JACQUET MODULES OF TATE COHOMOLOGY AND BASE CHANGE LIFTING

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Abstract. Let G be a connected reductive group defined over a p-adic field F. Let σ be an automorphism of G of prime order ℓ ($\ell \neq p$), and defined over F. Set $\Gamma = \langle\sigma\rangle$, and let G^σ be the subgroup of σ-fixed points of G. Let Π be a finite length $\overline{\mathbb{F}}_\ell$-representation (or an ℓ-modular representation) of $G(F) \rtimes \Gamma$. We prove a conjecture of Treumann and Venkatesh which predicts that the Tate cohomology $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is a finite length representation of $G^\sigma(F)$, for $i \in \{0, 1\}$. Let E/F be a cyclic Galois extension with $[E : F] = \ell$. We discuss the explicit computation of these Tate cohomology spaces when $G(F)$ is $\mathrm{GL}_n(E)$, the automorphism σ is induced by a generator of $\mathrm{Gal}(E/F)$, and Π is obtained as a base change lifting of a depth-zero supercuspidal representation of $\mathrm{GL}_n(F)$. The primary novelty from our previous work is that we treat the case where Π is possibly non-cuspidal. We also study the $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$-Tate cohomology groups of the mod-ℓ reduction of the unipotent cuspidal representation of $\mathrm{Sp}_4(\mathbb{F}_{q^\ell})$.

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1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$, and let G be a connected reductive algebraic group defined over F. Let ℓ be a prime number distinct from p. Let σ be an order ℓ automorphism of G, defined over F, and let Γ be the cyclic group generated by σ. We denote by H the connected component containing the identity element of G^σ–the subgroup of σ-fixed points of G. Let Π be an ℓ-modular smooth irreducible representation of $G(F)$ such that Π extends as a representation of $G(F) \rtimes \Gamma$. Treumann and Venkatesh, in their work on mod-ℓ functoriality ([TV16, Conjectures 6.3]), proposed some remarkable conjectures relating the ℓ-modular irreducible smooth representations of the p-adic groups $G(F)$ and $H(F)$. Their conjectures predict that the Tate cohomology $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is a finite length representation of $H(F)$, for $i \in \{0, 1\}$, and they also proposed that the subquotients of $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ must be related to Π via ℓ-modular functoriality in the sense of Langlands, and they called it the *linkage principle*. In our previous work [DN24], we proved that $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is a finite length representation for the case where $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$ and σ is the automorphism of G induced by a generator of the cyclic group $\mathrm{Gal}(E/F)$ of order ℓ, and we showed that the linkage principle holds when ℓ does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$.

In this article, we first prove the finiteness part of Treumann–Venkatesh conjectures. The main technique is to control the Jacquet modules and the twisted Jacquet modules of the Tate cohomology groups. We also use the fact that the subrepresentations of finitely generated ℓ-modular representations of a p-adic group are finitely generated; this is a deep result proved recently in [DHKM24]. Let (Π, W) be the base change lifting to $\mathrm{GL}_n(E)$ of an integral cuspidal absolutely irreducible representation (π, V) of $\mathrm{GL}_n(F)$ where V is a vector space over the maximal unramified extension $\mathcal{K}$ of $\mathbb{Q}_\ell$. Under the assumption that the mod-ℓ reduction of π is supercuspidal, we show that $\widehat{\mathrm{H}}^0(\mathrm{Gal}(E/F), \mathcal{L})$ is irreducible and a cuspidal representation, for any $\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)$-stable $\mathbb{Z}_\ell^{\mathrm{un}}$-lattice $\mathcal{L}$ in W. In the depth-zero case, using local methods, we obtain the precise description of Tate cohomology of a base change lift of such π. These results on depth-zero representations

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are obtained by studying the Tate cohomology of invariant lattices in Shintani lifts for GL_n over finite fields. We then discuss the linkage principal for generic depth-zero ℓ-modular representations of $\mathrm{GL}_n(E)$.

