-generic.

Consider the case where F/F_0 is unramified. Since the isotropic vector e_1 belongs to the lattice $\Lambda(1)$, we get that $ge_1 = av_1 + bv_2 + cv_3$, for some $a, c \in \mathfrak{p}_F$ and $b \in \mathfrak{p}_F$, with

$$\varpi a\sigma(a) + b\sigma(b) + \varpi c\sigma(c) = 0.$$

By a change of variable: $b' = \varpi b$, we have

$$a\sigma(a) + \varpi b'\sigma(b') + c\sigma(c) = 0.$$

Since $\nu_\Lambda(e_1) = 1$, the above equality implies that $a, c \in \mathfrak{o}_F^\times$. Now, we have

$$\begin{aligned}\nu_F(h(ge_1, \beta ge_1)) =& \nu_F(\beta_1\varpi a\sigma(a) + \beta_2(b\sigma(b) + \varpi c\sigma(c)))\\ &= \nu_F((\beta_1 - \beta_2)) + 1\end{aligned}$$

We note that $2\nu_F(\beta_1) = \nu_{\Lambda_1}(\beta_1)$ and $\nu_F(\beta_2) = \nu_{\Lambda_2}(\beta_2)$. We also have

$$-n = \nu_\Lambda(\beta) = \min\{\nu_{\Lambda_1}(\beta_1), \nu_{\Lambda_2}(\beta_2)\}.$$

Assume that $-n = \nu_{\Lambda_1}(\beta_1) \leq \nu_{\Lambda_2}(\beta_2)$. In this case, $n = 4m + 2r$, where m is an integer and $r \in \{0, 1\}$. Now, we have $(n/2)+ = 2m + r + 1$ and the group $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$ is equal to $U_{\mathrm{der}}(m)$. We may have two possibilities either $\nu_{\Lambda_2}(\beta_2) \leq \nu_F(\beta_1) = -(2m + r)$ or $\nu_{\Lambda_2}(\beta_2) \geq \nu_F(\beta_1) = 2m + r$. In the first case, we have

$$\nu_F(\beta_1 - \beta_2) + 1 = \nu_F(\beta_2) + 1 \leq -(2m + r) + 1 \leq -m.$$

In the second case, we have

$$\nu_F(\beta_1 - \beta_2) + 1 = \nu(\beta_1) + 1 = -(2m + r) + 1 \leq -m.$$

Hence, the character ψ_{β^g} on $P_{(n/2)+}(\Lambda)$ is non-trivial on $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$ and we obtain a contradiction to the equation (6.1).

Assume that $-n = \nu_{\Lambda_2}(\beta_2) \leq \nu_{\Lambda_1}(\beta_1)$ and set $n = 4m + r$, for some integer m and $0 \leq r \leq 3$. In this case, the group $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$ is equal to $U_{\mathrm{der}}(m)$. Since $\nu_{\Lambda_1}(\beta) \leq \nu_F(\beta_1)$, we get that

$$\nu_F(\beta_1 - \beta_2) + 1 = \nu_{\Lambda_2}(\beta_2) = -(4m + r) + 1 \leq -m.$$

The above inequality implies that the character ψ_β^g is non-trivial on the group $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$ and hence we get a contradiction to the equation (6.1). In every case, we get that the representation π is non-generic. $\square$

6.3. (V_2, h) **is isotropic.** In this part, we assume that (V_2, h) is isotropic. Note that the set $\mathfrak{X}_\beta(F_0)$ is non-empty. However, it turns out that every cuspidal representation in the set $\Pi_{\mathfrak{x}}$ is non-generic; the essential reason for this is that any cuspidal representation of $P_0(\Lambda_\beta)/P_1(\Lambda_\beta)$ is generic. The group $P_0(\Lambda_\beta)/P_1(\Lambda_\beta)$ is equal to $U(1,1)(k_F/k_{F_0}) \times U(1)(k_F/k_{F_0})$, when F/F_0 is unramified and is equal to $\mathrm{SL}_2(k_F) \times \{\pm 1\}$, when F/F_0 is ramified.

Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{G}$ such that B fixes the subspace $\langle e_1 \rangle$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Although $P^0(\Lambda)$ is a special parahoric subgroup of G, it is convenient to use the decomposition

$$G = \mathcal{I} W_G B$$

for Mackey-decompositions. Here, $\mathcal{I}$ is the Iwahori subgroup contained in the subgroups $P(\Lambda)$, where Λ varies over the two (representatives for G_β conjugacy classes) lattice sequences in 6.1.1 and 6.1.2 when F/F_0 is unramified and ramified respectively.

6.3.1. *Shallow elements.* To understand Mackey decompositions, we need to control the conjugation by elements in $\mathcal{I}$. For the present purposes, it is easy to understand the conjugation action of an element which is contained in a sufficiently small compact subgroup of $\mathcal{I}$. So, we introduce a measure of shallowness, relative to the group $P_{(n/2)+}(\Lambda)$, on the elements $u(x, y)$ and $\bar{u}(x, y)$ in $\mathcal{I} \cap U$ and $\mathcal{I} \cap U^w$ respectively (here, w is the non-trivial element in W_G).

We begin with defining an integer $d(\mathfrak{x}, w, x)$. The main purpose of the definition of $d(\mathfrak{x}, w, x)$ becomes apparent in Lemma 6.3.1. Let $n = 4m + 2r$, for some positive integer m and $r \in \{0, 1\}$. Consider the case where F/F_0 is unramified. For any $x, y \in F$ such that $x\sigma(x) + y + \sigma(y) = 0$, $w \in W_G$ and Λ a lattice sequence defined in 6.1.1 or in 6.1.2, we set $d(\mathfrak{x}, w, x)$ to be

$$d(\mathfrak{x}, w, x) = \begin{cases} \max\{1, m + 1 - \nu_F(x)\}, & \text{if } \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_{-1}, \\ \max\{0, m + r - \nu_F(x)\} & \text{if } \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}, w = \mathrm{id}, \\ \max\{2, m + r + 1 - \nu_F(x)\} & \text{if } \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}, w \neq \mathrm{id}\,. \end{cases} \tag{6.2}$$

If F/F_0 is ramified, then the lattice sequences Λ, defined in 6.1.2, up to G_β conjugacy, is the unique lattice sequence such that $P^0(\Lambda_{F[\beta]})$ is a maximal parahoric subgroup in G_β. Note that $\Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}$. We define the integer $d(\mathfrak{r}, w, x)$ as follows:

$$d(\mathfrak{r}, w, x) = \begin{cases} \max\{0, \lceil (m+r-1)/2 - \nu_F(x) \rceil\} & \text{if } w = \mathrm{id}, \\ \max\{1, \lceil (m+r)/2 - \nu_F(x) \rceil\} & \text{if } w \neq \mathrm{id}\,. \end{cases} \tag{6.3}$$

Note that $d(\mathfrak{r}, w, x)$ is a constant for $\nu_F(x) >> 0$, and we denote this constant by $d(\mathfrak{r}, w)$. For example, when F/F_0 is unramified, $\Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_E e_{-1}$, and $w \neq \mathrm{id}$ we have $d(\mathfrak{r}, w) = 2$. If F/F_0 is ramified and $w \neq \mathrm{id}$, we have $d(\mathfrak{r}, w) = 1$.

6.3.2. With these preliminaries we are ready to prove that any representation in the set $\Pi_\mathfrak{r}$ is non-generic. Let $\overline{\mathbf{U}^w}$ be the unipotent radical of the opposite Borel subgroup of $\mathbf{B}^w$ with respect to the torus $\mathbf{T}$, for all $w \in W_G$. Recall that $\mathbf{T}$ is the maximal F_0-split torus of $\mathbf{G}$ such that T stabilises the decomposition

$$V = \langle e_1 \rangle \oplus \langle e_0 \rangle \oplus \langle e_{-1} \rangle.$$

Lemma 6.3.1. *Let F/F_0 be any quadratic extension and let $\mathfrak{r}$ be a skew semisimple strata of type* **(C)** *such that (V_2, h) is isotropic. Let $u^- = \bar{u}(x, y)$ be an element of $\mathcal{I} \cap \overline{U^w}$, then we have*

$$U^w_{\mathrm{der}}(d(\mathfrak{r}, w, x)) \subseteq H^1(\Lambda, \beta)^{u^-} \cap U^w_{\mathrm{der}}.$$

Proof. We prove the lemma in the case where $w = \mathrm{id}$; the case where $w \neq \mathrm{id}$ is entirely similar. We have to show that $\{U_{\mathrm{der}}(d(\mathfrak{r}, \mathrm{id}, x))\}^{u^-}$ is contained in the group $H^1(\Lambda, \beta)$. First we note the following matrix identity

$$\bar{u}(x, y) u(0, a) \bar{u}(-x, -y - x\sigma(x)) = \begin{pmatrix} 1 - a(x\sigma(x) + y) & a\sigma(x) & a \\ ax(-y - x\sigma(x)) & 1 + ax\sigma(x) & ax \\ -ay(y + x\sigma(x)) & a\sigma(x) y & ay + 1 \end{pmatrix}. \tag{6.4}$$

Using the equality (6.4), for any element $u(0, a) \in U_{\mathrm{der}}(d(\mathfrak{r}, \mathrm{id}, x))$, we get that

$$\bar{u}(x, y) u(0, a) \bar{u}(-x, -y - x\sigma(x)) \in P_1(\Lambda_{F[\beta]}) P_{(n/2)+}(\Lambda).$$

The lemma now follows because we have

$$P_1(\Lambda_{F[\beta]}) P_{(n/2)+}(\Lambda) \subseteq H^1(\Lambda, \beta).$$

□

Lemma 6.3.2. *With the same assumptions and notations as in Lemma 6.3.1, for any skew semisimple character $\theta \in \mathcal{C}(\Lambda, 0, \beta)$ of the group $H^1(\Lambda, \beta)$, we have*

$$\mathrm{res}_{U^w_{\mathrm{der}}(d(\mathfrak{r}, w, x))}\, \theta^{u^-} = \psi_{\beta^{u^-}}\,.$$

Proof. Let $u^+ = u(0, a)$ be an element in $U_{\mathrm{der}}(d(\mathfrak{r}, w, x))$, and $u^- = \bar{u}(x, y)$ be any element as in Lemma 6.3.1. Assume that $u^- u^+ (u^-)^{-1} = g_1 g_2$, where $g_1 \in P_1(\Lambda_{F[\beta]})$ and $g_2 \in P_{(n/2)+}(\Lambda)$. From the matrix identity 6.4 and from the definition of $d(\mathfrak{r}, \mathrm{id}, x)$, the constant $1 + ax\sigma(x)$ belongs to $F^\times \cap P_{(n/2)+}(\Lambda)$. Hence, we get that $\mathbf{1}_{V_1} g \mathbf{1}_{V_1} \in F^\times \cap P_{(n/2)+}(\Lambda)$. This implies that the determinant of the element $\mathbf{1}_{V_2} g_2 \mathbf{1}_{V_2}$ is contained in $F^\times \cap P_{(n/2)+}(\Lambda)$. Now, from the definition of a simple character, we get that $\theta(u^- u^+ (u^-)^{-1})$ is equal to $\psi_\beta(u^- u^+ (u^-)^{-1})$ and we get the lemma. □

Lemma 6.3.3. *With the same assumptions in the lemma 6.3.1, the restriction*

$$\mathrm{res}_{J^0(\Lambda,\beta) \cap U^w_{\mathrm{der}}} (\kappa \otimes \tau)$$

is equivalent to a direct sum of non-trivial characters.