Ronchetti gave initial evidence to the linkage principle in the cyclic base change case of GL_n [Ron16], where he explicitly computed the Tate cohomology of a $\mathrm{Gal}(E/F)$-stable lattice in a depth-zero cuspidal representation of $\mathrm{GL}_n(E)$ obtained as a base change lift of $\mathrm{GL}_n(F)$. In two papers [Fen23] and [Fen24], Feng showed that, under mild restrictive hypothesis on the groups involved, the Genestier–Lafforgue correspondence and the Fargues–Scholze correspondence are compatible with the linkage principle. However, the explicit computation of Tate cohomology groups, in particular, admissibility and the non-vanishing of Tate cohomology $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$, genericity, are not well understood. We answer some of these questions in this article. In their paper [HV25, Section 5.0.3], Henniart and Vignéras verified Treumann and Venkatesh conjectures for $G(F) = \mathrm{GL}_2(F)$ and $G^\sigma(F) = \mathrm{SL}_2(F)$ with $\ell = 2$.

The following are our principal results. The first result of this article is to show that the Tate cohomology groups are finitely generated representations of $G^\sigma(F)$. We prove the following result.

Theorem 1. *Let G be a connected reductive algebraic group over a non-Archimedean local field F of residue characteristic p. Let $\Gamma \subset \mathrm{Aut}(G)$ be a subgroup of automorphisms of order ℓ, defined over F. Let (Π, V) be a finite length ℓ-modular representation of $G(F) \rtimes \Gamma$. Then the Tate cohomology $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is a finitely generated representation of $G^\sigma(F)$, for $i \in \{0, 1\}$.*

The above theorem is discussed in Section 2. The proof of Theorem 1 uses the ℓ-modular version of Schneider and Stuhler resolutions ([SS97, Theorem II.3.1]) of smooth irreducible representations as discussed in Vignéras's work [Vig97, Proposition 2.6] on ℓ-modular representations. In what follows, we use the hypothesis and notations of the above theorem.

To prove the admissibility of Tate cohomology groups, we prove that the Jacquet modules of $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ are controlled by the Tate cohomology of Jacquet modules of Π with respect to σ-stable parabolic subgroups. This is discussed in Section 3. Let P be a σ-stable parabolic subgroup of G. Then the connected component of P^σ, denoted by Q, is a parabolic subgroup of $H = (G^\sigma)^0$. Let N be the unipotent radical of P and let U be the unipotent radical of Q. Let ψ be a character of $N(F)$. We will show that the natural map of twisted Jacquet modules

$$\widehat{\mathrm{H}}^i(\Gamma, \Pi)_{U(F),\psi} \to \widehat{\mathrm{H}}^i(\Gamma, \Pi_{N(F),\psi}), \tag{1.1}$$

is an injection. This article, in essence, is about various applications of the above injective map. First of which is the following theorem.

Theorem 2. *Let G be a connected reductive algebraic group defined over a non-Archimedean local field F of residue characteristic p. Let $\Gamma \subset \mathrm{Aut}(G)$ be a subgroup of automorphisms of order ℓ, defined over F. Let (Π, V) be a finite length ℓ-modular representation of $G(F) \rtimes \Gamma$. Then the Tate cohomology $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is a finite length representation of $G^\sigma(F)$, for $i \in \{0, 1\}$.*

Although, F is assumed to be p-adic, the above two theorems are proved for any non-Archimedean local field F with residue characteristic p. We prove the above theorem using induction on the dimension of G. In the induction argument, we need some control on the action of the centre of G^σ along with the exponents in the Jacquet modules with respect to the parabolic subgroups of G^σ. We begin the proof of the theorem by reducing it to the case where the maximal F-split torus in the centre of $Z(G^\sigma)$ is contained in $Z(G)$; we call this the first reduction step. This helps us to control the action of $Z(G^\sigma(F))$ using the action of $Z(G(F))$ via the central character on the Jordan–Holder factors of the $G(F)$ representation Π.

An interesting corollary from this part of the proof is the following result: Let Π be an irreducible ℓ-modular representation of $\mathcal{G}(F)$ where $\mathcal{G}$ is a connected reductive group defined over F, and let

$s \in \mathcal{G}(F)$ be a regular element of order ℓ (the group $C_{\mathcal{G}}(s)^0$ is a torus in $\mathcal{G}$). Then the space $\widehat{\mathrm{H}}^i(\langle s\rangle, \Pi)$ is a finite dimensional ℓ-modular representation of $C_{\mathcal{G}}(s)(F)$. After this first reduction step, we can use (1.1) to prove inductively the finite length of the Tate cohomology spaces $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$. The basic step in this induction argument is the case where the connected component $(G^\sigma)^0$ has no proper F-rational parabolic subgroups. In this case, the first reduction step and Theorem 1 together imply that $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is a finite dimensional ℓ-modular representation of $G^\sigma(F)$.