Proof. We essentially follow ideas from [16, Theorem 2.6] and [4, Theorem 4.3]. We prove this lemma in the case where $w = \mathrm{id}$, and the other case is similar. If F/F_0 is unramified, then $P_0(\Lambda)/P_1(\Lambda)$ is isomorphic to

$$U(1,1)(k_F/k_{F_0}) \times U(1)(k_F/k_{F_0}).$$

If F/F_0 is ramified, then $P_0(\Lambda)/P_1(\Lambda)$ is isomorphic to

$$\mathrm{SL}_2(k_F) \times \{\pm 1\}.$$

Let $\tilde{J}_1$ be the group $(J^0(\Lambda,\beta)\cap U)J^1(\Lambda,\beta)$ and observe that we have $\tilde{J}_1$ is equal to $(J^0(\Lambda,\beta)\cap U_{\mathrm{der}})J^1(\Lambda,\beta)$. Note that the image of $J^0(\Lambda,\beta)\cap U$ in the quotient $P_0(\Lambda)/P_1(\Lambda)$ is its p-Sylow subgroup. Let $\tilde{H}_1$ be the group $(J^0(\Lambda,\beta)\cap U)H^1(\Lambda,\beta)$.

We observe that $\psi_\beta(u)=1$, for all $u\in U$. Let u be an element of $H^1(\Lambda,\beta)\cap U$. Write $u=g_2g_1g_1'$ where $g_2\in P_{(n/2)+}(\Lambda)$, $g_2\in P_1(\Lambda_{\beta_1})\cap U(V_2,h)$, and $g_1'\in P_1(\Lambda_{\beta_1})\cap U(V_1,h)$. Since $g_1'e_1=e_1$, we get that $g_2g_1e_1=e_1$. Hence, we get that $g_2g_1\in U$ and $g_1'=\mathrm{id}$. Now, the determinant of g_2^{-1} and g_1 are the same. With this observation, we get that

$$\mathrm{res}_{H^1(\Lambda,\beta)\cap U}\,\theta=\mathrm{res}_{H^1(\Lambda,\beta)\cap U}\,\psi_\beta=\mathrm{id}\,. \tag{6.5}$$

Let θ be a skew semisimple character of $H^1(\Lambda,\beta)$ and η be a Heisenberg lift of θ to the group $J^1(\Lambda,\beta)$. Using the equality (6.5), we define a character $\tilde{\theta}$ of the group $\tilde{H}_1$ by setting:

$$\tilde{\theta}(jh)=\theta(h),\ \text{for all } j\in J^0(\Lambda,\beta)\cap U,\ h\in H^1(\Lambda,\beta).$$

Using [16, Lemma 2.5], we get that the representation $\mathrm{ind}_{\tilde{H}_1\cap J^1(\Lambda,\beta)}^{J^1(\Lambda,\beta)}\,\tilde{\theta}$ is isomorphic to η.

Note that the group $J^1(\Lambda,\beta)\cap U_{\mathrm{der}}$ is equal to $H^1(\Lambda,\beta)\cap U_{\mathrm{der}}$ and we get that the representation

$$\mathrm{res}_{J^1(\Lambda,\beta)\cap U_{\mathrm{der}}}\,\eta\simeq\mathrm{res}_{H^1(\Lambda,\beta)\cap U_{\mathrm{der}}}\,\eta\simeq(\mathrm{id})^{\dim_{\mathbb{C}}\eta}.$$

This implies that η extends as a representation of $\tilde{J}_1$ such that $J^1(\Lambda,\beta)\cap U_{\mathrm{der}}$ acts trivially on this extension; let us denote this extension by $\tilde{\eta}$. By Frobenius reciprocity we get a map

$$\mathrm{ind}_{\tilde{H}_1}^{\tilde{J}_1}\,\tilde{\theta}\to\tilde{\eta}. \tag{6.6}$$

The representation η is irreducible. The dimension of the representation $\mathrm{ind}_{\tilde{H}_1}^{\tilde{J}_1}\,\tilde{\theta}$ is equal to $[J^1(\Lambda,\beta):H^1(\Lambda,\beta)]^{1/2}$. Hence, the map (6.6) is an isomorphism.

Let $[\lambda^m,n_1,0,\beta]$ be a skew semisimple stratum such that $\tilde{J}_1$ is equal to $(P_1(\Lambda_\beta^m)\cap U_{\mathrm{der}})J^1(\Lambda,\beta)$. Let θ_m be the skew semisimple character of $H^1(\Lambda^m,\beta)$ obtained as a transfer from the skew semisimple character θ of $H^1(\Lambda^m,\beta)$. We note that the groups $\tilde{H}_1$ and $H^1(\Lambda^m,\beta)$ have the Iwahori decomposition with respect to the pair (B,T). We then get that

$$\mathrm{Hom}_{\tilde{H}_1\cap H^1(\Lambda^m,\beta)}(\tilde{\theta},\theta_m)\neq 0.$$

This implies that

$$\mathrm{ind}_{\tilde{J}^1}^{P_1(\Lambda_m)}\,\tilde{\eta}\simeq\mathrm{ind}_{J^1(\Lambda^m,\beta)}^{P_1(\Lambda_m)}\,\eta_m.$$

From the uniqueness properties of β-extensions, we get that $\mathrm{res}_{\tilde{J}_1}\,\kappa\simeq\tilde{\eta}$. This shows that the representation

$$\mathrm{res}_{J^0(\Lambda,\beta)\cap U_{\mathrm{der}}}(\kappa\otimes\tau)$$

is a direct sum of non-trivial characters. □

The following lemma is a numerical verification to be used in the subsequent Lemma 6.3.5.

Lemma 6.3.4. *Let F/F_0 be any quadratic extension and let $x,y\in\mathfrak{o}_F$ such that $x\sigma(x)+y+\sigma(y)=0$. If $d(\mathfrak{r},w,x)>d(\mathfrak{r},w)$, then we have*

$$\nu_{F_0}(\delta(\beta_1-\beta_2)x\sigma(x))\leq -d(\mathfrak{r},w,x).$$

Proof. Assume that F/F_0 is unramified and $\Lambda(0)\cap V_2=\mathfrak{o}_Fe_1\oplus\mathfrak{o}_Fe_{-1}$. If $d(\mathfrak{r},w,x)>d(\mathfrak{r},w)$, then we get that $\nu_F(x)<m$, and this implies that

$$2\nu_F(x)-2m-r\leq-(m+1)+\nu_F(x).$$

Now, consider the case where $\Lambda(0)\cap V_2=\mathfrak{o}_Fe_1\oplus\mathfrak{p}_Fe_{-1}$ and assume that $d(\mathfrak{r},w,x)>d(\mathfrak{r},w)$. If $r=0$ and $w=\mathrm{id}$, then we get that $\nu_F(x)<m$ which implies that

$$2\nu_F(x)-2m\leq-m+\nu_F(x).$$

If $r=0$ and $w\neq\mathrm{id}$, then we get that $\nu_F(x)<m-1$ and hence

$$2\nu_F(x)-2m\leq-m-1+\nu_F(x).$$

If $r = 1$ and $w = \mathrm{id}$, then we have $\nu_F(x) < m+1$ and hence we get that

$$2\nu_F(x) - (2m+1) \leq -(m+1) + \nu_F(x).$$

Finally, we consider the case where $r = 1$ and $w \neq \mathrm{id}$; we then have $\nu_F(x) < m$. Hence, we get that

$$2\nu_F(x) - (2m+1) \leq -m - 1 + \nu_F(x).$$

Now, assume that F/F_0 is a ramified extension and $d(\mathfrak{x}, w, x) > d(\mathfrak{x}, w)$. Note that $\nu_{F/F_0}(\beta_1 - \beta_2) = -(2m+1)$, for some $m \in \mathbb{Z}$. If $w = \mathrm{id}$, then we have $\nu_{F/F_0}(x) < m/2$ and hence

$$2\nu_{F/F_0}(x) - (2m+1)/2 + 1/2 < -m/2 + \nu_{F/F_0}(x) \leq -d(\mathfrak{x}, w, x).$$

If $w \neq \mathrm{id}$, then we have $\nu_{F/F_0}(x) < (m-1)/2$ and we have

$$2\nu_{F/F_0}(x) - (2m+1)/2 + 1/2 < -(m+1)/2 + \nu_{F/F_0}(x) \leq -d(\mathfrak{x}, w, x).$$

From the above inequalities, in all exhaustive cases, gives the required inequality:

$$\nu_{F_0}(\delta(\beta_2 - \beta_1) x\sigma(x)) \leq -d(\mathfrak{x}, w, x)$$

□

Lemma 6.3.5. *Let F/F_0 be an unramified extension and let $\mathfrak{x}$ be a strata of type* **(C)** *such that (V_2, h) is isotropic. Every cuspidal representation in the set $\Pi_{\mathfrak{x}}$ is non-generic.*

Proof. Let π be a cuspidal representation in the set $\Pi_{\mathfrak{x}}$. Then we have $\pi \simeq \mathrm{ind}_{J^0(\Lambda,\beta)}^G(\kappa \otimes \tau)$, where $(J^0(\Lambda, \beta), \kappa \otimes \tau)$ is a Bushnell–Kutzko type contained in π. Assume that the representation π is generic. Then there exists a $g \in \mathcal{I}$, an element $w \in W_G$, and a character Ψ of U^w such that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^g \cap U^w}((\kappa \otimes \tau)^g, \Psi) \neq 0. \tag{6.7}$$

We write $g = p^+u^-$ such that $p^+ \in B^w \cap \mathcal{I}$ and $u^- \in \overline{U^w} \cap \mathcal{I}$, where $\overline{U^w}$ is the unipotent radical of the opposite Borel subgroup of B^w with respect to the torus T. From the expression (6.7), we get that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^{u^-} \cap U^w}((\kappa \otimes \tau)^{u^-}, \Psi') \neq 0, \tag{6.8}$$

for some character Ψ' of U^w. We set $e_w = we_1$ and $e_{-w} = we_{-1}$, we then have

$$h(u^-e_w, \beta u^- e_w) = \beta_1 x\sigma(x) + \beta_2 h(e_w + ye_{-w}, e_w + ye_{-w}) = (\beta_1 - \beta_2)x\sigma(x).$$

Hence, $\nu_F(\delta h(u^-e_w, \beta u^-e_w))$ is equal to $\nu_F(\delta(\beta_1 - \beta_2)x\sigma(x))$. If $d(\mathfrak{x}, w, x) > d(\mathfrak{x}, w)$, then Lemma 6.3.4 implies that

$$\nu_F(\delta(\beta_1 - \beta_2)x\sigma(x)) \leq -d(\mathfrak{x}, w, x).$$

Now, using Lemmas 6.3.2 and 2.6.1, we get that

$$\mathrm{res}_{U_{\mathrm{der}}(d(\mathfrak{x},w,x))}\, \theta = \psi_\beta^{u^-} \neq \mathrm{id}\,.$$

But, this is a contradiction to the assumption in (6.7). Hence, we obtain $d(\mathfrak{x}, w, x) = d(\mathfrak{x}, w)$ and this implies that $u^- \in H^1(\Lambda, \beta)$. We may as well assume that $u^- = \mathrm{id}$. The lemma now follows from Lemma 6.3.3. □

7. Non simple type (D) strata

7.1. **Inducing data.** A skew semisimple stratum $[\Lambda, n, 0, \beta]$, denoted by $\mathfrak{x}$, is of type **(D)** if the underlying splitting is of the form $V = V_1 \perp V_2 \perp V_3$ with $\dim_F V_i = 1$, for $1 \leq i \leq 3$. We use the notation W_i for the space $\oplus_{j \neq i} V_j$, for $1 \leq i \leq 3$.

When F/F_0 is unramified, we fix a vector $v_i \in V_i$ such that $\nu_{F_0}(h(v_i, v_i)) \in \{0, 1\}$, for $1 \leq i \leq 3$. When F/F_0 is ramified, we fix a vector $v_i \in V_i$ such that $\nu_{F_0}(h(v_i, v_i)) = 0$, for $1 \leq i \leq 3$. We denote by λ_i the inner product $h(v_i, v_i)$, for $1 \leq i \leq 3$. We have $\beta = \beta_1 + \beta_2 + \beta_3$, where $\beta_i = \mathbf{1}_{V_i}\beta\mathbf{1}_{V_i}$, for $1 \leq i \leq 3$. The lattice sequence Λ is uniquely determined by the element β and we have $e(\Lambda) = 2$. Let Λ_i be the $\mathfrak{o}_F$-lattice sequence $\Lambda \cap V_i$, for $1 \leq i \leq 3$.

Since, $\mathfrak{x}$ is a skew semisimple stratum of type **(D)**, we have

$$|\{\beta_i : \beta_i \neq 0, 1 \leq i \leq 3\}| \leq 1.$$

Without loss of generality, we assume that

$$\beta_1 \neq 0, \beta_2 \neq 0. \tag{7.1}$$

Moreover, we assume that

$$\begin{array}{ll} -n = \nu_{\Lambda_1}(\beta_1) \leq \nu_{\Lambda_2}(\beta_2) \leq 0 & \text{if } \beta_3 = 0, \\ -n = \nu_{\Lambda_1}(\beta_1) \leq \nu_{\Lambda_2}(\beta_2) \leq \nu_{\Lambda_3}(\beta_3) \leq 0 & \text{if } \beta_3 \neq 0. \end{array} \tag{7.2}$$

We have $e(\Lambda_i) = 2$, for $1 \leq i \leq 3$, and

$$\Lambda_i(-1) = \Lambda_i(0) = \mathfrak{o}_F v_i \text{ if } \nu_{F_0}(\lambda_i) = 0,$$
$$\Lambda_i(0) = \Lambda_i(1) = \mathfrak{o}_F v_i \text{ if } \nu_{F_0}(\lambda_i) = 1.$$

As the lattice sequence Λ depends on various possibilities on V_i, we will describe these lattice sequences and a Witt-basis, which gives a splitting for Λ, as required in each individual case. However, the group $J^0(\Lambda, \beta)$ and $H^1(\Lambda, \beta)$ are given by

$$P_0(\Lambda_\beta)P_{(q_2/2)}(\Lambda_{\beta_1})P_{(n/2)}(\Lambda)$$

and

$$P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda_{\beta_1})P_{(n/2)+}(\Lambda)$$

respectively.

7.2. **Criterion for non-emptiness of the set $\mathfrak{X}_\beta(F_0)$.** When F/F_0 is unramified, the non-emptiness of the set $\mathfrak{X}_\beta(F_0)$ depends only on the integers $\{\nu_F(\beta_i), \nu_{F_0}(\lambda_i) : 1 \leq i \leq 3\}$. This will be made precise in the following lemmas. However, when F/F_0 is ramified, one requires more information on $\{\beta_1, \beta_2, \beta_3\}$ to determine whether the set $\mathfrak{X}_\beta(F_0)$ empty or not. In the case where F/F_0 is ramified we will not make these conditions explicit, but we will show that a cuspidal representation in $\Pi_\mathfrak{x}$ is generic if and only if $\mathfrak{X}_\beta(F_0)$ is non-empty.

Lemma 7.2.1. *Let F/F_0 be an unramified extension and let $\mathfrak{x}$ be a stratum of type* **(D)** *such that (W_i, h) is isotropic, for $1 \leq i \leq 3$. The set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if $\nu_F(\beta_1) - \nu_F(\beta_2)$ is an even integer.*

Proof. Since the extension F/F_0 is unramified, we may assume that $\lambda_i = 1$, for $1 \leq i \leq 3$. The set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if there exists a non-zero vector $v \in V$ such that

$$h(v, v) = h(v, \beta v) = 0. \tag{7.3}$$

Let $v = av_1 + bv_2 + cv_3$, for some $a, b, c \in F$. From the equation (7.3), we get that

$$a\sigma(a) + b\sigma(b) + c\sigma(c) = 0 \text{ and } \beta_1 a\sigma(a) + \beta_2 b\sigma(b) + \beta_3 c\sigma(c) = 0. \tag{7.4}$$

Using the assumption (7.2) on the skew semisimple stratum $\mathfrak{x}$ we get that $(1-\beta_2\beta_1^{-1}) \in \mathfrak{o}_F^\times$ and $(1-\beta_3\beta_2^{-1}) \in \mathfrak{o}_F^\times$. The set of equations (7.4) imply that

$$a\sigma(a) = -c\sigma(c)\beta_2\beta_1^{-1}(1 - \beta_3\beta_2^{-1})(1 - \beta_2\beta_1^{-1})^{-1}.$$

Hence, we get that $\nu_F(\beta_2) - \nu_F(\beta_1)$ is an even integer. Conversely, if $\nu_F(\beta_2) - \nu_F(\beta_1)$ is even, we find can find a non-zero tuple $(a, b, c) \in F^3$ satisfying the equalities in equation (7.4); therefore, the set $\mathfrak{X}_\beta(F_0)$ is non-empty. ☐

Lemma 7.2.2. *Let F/F_0 be an unramified extension and let $\mathfrak{x}$ be a stratum of type* **(D)** *such that (W_i, h) is anisotropic, for some i, $1 \leq i \leq 3$. The set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if $\nu_F(\lambda_2) = \nu_F(\lambda_3) = 1$ and $\nu_F(\beta_1) - \nu_F(\beta_2)$ is an odd integer.*

Proof. Recall that the determinant of (V, h) is the trivial class in $F_0^\times/\mathrm{Nr}_{F/F_0}(F^\times)$. With the hypothesis on the spaces W_i, there exists an unique $i \in \{1, 2, 3\}$ such that $\nu_F(\lambda_i) = 0$. The set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if the following equations

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0 \text{ and } \beta_1\lambda_1 a\sigma(a) + \beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c) = 0 \tag{7.5}$$

have a non-trivial simultaneous solution. Recall that $(1 - \beta_2\beta_1^{-1}) \in \mathfrak{o}_F^\times$ and $(1 - \beta_3\beta_2^{-1}) \in \mathfrak{o}_F^\times$. Assume that there exists a non-trivial simultaneous solution to the equations in (7.5). Then, we get that

$$(1 - \beta_2\beta_1^{-1})\lambda_2 b\sigma(b) + (1 - \beta_3\beta_1^{-1})\lambda_3 c\sigma(c) = 0.$$

This implies that $\nu_F(\lambda_2) = \nu_F(\lambda_3)$. From the assumption on the spaces W_i, for $1 \le i \le 3$, we get that $\nu_F(\lambda_2) = \nu_F(\lambda_3) = 1$ and $\nu_F(\lambda_1) = 0$. From the equation (7.5) we have

$$\lambda_1 a\sigma(a) = \lambda_3 c\sigma(c)\beta_2\beta_1^{-1}(1-\beta_3\beta_2^{-1})/(1-\beta_2\beta_1^{-1}).$$

Hence, $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if (W_1, h) is isotropic and $\nu_F(\beta_2) - \nu_F(\beta_1)$ is an odd integer. □

7.3. **Estimating the valuation of $h(gv, \beta gv)$.** As observed in the previous sections, our approach to show non-genericity is by showing an appropriate inequality on the function sending $g \in P(\Lambda)$ to $\nu_F(h(gv, gv))$, where v is a well chosen isotropic vector with respect to $P(\Lambda)$. Hence, we need some technical lemmas to understand the growth of this function.

Lemma 7.3.1. *Let F/F_0 be an unramified extension and let $\mathfrak{x}$ be a skew semisimple stratum of type* **(D)** *such that $\mathfrak{X}_\beta(F_0)$ is the empty set. Let v be an isotropic vector in V, and let $g \in G$. Assume that $gv = av_1 + bv_2 + cv_3$ for some $a, b, c \in F$. If $a \neq 0$ and $\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c) \neq 0$, then we have*

$$\nu_F(h(gv, \beta gv)) = \min\{\nu_F(\beta_1\lambda_1 a\sigma(a)), \nu_F(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c))\}.$$

Remark 7.3.2. *Since the set $\mathfrak{X}_\beta(F_0)$ is the empty-set, we get that $h(v, \beta v) \neq 0$, for all isotropic vectors $v \in V$.*

Proof of Lemma 7.3.1. Before we begin the proof, it is useful to recall that the determinant of (V, h) is the trivial class in $F_0^\times/\mathrm{Nr}_{F/F_0}(F^\times)$. Since v is an isotropic vector, we get that

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0. \tag{7.6}$$

Observe that rescaling the constants a, b, c does not effect the lemma. Hence, rescaling a, b and c, if necessary, we assume that $a, b, c \in \mathfrak{o}_F$ and the $\mathfrak{o}_F$-ideal (a, b, c) is equal to $\mathfrak{o}_F$.

First consider the case where (W_i, h) is isotropic, for $1 \le i \le 3$. In this case we have $\nu_{F_0}(\lambda_i) = 0$, for $1 \le i \le 3$ and $\nu_F(\beta_1) - \nu_F(\beta_2)$ is an odd integer. If $\nu_F(a) = 0$, we get that

$$\begin{aligned}
&\nu_F(\beta_1\lambda_1 a\sigma(a) + \beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c))\\
=&\nu_F(\beta_1) + \nu_F(\lambda_1 a\sigma(a) + \beta_2\beta_1^{-1}\lambda_2 b\sigma(b) + \beta_3\beta_1^{-1}\lambda_3 c\sigma(c))\\
=&\nu_F(\beta_1) \le \nu_F(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c)).
\end{aligned}$$

Thus we prove the lemma in the case where $\nu_F(a) = 0$. Consider the case where $a \in \mathfrak{p}_F$; we necessarily have $b, c \in \mathfrak{o}_F^\times$. We have

$$\nu_F(\beta_2\beta_1^{-1}\lambda_2 b\sigma(b) + \beta_3\beta_1^{-1}\lambda_3 c\sigma(c)) = \nu_F(\beta_2\beta_1^{-1}) + \nu_F(\lambda_2 b\sigma(b) + \lambda_3\beta_3\beta_2^{-1}c\sigma(c)).$$

Since $\mathfrak{x}$ is a skew semisimple stratum, we get that $1 - \beta_3\beta_2^{-1} \in \mathfrak{o}_F^\times$. This implies that $\nu_F(\lambda_2 b\sigma(b) + \lambda_3\beta_3\beta_2^{-1}c\sigma(c)) = 0$. Observe that $\nu_F(a\sigma(a))$ is an even integer. Therefore, we conclude that

$$\nu_F(\beta_1\lambda_1 a\sigma(a)) \neq \nu_F(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c)),$$

and this proves the lemma in the case where (W_i, h) is isotropic, for $1 \le i \le 3$.