For the next result, proved in Section 4, we assume that G is $\mathrm{Res}_{E/F}\mathrm{GL}_n$, where E/F is a finite Galois extension of degree ℓ and σ is an automorphism of G induced by a generator of the Galois group $\mathrm{Gal}(E/F)$. Fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Let $\mathcal{K}$ be the maximal unramified extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, and let Λ be the ring of integers of $\mathcal{K}$. Let π be an absolutely irreducible, integral, cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$ such that the mod-ℓ reduction of π is a supercuspidal. The base change lifting to $\mathrm{GL}_n(E)$ of the ℓ-adic representation $\pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ is defined over $\mathcal{K}$, and let Π be the $\mathcal{K}$-representation of $\mathrm{GL}_n(E)$ such that $\Pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ is the base change lift of $\pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$. Note that $\Pi^\sigma \simeq \Pi$ and it is integral, but Π is not necessarily a cuspidal representation. The representation Π extends uniquely to a representation of $\mathcal{K}[\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)]$. We have

Theorem 3. *With the Hypothesis on Π and π as above, let $\mathcal{L}$ be a $\Lambda[\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)]$-stable lattice in Π. Then, the Tate cohomology $\widehat{\mathrm{H}}^0(\mathrm{Gal}(E/F), \mathcal{L})$ is an ℓ-modular irreducible cuspidal representation of $\mathrm{GL}_n(F)$ and $\widehat{\mathrm{H}}^1(\mathrm{Gal}(E/F), \mathcal{L})$ is trivial.*

In the proof of the above result, we use the fact ([DN24, Proposition 6.3]) that the zeroth $\mathrm{Gal}(E/F)$-Tate cohomology of a $\Lambda[\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)]$-stable lattice in Π has a unique generic subquotient. In fact, this genericity result holds in greater generality. In our previous work on GL_n, we used the Kirillov model to prove the genericity of Tate cohomology for the action of $\mathrm{Gal}(E/F)$, and generalising this result we prove in Theorem 4.1.1 about the genericity of Tate cohomology of ℓ-modular representations in Section 4. The main technique for Theorem 4.1.1 is Rodier's results on Whittaker models and their integral versions due to Vignéras [Vig04]. Note that the proof of [DN24, Proposition 6.3] works for all non-Archimedean local fields F of residue characteristic p. However, we point out that the genericity result in this article is proved only for p-adic groups.

To explain this genericity result, let G be a quasi-split connected reductive group defined over F. Let Π be a smooth irreducible ℓ-modular representation of $G(F)$ such that $\Pi \simeq \Pi^\sigma$. It can be shown that Π extends uniquely to a representation of $G(F) \rtimes \Gamma$. We assume that there exists a σ-stable Borel subgroup B of G defined over F, and let N be the unipotent radical of B. We set $U = N^\sigma$ to be the unipotent radical of the connected component of the idendity of B^σ. We also assume that there exists a non-degenerate character $\psi : N(F) \to \overline{\mathbb{F}}_\ell^\times$ such that $\psi^\sigma = \psi$ and ψ restricts to a non-degenerate character on $U(F)$. With these two assumptions on the group G, we prove that if Π is $(N(F), \psi)$-generic, then the $H(F)$-representation $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is non-zero and it has a unique irreducible $(U(F), \psi)$-generic subquotient (Theorem 4.1.1); we also prove an integral version of this result. We then prove Theorem 3, using the injection (1.1) and [DN24, Proposition 6.3]. We show that the Jacquet-modules of $\mathcal{L}$, as in the above theorem, with respect to any σ-stable proper parabolic subgroup is a free Γ-module.

In Section 5, we study the Tate cohomology of invariant lattices in representations obtained as Shintani liftings to $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ of representations of the finite group $\mathrm{GL}_n(\mathbb{F}_q)$. The analogue of Theorem 3 (Proposition 5.3.4) is first proved for Π–which is obtained as a Shintani lifting to $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ of a cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ with supercuspidal mod-ℓ reduction; we use the Deligne–Lusztig character formula to show that, for any $\Lambda[\mathrm{GL}_n(\mathbb{F}_{q^\ell}) \rtimes \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)]$-stable lattice $\mathcal{L}$ in Π, the space $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is isomorphic to the Frobenius twist of the mod-ℓ reduction of π (Theorem 5.4.3). In section 6, we use the results in section 5 to show the linkage principle for depth-zero generic representations for p-adic groups for all $\ell \neq 2$. Essentially, we show that the generic constituent of Tate cohomology can be described if the linkage principle is established for absolutely cuspidal integral $\mathcal{K}$-representations with supercuspidal ℓ-modular reduction, and in the depth-zero case we establish this using our results on Shintani lifting. The main theorem for section 6 is Theorem 6.4.3.