Assume that (W_i, h) is anisotropic for some $1 \le i \le 3$. Using Lemma 7.2.2, the set $\mathfrak{X}_\beta(F_0)$ is empty in either of the following cases: case **(I)** where $\nu_F(\lambda_2) \neq \nu_F(\lambda_3)$, case **(II)** where $\nu_F(\lambda_2) = \nu_F(\lambda_3)$ and $\nu_F(\beta_1) - \nu_F(\beta_2)$ is an even integer. We first assume that $\nu_{F_0}(\lambda_2) = 0$, and we get that $\nu_{F_0}(\lambda_1) = \nu_{F_0}(\lambda_3) = 1$. We may also assume that $\lambda_2 = 1$ and $\lambda_1 = \lambda_3 = \varpi$. Using equation (7.6), we get that $b \in \mathfrak{p}_F$, and set $b = \varpi b'$, for some $b' \in \mathfrak{o}_F$. Since we have

$$\varpi a\sigma(a) + \varpi^2 b'\sigma(b') + \varpi c\sigma(c) = 0,$$

we get that $a, c \in \mathfrak{o}_F^\times$. We now have

$$\nu_F(h(ge_1, ge_1)) = \nu_F(\beta_1) + \nu_F((1-\beta_2\beta_1^{-1})\varpi^2 b'\sigma(b') + (1-\beta_3\beta_1^{-1})\varpi c\sigma(c))$$

and note that

$$\nu_F((1-\beta_2\beta_1^{-1})\varpi^2 b'\sigma(b') + (1-\beta_3\beta_1^{-1})\varpi c\sigma(c)) = 1.$$

Hence, we get that $\nu_F(h(ge_1, ge_1)) = \nu_F(\beta_1) + 1$. Note that $\nu_F(\beta_2\varpi^2 b'\sigma(b') + \beta_3\varpi c\sigma(c))$ is equal to $\nu_F(\beta_2) + 1$. Hence, the lemma follows in case **(I)**, from the observation that

$$\nu_F(\beta_1\lambda_1 a\sigma(a)) = \nu_F(\beta_1\varpi a\sigma(a)) = \nu_F(\beta_1) + 1 \le \nu_F(\beta_2) + 1.$$

The case where $\nu_F(\lambda_2) = 1$–in which case $\nu_F(\lambda_1) = \nu_F(\lambda_2) = 1$ and $\nu_F(\lambda_3) = 0$–is entirely similar.

Assume that we are in case **(II)**. In this case we may assume that $\lambda_2 = \lambda_3 = \varpi$ and $\lambda_1 = 1$. Then the equation (7.6) implies that $b, c \in \mathfrak{o}_F^\times$ and $a \in \mathfrak{p}_F$. We now have

$$h(gv, \beta gv) = \beta_1\{a\sigma(a) + \beta_2\beta_1^{-1}\varpi(b\sigma(b) + \beta_3\beta_2^{-1}c\sigma(c))\}.$$

Since $\mathfrak{x}$ is a skew semisimple strata and $b, c \in \mathfrak{o}_F^\times$, we get that $\nu_F((b\sigma(b) + \beta_3\beta_2^{-1}c\sigma(c))) = 0$. The lemma now follows from the observation that the integer $\nu_F(\lambda_1 a\bar{a})$ is even and the integer $\nu_F(\beta_2\beta_1^{-1}\varpi_{F_0}(b\sigma(b) + \beta_3\beta_2^{-1}c\sigma(c)))$ is always odd. □

Lemma 7.3.3. *Let F/F_0 be a ramified extension and let $\mathfrak{x}$ be a stratum of type* **(D)** *such that $\mathfrak{X}_\beta(F_0)$ is the empty set. Let v be any isotropic vector in V and let $g \in G$. Assume that $gv = av_1 + bv_2 + cv_3$ for some $a, b, c \in F$. If $a \neq 0$ and $\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c) \neq 0$, then we have*

$$\nu_F(h(gv, \beta gv)) = \min\{\nu_F(\beta_1\lambda_1 a\sigma(a)), \nu_F(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c))\}.$$

Proof. Since v is an isotropic vector, we get that

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0. \tag{7.7}$$

Rescaling the constants a, b, and c, if necessary, we assume that $a, b, c \in \mathfrak{o}_F$ and the $\mathfrak{o}_F$-ideal (a, b, c) is equal to $\mathfrak{o}_F$.

Since $\mathfrak{x}$ is a skew semisimple stratum, using the assumptions in (7.2), we get that $(1 - \beta_i\beta_j^{-1}) \in \mathfrak{o}_F^\times$, for $1 \leq j < i \leq 3$. The set $\mathfrak{X}_\beta(F_0)$ is empty in one of the two cases

(1) The case where
$$-(1 - \beta_2\beta_1^{-1})(1 - \beta_3\beta_1^{-1})^{-1}\lambda_2\lambda_3^{-1} \notin \mathrm{Nr}_{F/F_0}(F^\times)$$
(2) The case where
$$-(1 - \beta_2\beta_1^{-1})(1 - \beta_3\beta_1^{-1})^{-1}\lambda_2\lambda_3^{-1} \in \mathrm{Nr}_{F/F_0}(F^\times)$$
and
$$\lambda_3/\lambda_1\beta_2\beta_1^{-1}(1 - \beta_3\beta_2^{-1})(1 - \beta_2\beta_1^{-1})^{-1} \notin \mathrm{Nr}_{F/F_0}(F^\times). \tag{7.8}$$

In Case (1), unless $b, c \in \mathfrak{p}_F$, we have

$$(1 - \beta_2\beta_1^{-1})\lambda_2 b\sigma(b) + (1 - \beta_3\beta_1^{-1})\lambda_3 c\sigma(c) \notin \mathfrak{p}_F. \tag{7.9}$$

Hence, we get that

$$\nu_F(h(gv, \beta gv)) = \min\{\nu_F(\beta_1\lambda_1 a\sigma(a)), \nu_F(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c))\}.$$

Now consider Case (2) and assume that

$$(1 - \beta_2\beta_1^{-1})\lambda_2 b\sigma(b) + (1 - \beta_3\beta_1^{-1})\lambda_3 c\sigma(c) \in \mathfrak{p}_F; \tag{7.10}$$

if the condition (7.10) is false we get the lemma immediately. If $\nu_F(\beta_1) = \nu_F(\beta_2)$, then we have

$$-(1 - \beta_2\beta_1^{-1})\beta_2^{-1}\beta_1\lambda_1 a\sigma(a) + (1 - \beta_3\beta_2^{-1})\lambda_3 c\sigma(c) \in \mathfrak{p}_F$$

But, the above containment is a contradiction to the second condition in case (2), the equation (7.8). Hence, we get that $\nu_F(\beta_1) \neq \nu_F(\beta_2)$. If $a \in \mathfrak{o}_F^\times$, the valuation of $h(gv, \beta gv)$ is equal to

$$\nu_F(\beta_1) + \nu_F(\lambda_1 a\sigma(a) + \beta_2\beta_1^{-1}(\lambda_2 b\sigma(b) + \beta_3\beta_2^{-1}\lambda_3 c\sigma(c))) = \nu_F(\beta_1).$$

Note that the lemma follows from the observation that

$$\nu_F(\beta_1 a\sigma(a)) \leq \nu_F(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c)).$$

We consider the case where $a \in \mathfrak{p}_F$ and this implies that $b, c \in \mathfrak{o}_F^\times$. Thus we have $\lambda_2 b\sigma(b) + \beta_3\beta_2^{-1}\lambda_3 c\sigma(c) \in \mathfrak{o}_F^\times$. Suppose

$$\lambda_1 a\sigma(a) + \beta_2\beta_1^{-1}(\lambda_2 b\sigma(b) + \beta_3\beta_2^{-1}\lambda_3 c\sigma(c)) \in \mathfrak{p}_F^{2\nu_F(a)+1},$$

then we have

$$\beta_1\beta_2^{-1}\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \beta_3\beta_2^{-1}\lambda_3 c\sigma(c) \in \mathfrak{p}_F.$$

Using the equality (7.7), we get that

$$(1 - \beta_1\beta_2^{-1})\lambda_1 a\sigma(a) + (1 - \beta_3\beta_2^{-1})\lambda_3 c\sigma(c) \in \mathfrak{p}_F.$$

This is a contradiction to the condition (7.8). Hence, we obtain

$$\nu(h(gv, \beta gv)) = \min\{\nu_F(\beta_1\lambda_1 a\sigma(a)), \nu_F(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c))\}.$$

□

7.4. **Generic cuspidal representations of type (D).**

Lemma 7.4.1. *Let F/F_0 be any quadratic extension and let $\mathfrak{x}$ be a stratum of type* **(D)** *such that $\mathfrak{X}_\beta(F_0)$ is non-empty. Every cuspidal representation contained in the set $\Pi_\mathfrak{x}$ is generic.*

Proof. Let θ be any semisimple character in $\mathcal{C}(\Lambda, 0, \beta)$. Let $\mathbf{U}$ be the unipotent radical of a Borel subgroup in the set $\mathfrak{X}_\beta(F_0)$. We will first show that

$$\mathrm{res}_{H^1(\Lambda,\beta)\cap U}\, \theta = \psi_\beta.$$

If $q_1 = q_2$, then the group $H^1(\Lambda, \beta)$ is equal to $P_1(\Lambda_\beta)P_{q_1/2}(\Lambda)$ and from Lemma 3.6.1, we get that $H^1(\Lambda, \beta) \cap U$ is equal to $P_{q_1/2+}(\Lambda) \cap U$. In the case where $q_1 > q_2$, the group $H^1(\Lambda, \beta)$ is equal to

$$P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda_{\beta_1})P_{(q_1/2)+}(\Lambda).$$

Let H' be the group $P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda)$. From Lemma 3.6.1 we get that the group $H' \cap U$ is equal to $P_{(q_2/2)+}(\Lambda) \cap U$ and hence, $H^1(\Lambda, \beta) \cap U$ is equal to $P_{(q_2/2)+}(\Lambda_{\beta_1})P_{(q_1/2)+}(\Lambda) \cap U$.

Let $v = av_1 + bv_2 + cv_3$ be an isotropic vector fixed by U. Since $\mathfrak{x}$ is a skew semisimple stratum of type **(D)**, we get that $a \neq 0$. Assume that $g_1 g_2 \in U$, for some $g_1 \in P_{(q_2/2)+}(\Lambda_{\beta_1})$ and $g_2 \in P_{(q_1/2)+}(\Lambda)$. From the equality

$$g_1(av_1 + bv_2 + cv_3) = a'av_1 + g_1(bv_2 + cv_3) = g_2(av_1 + bv_2 + cv_3)$$

we get that

$$\mathbf{1}_{\langle v_1\rangle} g_1 \mathbf{1}_{\langle v_1\rangle} = a' \in F^\times \cap P_{(q_1/2)+}(\Lambda).$$

Which implies that the determinant of $\mathbf{1}_{W_1} g_1 \mathbf{1}_{W_1}$ belongs to $F^\times \cap P_{(q_1/2)+}(\Lambda)$. Hence, the simple character $\theta(g_1g_2)$ is equal to $\psi_\beta(g_1g_2)$. Now, the lemma follows from Proposition 3.6.2. □

7.5. **Non-generic cuspidal representations of type (D).**

We will show that $\pi \in \Pi_\mathfrak{x}$ is non-generic if and only if the set $\mathfrak{X}_\beta(F_0)$ is empty. We will devide the proof into several cases beginning with the easier case where (W_1, h) is anisotropic; in which case we will show that ψ_β^g is non-trivial on $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$, for all $g \in P(\Lambda)$. In the case where (W_1, h) is isotropic, the method of proof is more involved and we had to deal with conjugation of some shallow elements in the group $P(\Lambda)$.

7.5.1. *The case where (W_1, h) is anisotropic.*

Lemma 7.5.1. *Let F/F_0 be a quadratic extension and let $\mathfrak{x}$ be a skew semisimple stratum of type* **(D)** *such that (W_1, h) is anisotropic. If the set $\mathfrak{X}_\beta(F_0)$ is empty, then every representation in the set $\Pi_\mathfrak{x}$ is non-generic.*

Proof. Let us begin with the case where F/F_0 is ramified. In this case, $e(\Lambda) = 2$, and we have

$$\Lambda(-1) = \Lambda(0) = \mathfrak{o}_F v_1 \oplus \mathfrak{o}_F v_2 \oplus \mathfrak{o}_F v_3.$$

It is possible that neither of the spaces (W_2, h) and (W_3, h) are isotropic. However, there exists an orthogonal $\mathfrak{o}_F$-basis, $(\tilde{v}_2, \tilde{v}_3)$, for the lattice $\mathfrak{o}_F v_2 \oplus \mathfrak{o}_F v_3$ such that $\langle v_1, \tilde{v}_2\rangle$ is isotropic. Using arguments similar to Lemma 5.1.2, we can we choose a Witt-basis (e_1, e_{-1}) of $\langle v_1, \tilde{v}_2\rangle$ and an unit vector $e_0 \in \langle v_1, \tilde{v}_2\rangle^\perp$ such that the Witt-basis (e_1, e_0, e_{-1}) provides a splitting for the lattice sequence Λ. In the basis (e_1, e_0, e_{-1}), we have

$$\Lambda(-1) = \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}.$$

Let $\mathbf{U}$ be the unipotent radical of the Borel subgroup, $\mathbf{B}$, such that $\langle e_1\rangle$ is fixed by B. We set

$$-\nu_\Lambda(\beta) = 4m + 2,$$

for some integer m. Assume that $\pi \in \Pi_\mathfrak{x}$ is a generic representation. Let $(J^0(\Lambda, \beta), \kappa)$ be a Bushnell–Kutzko type contained in π. There exists a $g \in P(\Lambda)$, and a non-trivial character Ψ of U such that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^g\cap U}(\kappa^g, \Psi) \neq 0.$$

Since g normalises $P_r(\Lambda)$, for $r > 0$, we get that the group $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$ is equal to $U_{\mathrm{der}}(\lceil m/2 \rceil)$. Let $ge_1 = av_1 + bv_2 + cv_3$ for some $a, b, c \in \mathfrak{o}_F$. Then we have

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0.$$

If $a \in \mathfrak{p}_F$, then $b, c \in \mathfrak{p}_F$ as (W_1, h) is anisotropic. Since $\nu_\Lambda(e_1) = 0$, we get that $a \in \mathfrak{o}_F^\times$. Now, we have

$$\begin{aligned} h(ge_1, \beta ge_1) &= \beta_1(\lambda_1 a\sigma(a) + \lambda_2\beta_2\beta_1^{-1} b\sigma(b) + \lambda_3\beta_3\beta_1^{-1} c\sigma(c)) \\ &= \beta_1(\lambda_1(1 - \beta_3\beta_1^{-1}) a\sigma(a) + \lambda_2\beta_2\beta_1^{-1}(1 - \beta_3\beta_2^{-1}) b\sigma(b)). \end{aligned}$$

If $\nu_F(\beta_2) > \nu_F(\beta_1)$, then we get that $\nu_F(h(ge_1, \beta ge_1))$ is equal to $\nu_F(\beta_1)$. Assume that $\nu_F(\beta_1 = \nu_F(\beta_2)$. Since, both the constants b, c cannot be in $\mathfrak{p}_F$, without loss of generality, we assume that $b \in \mathfrak{o}_F^\times$. If

$$\lambda_1(1 - \beta_3\beta_1^{-1}) a\sigma(a) + \lambda_2\beta_2\beta_1^{-1}(1 - \beta_3\beta_2^{-1}) b\sigma(b) \in \mathfrak{p}_F,$$

then we get that

$$\overline{-\lambda_2\lambda_3^{-1} b\sigma(b)(1 - \beta_2\beta_1^{-1})(1 - \beta_3\beta_1^{-1})^{-1}} \in (k_F^\times)^2;$$

therefore, we get a contradiction to the assumption that the set $\mathfrak{X}_\beta(F_0)$ is empty. Hence, we have

$$\nu_{F/F_0}(\delta h(ge_1, \beta ge_1)) = \nu_{F/F_0}(\beta_1) + 1/2 = m + 1.$$

Thus we get that

$$\nu_{F/F_0}(h(ge_1, \beta ge_1)) \leq -\lceil m/2 \rceil.$$

Now, Lemma 2.6.1 implies that the character ψ_β^g is non-trivial on the group $P_{(n/2)+}(\Lambda) \cap U_{\mathrm{der}}$, and we get a contradiction to the assumption that π is generic.

Consider the case where F/F_0 is unramified. In this case, we may assume that $\lambda_1 = \varpi$ and $(\lambda_2, \lambda_3) \in \{(\varpi, 1), (1, \varpi)\}$. So we define $\tilde{v}_3$ to be the vector in the set $\{v_2, v_3\}$ with $h(\tilde{v}_3, \tilde{v}_3) = \varpi$, and the remaining vector in the set $\{v_2, v_3\}$ is denoted by $\tilde{v}_2$. The notation $\tilde{\beta}_i$ will be used for $\mathbf{1}_{\langle \tilde{v}_i \rangle}\beta\mathbf{1}_{\langle \tilde{v}_i \rangle}$, for $i \in \{2, 3\}$. The period 2 lattice sequence Λ is given by

$$\Lambda(0) = \mathfrak{o}_F v_1 \oplus \mathfrak{o}_F \tilde{v}_2 \oplus \mathfrak{o}_F \tilde{v}_3 \text{ and } \Lambda(1) = \mathfrak{o}_F v_1 \oplus \mathfrak{p}_F \tilde{v}_2 \oplus \mathfrak{o}_F \tilde{v}_3.$$

Let e_0 be the vector $\tilde{v}_2$. Since the space $\langle v_1, \tilde{v}_3 \rangle$ is isotropic, and there exists a Witt-basis (e_1, e_{-1}) for the space $\langle v_1, \tilde{v}_3 \rangle$ such that

$$\Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{o}_F e_{-1}.$$

Let $\mathbf{U}$ be the unipotent radical of the Borel subgroup, $\mathbf{B}$, such that $\langle e_1 \rangle$ is fixed by B. Assume that $\pi \in \Pi_{\mathfrak{r}}$, containing a Bushnell–Kutzko type $(J^0(\Lambda, \beta), \kappa)$, is a generic representation. There exists a $g \in P(\Lambda)$ and a non-trivial character Ψ of U such that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^g \cap U}(\kappa^g, \Psi) \neq 0.$$

Note that $\nu_\Lambda(e_1) = 1$ and hence we get that $ge_1 = av_1 + \varpi b\tilde{v}_2 + c\tilde{v}_3$, for some $a, b, c \in \mathfrak{o}_F$. Since e_1 is an isotropic vector, we get that

$$a\sigma(a) + \varpi b\sigma(b) + c\sigma(c) = 0.$$

From the above equality and the fact that $\nu_\Lambda(e_1) = 1$, we get that $a, c \in \mathfrak{o}_F^\times$. We set $n = 4m + 2r$, for some integer m and $r \in \{0, 1\}$. The valuation of $h(ge_1, \beta ge_1)$ is equal to

$$\nu_F(\beta_1) + \nu_F(a\sigma(a) + \tilde{\beta}_2\beta_1^{-1} b\sigma(b) + \tilde{\beta}_3\beta_1^{-1} c\sigma(c)) = \nu_F(\beta_1) = -(2m + r).$$

We observe that g normalises the group $P_{(n/2)+}(\Lambda)$; therefore, we get that $P_{(n/2)+}(\Lambda)^g \cap U_{\mathrm{der}}$ is equal to $U_{\mathrm{der}}(m)$. Since $\nu_F(h(ge_1, \beta ge_1)) \leq -m$, using Lemma 2.6.1, we get that the character ψ_β^g is non-trivial on the group $P_{(n/2)+}(\Lambda)^g \cap U_{\mathrm{der}}$. Thus we obtain a contradiction to the assumption on the genericity of the representation $\pi \in \Pi_{\mathfrak{r}}$. $\square$

7.5.2. *The case where (W_1, h) is isotropic.*

Lattice sequences: We begin with the description of the lattice sequence Λ. Let (e_1, e_{-1}) be a Witt-basis for the space (W_1, h), and e_0 be an unit vector in V_1. Let $\mathbf{U}$ be the unipotent radical of the Borel subgroup, $\mathbf{B}$, such that $\langle e_1\rangle$ is fixed by B. Let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$, contained in $\mathbf{B}$ such that T fixes the decomposition

$$\langle e_1\rangle \oplus \langle e_0\rangle \oplus \langle e_{-1}\rangle.$$

If F/F_0 is unramified and $h(v_2, v_2) = h(v_3, v_3) = \varpi$, then $e(\Lambda) = 2$ and

$$\Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

If F/F_0 is unramified and $h(v_2, v_2) = h(v_3, v_3) = 1$, then $e(\Lambda) = 2$ and

$$\Lambda(-1) = \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}.$$

If F/F_0 is ramified, then $e(\Lambda) = 2$ and

$$\Lambda(-1) = \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}.$$

Let $\mathcal{I}$ be the Iwahori subgroup defined, in its matrix form with respect to the basis (e_1, e_0, e_{-1}), by

$$\begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix} \cap G.$$

Recall that we use the notation q_i for the integer $-\nu_{\Lambda_i}(\beta_i)$, for $1 \leq i \leq 3$. We set $q_1 = 4m_1 + 2r_1$ and $q_2 = 4m_2 + 2r_2$, for some integers m_1, m_2 and $r_1, r_2 \in \{0, 1\}$. Also recall our convention that $q_1 \geq q_2$.

Shallow elements: For the purpose of understanding linear functionals supported on $J^0(\Lambda, \beta)uU$, for $u \in \mathcal{I}$, we need a certain measure of depth of an element u with respect to the group $P_{(n/2)+}(\Lambda)$. Let $x, y \in \mathfrak{o}_F$ be two elements such that $x\sigma(x) + y + \sigma(y) = 0$, and let w be an element in W_G. The function $d(\mathfrak{r}, w, x)$, defined below, measures the depth of an element $u(x, y)^w$ with respect to $P_{(n/2)+}(\Lambda)$. When F/F_0 is an unramified extension, we set

$$d(\mathfrak{r}, w, x) = \begin{cases} \max\{m_2 + 1, m_1 + 1 - \nu_F(x)\}, & \text{if } \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_{-1}, \\ \max\{m_2, m_1 + r_1 - \nu_F(x)\} & \text{if } \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}, w = \mathrm{id}, \\ \max\{m_2 + 2, m_1 + r_1 + 1 - \nu_F(x)\} & \text{if } \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}, w \neq \mathrm{id}. \end{cases} \tag{7.11}$$

When F/F_0 is a ramified extension, the lattice sequences Λ is uniquely determined (see subsection 7.5.2). We define the integer $d(\mathfrak{r}, w, x)$ as follows:

$$d(\mathfrak{r}, w, x) = \max\{\lceil m_2/2\rceil, \lceil m_1/2 - \nu_F(x)\rceil\} \tag{7.12}$$

Lemma 7.5.2. *Let $\mathfrak{r}$ be a skew semisimple stratum of type* **(D)**. *Let w be an element of W_G. Let $u = \dot{u}(x, y)$ be an element of $\mathcal{I} \cap \overline{U^w}$, where $\overline{\mathbf{U}^w}$ is the unipotent radical of the opposite Borel subgroup of $\mathbf{B}^w$ with respect to $\mathbf{T}$. We have*

$$U^w_{\mathrm{der}}(d(\mathfrak{r}, w, x)) \subseteq H^1(\Lambda, \beta)^u \cap U^w_{\mathrm{der}}.$$

Proof. We will prove the above lemma when $w = \mathrm{id}$, and the proof is entirely similar in the case where $w \neq \mathrm{id}$. The group $H^1(\Lambda, \beta)$ is equal to

$$P_1(\Lambda_{F[\beta]}) P_{(q_2/2)+}(\Lambda_{F[\beta_1]}) P_{(n/2)+}(\Lambda).$$

Let $u = \bar{u}(x, y)$ be an element of $\mathcal{I} \cap \bar{B}$. Using the the matrix identity

$$\bar{u}(x, y) u(0, a) \bar{u}(-x, -y - x\sigma(x)) = \begin{pmatrix} 1 - a(x\sigma(x) + y) & a\sigma(x) & a \\ ax(-y - x\sigma(x)) & 1 + ax\sigma(x) & ax \\ -ay(y + x\sigma(x)) & a\sigma(x)y & ay + 1 \end{pmatrix} \tag{7.13}$$

and the definition of $d(\mathfrak{r}, \mathrm{id}, x)$, we see that the group $\{U_{\mathrm{der}}(d(\mathfrak{r}, \mathrm{id}, x))\}^u$ is contained in the group $H^1(\Lambda, \beta)$. $\square$

Lemma 7.5.3. *With the same notations as in Lemma 7.5.2, and for any skew semisimple character θ in $\mathcal{C}(\Lambda, 0, \beta)$ we have*

$$\mathrm{res}_{U^w_{\mathrm{der}}(d(\mathfrak{r}, w, x))}\, \theta^u = \psi^u_\beta.$$

Proof. We prove this in the case where $\Lambda \cap W_1$ equal to $\mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}$ and $w \neq \mathrm{id}$. The rest of the cases are similar and are simpler. Let u^+ be the element $u(x,y) \in \mathcal{I} \cap U$, and let $u = \bar{u}(0,a)$ be an element in the group $U^w_{\mathrm{der}}(d(\mathfrak{x}, w, x))$. Using the identity (7.13), we get that the element $u^+ u (u^+)^{-1}$ is of the form $g_1 g_2$, where $g_1 \in P_{(q_2/2)+}(\Lambda_{F[\beta_1]})$ and $g_2 \in P_{(n/2)+}(\Lambda)$. From the definition of the integer $d(\mathfrak{x}, w, x)$, we get that $1 + ax\sigma(x) \in F^\times \cap P_{(n/2)+}(\Lambda)$; therefore, we get that $\mathbf{1}_{V_1} g_1 \mathbf{1}_{V_1}$ belongs to $F^\times \cap P_{(n/2)+}(\Lambda)$. Hence, the determinant of $\mathbf{1}_{W_1} g_1 \mathbf{1}_{W_1}$ belongs to $F^\times \cap P_{(n/2)+}(\Lambda)$. The lemma follows from the defining property, [6, Definition 3.2.3(a)], of the character θ. $\square$

Lemma 7.5.4. *Let F/F_0 be a any quadratic extension and $\mathfrak{x} = [\Lambda, n, 0, \beta]$ be a stratum of type* **(D)** *such that $\mathfrak{X}_\beta(F_0)$ is the empty set. Every cuspidal representation contained in the set $\Pi_{\mathfrak{x}}$ is non-generic.*

Proof. Let π be a cuspidal representation in the set $\Pi_{\mathfrak{x}}$. Let $(J^0(\Lambda,\beta), \kappa)$ be a Bushnell–Kutzko's type contained in the representation π. Assume that the representation π is generic. Then there exists a $w \in W_G$, an element $u = \dot{u}(x,y) \in \mathcal{I} \cap \overline{U^w}$, and a non-trivial character Ψ of U such that

$$\mathrm{Hom}_{J^0(\Lambda,\beta)^u \cap U^w}(\kappa^u, \Psi^w) \neq 0.$$

In particular, the above identity implies that

$$\mathrm{Hom}_{H^1(\Lambda,\beta)^u \cap U^w_{\mathrm{der}}}(\theta^u, \mathrm{id}) \neq 0, \tag{7.14}$$

where θ is the skew semisimple character contained in $\mathrm{res}_{H^1(\Lambda,\beta)} \kappa$.

Consider the case where F/F_0 is any quadratic extension and $\Lambda \cap W_1$ is equal to $\mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_{-1}$. In this case, we have $\nu_\Lambda(e_w) = 0$; therefore, we get that $ue_w = bv_2 + xv_1 + cv_3$, for some $b, c \in \mathfrak{o}_F$. If $\nu_{F/F_0}(x) = 0$, then we have

$$\nu_{F/F_0}(h(ue_w, \beta ue_w)) = \nu_{F/F_0}(\beta_1).$$

If $\nu_{F/F_0}(x) > 0$, then we have $b, c \in \mathfrak{o}_F^\times$. Since $\mathfrak{x}$ is skew semisimple stratum, we have $(1 - \beta_2\beta_3^{-1}) \in \mathfrak{o}_F^\times$. From the assumption that $\nu_{F/F_0}(x) > 0$, we get that $\lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) \in \mathfrak{p}_F$. Hence, we have

$$\nu_{F/F_0}(\lambda_2 b\sigma(b) + \beta_3\beta_2^{-1}\lambda_3 c\sigma(c)) = 0.$$

Using Lemmas 7.3.1 and 7.3.3, we get that

$$\begin{aligned}
\nu_{F/F_0}(h(ue_w, \beta ue_w)) &= \min\{\nu_{F/F_0}(\beta_1) + 2\nu_{F/F_0}(x), \nu_{F/F_0}(\beta_2\lambda_2 b\sigma(b) + \beta_3\lambda_3 c\sigma(c))\} \\
&= \min\{\nu_{F/F_0}(\beta_1) + 2\nu_{F/F_0}(x), \nu_{F/F_0}(\beta_2) + \nu_{F/F_0}(\lambda_2 b\sigma(b) + \beta_3\beta_2^{-1}\lambda_3 c\sigma(c))\} \\
&= \min\{\nu_{F/F_0}(\beta_1) + 2\nu_{F/F_0}(x), \nu_{F/F_0}(\beta_2)\}.
\end{aligned}$$

Consider the case where F/F_0 is unramified and $\Lambda \cap W_1$ is equal to $\mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}$. In this case, we have $\nu_\Lambda(e_{\pm 1}) = \pm 1$; therefore, we get that

$$ue_w = \begin{cases} xv_1 + bv_2 + cv_3, & b, c \in \mathfrak{o}_F, \text{ if } w = \mathrm{id} \\ xv_1 + bv_2 + cv_3, & b, c \in \mathfrak{p}_F^{-1}, \text{ if } w \neq \mathrm{id}. \end{cases} \tag{7.15}$$

If $w = \mathrm{id}$, we observe that $b\sigma(b) + c\sigma(c) \in \mathfrak{p}_F$; which together with $\nu_F(e_1) = 1$ implies that $b, c \in \mathfrak{o}_F^\times$. Since, $\mathfrak{x}$ is skew semisimple we have get that $b\sigma(b) + \beta_3\beta_2^{-1} c\sigma(c) \in \mathfrak{o}_F^\times$. From Lemma 7.3.1, we get that

$$\nu_F(h(ue_1, ue_1)) = \min\{\nu_F(\beta_1) + 2\nu_F(x), \nu_F(\beta_2) + 1\}.$$

If $w \neq \mathrm{id}$, then similar arguments as above imply that $\nu_F(b) = \nu_F(c) = -1$, and $\nu_F(b\sigma(b) + \beta_3\beta_2^{-1} c\sigma(c)) = -2$. Hence, using lemma we get that

$$\nu_F(h(ue_1, ue_1)) = \min\{\nu_F(\beta_1) + 2\nu_F(x), \nu_F(\beta_2) - 1\}.$$

Claim 1. *We claim that*

$$\nu_{F/F_0}(\delta h(ue_w, ue_w)) \leq -d(\mathfrak{x}, w, x).$$

Assuming Claim 1 we complete the proof of the lemma. Using Lemma 2.6.1, we get that the character ψ_β^u is non-trivial on the group $U_{\mathrm{der}}(d(\mathfrak{x}, w, x))$. Thus, we get a contradiction to the equation (7.14), and hence, the cuspidal representation π is non-generic.

Proof of Claim 1. **Case 1:** First consider the case where F/F_0 is unramified and

$$\Lambda(0) \cap W_1 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_{-1}.$$

The integer $\nu_F(h(ue_1, ue_1))$ is equal to $\min\{-(2m_2+r_2), -(2m_1+r_1)+2\nu_F(x)\}$. Recall that $-d(\mathfrak{x}, w, x)$ is equal to $\min\{-m_2-1, -m_1-1+\nu_F(x)\}$. Observe that $-(2m_2+r_2) \leq -m_2-1$, unless $m_2+r_2=0$; since the stratum $\mathfrak{x}$ is a skew semisimple stratum of type **(D)**, we have $m_2+r_2>0$ (see the assumption in the equation (7.1)). Assume that $-(2m_2+r_2)$ is equal to $\min\{-(2m_2+r_2), -(2m_1+r_1)+2\nu_F(x)\}$. Then we have

$$\nu_F(x) \geq m_1 - m_2 + (r_1-r_2)/2 = (m_1+1) - (2m_2+r_2) + (m_2 - 1 + (r_1+r_2)/2)).$$

Using Lemma 7.2.1, we get that $r_1 - r_2$ is an odd integer. Hence, we get that $m_2 - 1 + (r_1+r_2)/2) \geq -1/2$. Therefore, we have

$$\nu_F(x) - m_1 - 1 \geq -(2m_2+r_2).$$

Assume that $-(2m_1+r_1)+2\nu_F(x)$ is equal to $\min\{-(2m_2+r_2), -(2m_1+r_1)+2\nu_F(x)\}$. We have

$$\nu_F(x) \leq m_1 - m_2 + (r_1-r_2)/2 \leq m_1 + r_1 - 1 - (m_2 + (r_1+r_2)/2 - 1).$$

Since, r_1+r_2 is odd, we get that $m_2+(r_1+r_2)/2-1 \geq -1/2$. Hence, we the inequality $\nu_F(x) \leq m_1+r_1-1$ and we deduce that $-(2m_1+r_1)+2\nu_F(x) \leq -(m_1+1)+\nu_F(x)$. Finally, using the inequality

$$-(2m_1+r_1)+2\nu_F(x) \leq -(2m_2+r_2) \leq -(m_2+1),$$

we complete the verification of our claim in the present case.

Case 2: Let us consider the case where F/F_0 is unramified and

$$\Lambda(0) \cap W_1 = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1}.$$

In this case, we have $r_1 = r_2$. First, assume that $w = \mathrm{id}$. Then we have

$$-d(\mathfrak{x}, \mathrm{id}, x) = \min\{-m_2, -(m_1+r_1)+\nu_F(x)\},$$
$$\nu_F(h(ue_1, ue_1)) = \min\{-(2m_2+r_2)+1, -(2m_1+r_1)+2\nu_F(x)\}.$$

If $-(2m_2+r_2)+1$ is equal to $\nu_F(h(ue_1, ue_1))$, then we get that

$$\nu_F(x) - m_1 - r_1 \geq -m_2 - (r_1+r_2)/2 + 1/2 \geq -(2m_2+r_2) + 1 + (m_2 + (r_2-r_1)/2 - 1/2).$$

Since $m_2 + (r_2-r_1)/2 - 1/2 \geq -1/2$, we get that

$$\nu_F(x) - m_1 - r_1 \geq -(2m_2+r_2)+1.$$

If $-(2m_1+r_1)+2\nu_F(x)$ is equal to $\nu_F(h(ue_1, ue_1))$, then we that

$$\nu_F(x) \leq m_1 - m_2 + 1/2 \leq m_1 + 1/2.$$

Since $\nu_F(x)$ is an integer, we get that $\nu_F(x) \leq m_1$. Therefore, we get that

$$-(2m_1+r_1)+2\nu_F(x) \leq -(m_1+r_1)+\nu_F(x).$$

Hence, in the case where $w = \mathrm{id}$ we get that

$$\min\{-(2m_2+r_2)+1, -(2m_1+r_1)+2\nu_F(x)\} \leq -d(\mathfrak{x}, w, x).$$

Let us continue with the case considered in the previous paragraph but with $w \neq \mathrm{id}$. We have

$$-d(\mathfrak{x}, \mathrm{id}, x) = \min\{-(m_2+2), -(m_1+r_1+1)+\nu_F(x)\},$$
$$\nu_F(h(ue_{-1}, ue_{-1})) = \min\{-(2m_2+r_2)-1, -(2m_1+r_1)+2\nu_F(x))\}.$$

If $-(2m_2+r_2)-1$ is equal to $\nu_F(h(ue_{-1}, ue_{-1}))$, then we get that

$$\nu_F(x) \geq m_1 - m_2 - 1/2.$$

From this we get that $\nu_F(x) \geq m_1$; therefore, we have

$$-(m_1+r_1+1)+\nu_F(x) \geq -(2m_2+r_2)-1.$$

If $-(2m_1+r_1)+2\nu_F(x)$ is equal to $\nu_F(h(ue_{-1}, ue_{-1}))$, then we get that $\nu_F(x) \leq m_1 - m_2 - 1/2$. Since $\nu_F(x)$ is an integer, we get that $\nu_F(x) \leq m_1 - 1$. Hence, we complete the verification of the claim in the present case.

Case 3: Let us consider the case where F/F_0 is a ramified extension. In this case, we have

$$-d(\mathfrak{x}, w, x) = \min\{-\lceil m_2/2\rceil, -\lceil m_1/2 - \nu_{F/F_0}(x)\rceil\},$$
$$\nu_{F/F_0}(\delta h(ue_w, ue_w)) = \min\{-m_2, -m_1 + 2\nu_{F/F_0}(x)\}.$$

We clearly have $-m_2 \leq -\lceil m_2/2\rceil$, for $m_2 \geq 0$. Now, assume that $-m_2$ is equal to $\nu_{F/F_0}(\delta h(ue_w, ue_w))$. Then we have

$$m_2 \geq m_2/2 \geq m_1/2 - \nu_{F/F_0}(x).$$

From the above inequality we get that $m_2 \geq \lceil m_1/2 - \nu_{F/F_0}(x)\rceil$. We assume that $-m_1 + 2\nu_{F/F_0}(x)$ is equal to $\nu_{F/F_0}(\delta h(ue_w, ue_w))$. Then we get that $\nu_{F/F_0}(x) \leq m_1/2 - m_2/2$. Hence, we have $\nu_{F/F_0}(x) \leq m_1/2$ and this is equivalent to the inequality

$$-m_1 + 2\nu_{F/F_0}(x) \leq -\lceil m_1/2 - \nu_{F/F_0}(x)\rceil.$$

With this we complete the verification of the claim in all cases. $\square$

$\square$

8. The depth-zero case.

When F/F_0 is unramified, the classification of generic depth-zero cuspidal representations of G can be deduced from the general work of DeBacker–Reeder in the article [7, Section 6]. From their results, generic depth-zero cuspidal representations are precisely the representations of the form

$$\mathrm{ind}_{P_0(\Lambda)}^{G} \sigma,$$

where $P_0(\Lambda)$ is a parahoric subgroup such that $P_0(\Lambda)/P_1(\Lambda)$ is isomorphic to $U(2,1)(k_F/k_{F_0})$, and σ is the inflation of a cuspidal generic representation of $P_0(\Lambda)/P_1(\Lambda)$. Now, we assume that F/F_0 is a ramified extension and consider a cuspidal representation of G, isomorphic to

$$\mathrm{ind}_{P_0(\Lambda)}^{G} \sigma, \tag{8.1}$$

where $P^0(\Lambda)$ is a maximal parahoric subgroup of G, and σ is the inflation of a cuspidal representation of $P_0(\Lambda)/P_1(\Lambda)$.

If F/F_0 is ramified, the groups $P_0(\Lambda)/P_1(\Lambda)$ is the k_F-rational points of a disconnected reductive group over k_F. An irreducible representation σ of $P_0(\Lambda)/P_1(\Lambda)$ is called a cuspidal representation if $\mathrm{res}_{P^0(\Lambda)/P_1(\Lambda)}\, \sigma$ is a direct sum of cuspidal representations. An irreducible representation σ of $P_0(\Lambda)/P_1(\Lambda)$ is called generic if and only if its restriction to a p-Sylow subgroup, say H, contains a non-trivial character of H.

Let (e_1, e_0, e_{-1}) be any Witt-basis for (V, h) then up to G conjugation there are two lattice sequences Λ_1 and Λ_2 such that $P^0(\Lambda_i)$ is a maximal parahoric subgroup, for $i \in \{1,2\}$. We have $e(\Lambda_i) = 2$, for $i \in \{1,2\}$, and

$$\Lambda_1(-1) = \Lambda_1(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1},$$
$$\Lambda_2(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda_2(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{G}$ such that $\langle e_1\rangle$ is fixed by $\mathbf{B}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. The groups $P^0(\Lambda_1)$ and $P^0(\Lambda_2)$ are special maximal compact subgroups of G, and we have the Iwasawa decomposition

$$G = P_0(\Lambda_i)B,$$

for $i \in \{1,2\}$. The representation of the form (8.1) is generic if and only if

$$\mathrm{Hom}_{P_0(\Lambda)\cap U}(\sigma, \Psi) \neq 0,$$

for some character Ψ of U.

Lemma 8.0.1. *Let F/F_0 be a ramified quadratic extension. A depth-zero cuspidal representation π of G is generic if and only if*

$$\pi \simeq \mathrm{ind}_{P_0(\Lambda_1)}^{G} \sigma,$$

where σ is a generic cuspidal representation of $P_0(\Lambda_1)/P_1(\Lambda_1)$.

Proof. Let π be a depth zero cuspidal representation isomorphic to $\mathrm{ind}_{P_0(\Lambda_1)}^{G}\sigma$. The image of $U\cap P_0(\Lambda_1)$ in the quotient $P_0(\Lambda_1)/P_1(\Lambda_1)$ is the pro-p Sylow subgroup of $P_0(\Lambda_1)/P_1(\Lambda_1)$. Note that $P_0(\Lambda_1)\cap U$ is equal to

$$\{u(x,y) : x,y\in F, y+\sigma(y)+x\sigma(x)=0, \nu_{F/F_0}(y)\geq 0\},$$

and the group $P_1(\Lambda_1)\cap U$ is equal to

$$\{u(x,y) : x,y\in F, y+\sigma(y)+x\sigma(x)=0, \nu_{F/F_0}(y)\geq 1/2\}.$$

The quotient $(P_0(\Lambda_1)\cap U)/(P_1(\Lambda_1)\cap U)$ is isomorphic to $\{u(x,-x^2/2) : x\in k_F\}$. Let Ψ be any non-trivial character of U such that $\mathrm{res}_{P_0(\Lambda)\cap U}\Psi\neq \mathrm{id}$ and $\mathrm{res}_{P_1(\Lambda)\cap U}\Psi=\mathrm{id}$. For such a character Ψ, the space

$$\mathrm{Hom}_{P_0(\Lambda)\cap U}(\sigma,\Psi)\neq 0$$

if and only if σ is the inflation of a generic cuspidal representation of $P_0(\Lambda_1)/P_1(\Lambda_1)$. Hence, π is generic if and only if σ is generic.

Let π be a depth-zero cuspidal representation of the form $\pi\simeq \mathrm{ind}_{P_0(\Lambda_2)}^{G}\sigma$. Assume that π is generic, then we get that

$$\mathrm{Hom}_{P_0(\Lambda_2)\cap U_{\mathrm{der}}}(\sigma,\mathrm{id})\neq 0. \tag{8.2}$$

The image of the group $P_0(\Lambda_2)\cap U_{\mathrm{der}}$ in the quotient $P_0(\Lambda_2)/P_1(\Lambda_2)\simeq \mathrm{SL}_2(k_F)\times\{\pm 1\}$, is a p-Sylow subgroup, say H. Note that $\mathrm{res}_H\,\sigma$ is a direct sum of non-trivial characters of H. Thus we get a contradiction to the condition to the equation (8.2). Hence, the representation π is non-generic. □

9. Main theorem

In this section, we combine the results obtained so far in the following theorem. Recall the following notation: if W is a non-degenerate subspace of V, then $\mathbf{1}_W$ is the projection onto W with kernel $W^{\perp}$.

Theorem 9.0.1. *Let F be a non-Archimedean local field with odd residue characteristic. Let $\mathfrak{x}=[\Lambda,n,0,\beta]$ be a skew semisimple stratum and let $\Pi_{\mathfrak{x}}$ be the set of cuspidal representations containing a Bushnell–Kutzko type of the form $(J^0(\Lambda,\beta),\lambda)$. The cuspidal representations in the set $\Pi_{\mathfrak{x}}$ are either all generic or all non-generic. Furthermore, the following holds.*

- *(A) Let $\mathfrak{x}$ is a skew simple stratum, i.e., the case where $F[\beta]$ is a degree 3 field extension of F. Then the set $\mathfrak{X}_\beta(F_0)$ is non-empty, and every representation contained in the set $\Pi_{\mathfrak{x}}$ is generic.*
- *(B) Let $\mathfrak{x}$ be a skew semisimple stratum with the underlying splitting $V=V_1\perp V_2$ such that $\dim_F V_i=i$, for $i\in\{1,2\}$. Assume that $\beta=\beta_1+\beta_2$, where β_i is equal to $\mathbf{1}_{V_i}\beta\mathbf{1}_{V_i}$, $\sigma_h(\beta_i)=-\beta_i$, $F[\beta_2]$ is a degree 2 field extension of F. Let q_i be the integer $\nu_{\Lambda_i}(\beta_i)$, for $i\in\{1,2\}$. If $q_1>q_2$, then a cuspidal representation in the set $\Pi_{\mathfrak{x}}$ is generic if and only if (V_2,h) is isotropic. If $q_2>q_1$, then a cuspidal representation in the set $\Pi_{\mathfrak{x}}$ is generic if and only if (V_2,h) is anisotropic. In part (B), a cuspidal representation in the set $\Pi_{\mathfrak{x}}$ is generic if and only if the set $\mathfrak{X}_\beta(F_0)$ is non-empty.*
- *(C) Let $\mathfrak{x}$ be a skew semisimple stratum with the underlying splitting $V=V_1\perp V_2$ such that $\dim_F V_i=i$, for $i\in\{1,2\}$. We assume that $\beta=\beta_1+\beta_2$, where β_i is equal to $\mathbf{1}_{V_i}\beta\mathbf{1}_{V_i}$, $\beta_i\in F$, and $\sigma(\beta_i)=-\beta_i$, for $i\in\{1,2\}$. Every representation in the set $\Pi_{\mathfrak{x}}$ is non-generic. The set $\mathfrak{X}_\beta(F_0)$ is non-empty if and only if (V_2,h) is isotropic.*
- *(D) Let $\mathfrak{x}$ be a skew semisimple stratum with the underlying splitting $V=V_1\perp V_2\perp V_3$. Then a representation in the set $\Pi_{\mathfrak{x}}$ is generic if and only if $\mathfrak{X}_\beta(F_0)$ is non-empty.*

In part (B) of the above theorem, we have $q_1\neq q_2$ (see paragraph 5.3). In the case where the underlying splitting of a skew semisimple strata $\mathfrak{x}$ is equal to $V=V_1\perp V_2\perp V_3$, it is fairly easy to determine the necessary and sufficient conditions on β for the non-emptiness of the set $\mathfrak{X}_\beta(F_0)$. Assume that $\beta_i=\mathbf{1}_{V_i}\beta\mathbf{1}_{V_i}$, for $1\leq i\leq 3$. When F/F_0 is unramified, the non-emptiness of $\mathfrak{X}_\beta(F_0)$ depends only on the set of integers $\{\nu_F(\beta_1),\nu_F(\beta_2),\nu_F(\beta_3)\}$ and the isomorphism classes of (V_i,h), for $1\leq i\leq 3$. We refer to Lemmas 7.2.1 and 7.2.2 for these results. However, in the case where F/F_0 is ramified, one requires more invariants on β to determine whether $\mathfrak{X}_\beta(F_0)$ is empty or not. Since these invariants are not the natural invariants attached to a stratum, we did not make them explicit. For details, we refer to the proof of Lemma 7.3.3.

Proof of Theorem 9.0.1. We indicate the precise references to proofs of various parts enumerated in the theorem. The first part, case (A), follows from Proposition 4.1.2. In case (B), when $q_1 > q_2$, the corresponding statements are proved in Lemmas 5.6.5, 5.7.1, and 5.7.2. In case (B), when $q_2 > q_1$, the corresponding statements are proved in Lemmas 5.6.1, 5.6.2, and 5.7.4. The statements in case (C) are proved in Lemmas 6.2.1 and 6.3.5. The statements in case (D) are proved in 7.4.1, 7.5.1 and 7.5.4. Now, the main statement of the theorem on genericity or non-genericity of all representations in $\Pi_{\mathfrak{x}}$ follows from statements in cases $(A), (B), (C)$, and (D). $\square$

Appendix A. Appendix: Filtration of U_{der} induced by lattice sequences

In this section we fix some representatives for G-conjugacy classes of self-dual lattice sequences on V and describe $a_n(\Lambda)$, for $n \in \mathbb{Z}$. Then we use them to determine $U_{\mathrm{der}} \cap a_n(\Lambda)$, for $n \in \mathbb{Z}$. These calculations are used in showing certain representations are non-generic.

A.1. **The unramified case:** We begin with the case where F/F_0 is unramified. Let Λ_1 be the lattice sequence of periodicity 2 and

$$\Lambda_1(-1) = \Lambda_1(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}.$$

The filtration $\{a_n(\Lambda) \mid n \in \mathbb{Z}\}$ of $\mathrm{End}_F(V)$ is given by

$$a_{2m-1}(\Lambda_1) = \varpi^m \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix} \cap \mathfrak{g} \text{ and } a_{2m}(\Lambda_1) = \varpi^m \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix} \cap \mathfrak{g}, \tag{A.1}$$

for all $m \in \mathbb{Z}$. Let Λ_2 be a period 2 lattice sequence given by

$$\Lambda_2(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda_2(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

The filtration $\{a_n(\Lambda_2) \mid n \in \mathbb{Z}\}$ is given by:

$$a_{2m}(\Lambda_2) = \varpi^m \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{p}_F^{-1} \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix} \cap \mathfrak{g} \text{ and } a_{2m+1}(\Lambda_2) = \varpi^m \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F^2 & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix} \cap \mathfrak{g}, \tag{A.2}$$

for all $m \in \mathbb{Z}$. Let Λ_3 be the lattice sequence of period 4 given by

$$\Lambda_3(-1) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1},\ \Lambda_3(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1},$$
$$\Lambda_3(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1},\ \ \Lambda_3(2) = \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}.$$

The filtration $\{a_n(\Lambda_3) \mid n \in \mathbb{Z}\}$ on $\mathfrak{g}$ is given by:

$$a_{4m+r}(\Lambda_3) = \begin{cases} \varpi^m \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix} \cap \mathfrak{g} \text{ if } r = 0, \\ \varpi^m \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix} \cap \mathfrak{g} \text{ if } r = 1, \\ \varpi^m \begin{pmatrix} \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix} \cap \mathfrak{g} \text{ if } r = 2, \\ \varpi^m \begin{pmatrix} \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix} \cap \mathfrak{g} \text{ if } r = 3. \end{cases} \tag{A.3}$$

Although, there is a lattice sequence, say Λ_4, with period 6, we do not need to write it down explicitly. This corresponds to type **(A)** strata and in this case all representations are generic. The filtration $\{U_{\mathrm{der}} \cap a_n(\Lambda_1) \mid n \in \mathbb{Z}\}$ is given by:

$$U_{\mathrm{der}} \cap a_{2m-1}(\Lambda_1) = U_{\mathrm{der}} \cap a_{2m}(\Lambda_1) = U_{\mathrm{der}}(m),$$

for $m \in \mathbb{Z}$. The filtration $\{U_{\mathrm{der}} \cap a_n(\Lambda_2) \mid n \in \mathbb{Z}\}$ is given by

$$U_{\mathrm{der}} \cap a_{2m}(\Lambda_2) = U_{\mathrm{der}}(m-1) \text{ and } U_{\mathrm{der}} \cap a_{2m+1}(\Lambda_2) = U_{\mathrm{der}}(m).$$

The filtration Λ_3 is given by

$$U_{\mathrm{der}} \cap a_{4m+r}(\Lambda_2) = \begin{cases} U_{\mathrm{der}}(m) \text{ if } r = 0, \\ U_{\mathrm{der}}(m) \text{ if } r = 1, \\ U_{\mathrm{der}}(m) \text{ if } r = 2, \\ U_{\mathrm{der}}(m+1) \text{ if } r = 3. \end{cases} \tag{A.4}$$

A.2. **The ramified case:** Now, assume that F/F_0 is a ramified extension and Λ_1 and Λ_2 be the lattice sequence of period 2 given by

$$\Lambda_1(-1) = \Lambda(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{o}_F e_{-1}.$$

and

$$\Lambda_2(0) = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_0 \oplus \mathfrak{p}_F e_{-1} \text{ and } \Lambda_2(1) = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_0 \oplus \mathfrak{p}_F e_{-1}$$

The filtration $\{a_n(\Lambda_1) \mid n \in \mathbb{Z}\}$ is similar to the filtration in (A.1). The flitration $\{a_n(\Lambda_2) \mid n \in \mathbb{Z}\}$, in this case, is similar to the filtration in (A.2). We will not require to write the filtrations $\{a_n(\Lambda') \mid n \in \mathbb{Z}\}$ for which $P^0(\Lambda')$ is an Iwahori subgroup of G. The filtration $\{U_{\mathrm{der}} \cap a_n(\Lambda_1) \mid n \in \mathbb{Z}\}$ is given by

$$U_{\mathrm{der}} \cap a_{2m-1}(\Lambda_1) = U_{\mathrm{der}} \cap a_{2m}(\Lambda_1) = U_{\mathrm{der}}([m/2]),$$

for all $m \in \mathbb{Z}$. The filtration $\{U_{\mathrm{der}} \cap a_n(\Lambda_2) \mid n \in \mathbb{Z}\}$ is given by

$$U_{\mathrm{der}} \cap a_{2m-1}(\Lambda_2) = U_{\mathrm{der}} \cap a_{2m}(\Lambda_2) = U_{\mathrm{der}}([(m-1)/2]),$$

for any $m \in \mathbb{Z}$.

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