We end this article by considering the example of θ_{10} in Section 7, the unipotent cuspidal irreducible representation of the finite symplectic group $\mathrm{Sp}_4(\mathbb{F}_q)$. This example demonstrates linkage principle in the case of non-generic representaions. Let π_n be the unique cuspidal unipotent representation of $\mathrm{Sp}_4(\mathbb{F}_{q^n})$. Note that the mod-$\ell$ reduction of π_n is irreducible and it is denoted by $r_\ell(\pi_n)$. For a vector space W over the field $\overline{\mathbb{F}}_\ell$, we denote by $W^{(\ell)}$ the $\overline{\mathbb{F}}_\ell$-vector space where the scalar action is twisted by the Frobenius automorphism of the field $\overline{\mathbb{F}}_\ell$. We have

Theorem 4. *Let ℓ be an odd prime number. Let π_n be the unipotent cuspidal representation of $\mathrm{Sp}_4(\mathbb{F}_{q^n})$. Then, we have the following isomorphism of $\mathrm{Sp}_4(\mathbb{F}_q)$ representations:*

$$\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), r_\ell(\pi_\ell)) \simeq r_\ell(\pi_1)^{(\ell)}.$$

Moreover, the Tate cohomology $\widehat{\mathrm{H}}^1(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), r_\ell(\pi_\ell))$ is trivial.

The above theorem seems to suggest that for all finite classical groups G admitting a cuspidal unipotent representation, the zeroth Tate cohomology space for the action of the Galois group $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$ on the mod-ℓ reduction of a cuspidal unipotent representation of $G(\mathbb{F}_{q^\ell})$ is isomorphic to the Frobenius twist of the mod-ℓ reduction of the cuspidal unipotent representation of $G(\mathbb{F}_q)$.

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Notations.

(1) In this article, we assume that F is p-adic and here are some standard notations: the ring of integers of F is $\mathfrak{o}_F$, the maximal ideal of $\mathfrak{o}_F$ is $\mathfrak{p}_F$, and the residue field $\mathfrak{o}_F/\mathfrak{p}_F$ is k_F. The field k_F has q_F elements and q_F is a power of a prime p. Let $\mathrm{val}_F : F^\times \to \mathbb{Z}$, be the normalised valuation of F.

(2) For coefficients of representations we use the following: We fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and let $\overline{\mathbb{Z}}_\ell$ be the ring of integers of $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{K}$ be the maximal unramified extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, and let Λ be the ring of integers of $\mathcal{K}$. We fix $q_F^{1/2} \in \mathcal{K}$ a square root of q_F. For $x \in \Lambda$, the symbol $\overline{x}$ stands for the mod-ℓ reduction of x.

(3) Let G be a connected reductive algebraic group defined over F. Let $\sigma \in \mathrm{Aut}(G)$, defined over F, be an element of prime order ℓ and $\ell \neq p$. Let Γ be the group generated by σ. Let H be the connected component of the identity element of G^σ–the fixed points of σ, note that H is a reductive group (see [PY02, Theorem 2.1]). We have the tuple of notations

$$(G, \Gamma, \sigma, G^\sigma, H) \tag{1.2}$$

associated with G and σ. The connected component of the identity of an algebraic group $\mathcal{G}$ is denoted by $\mathcal{G}^0$.

(4) A representation (π, V) of a locally pro-finite group $\mathcal{G}$ is said to be smooth if every vector $v \in V$ is fixed by a compact open subgroup of $\mathcal{G}$. In this article, our representation are defined over K-vector spaces where K is one of the fields $\{\overline{\mathbb{Q}}_\ell, \mathcal{K}, \overline{\mathbb{F}}_\ell\}$. When $K = \overline{\mathbb{Q}}_\ell$ (resp. $\overline{\mathbb{F}}_\ell$), we say that the representation is ℓ-adic (resp. ℓ-modular). Let (ρ, U) be a representation of any group $\mathcal{G}$ over an L-vector space U. For $\gamma \in \mathrm{Aut}(L)$, we denote by $\rho^{(\gamma)}$

the representation of $\mathcal{G}$ on the L-vector space space U, where the action of L is twisted by γ, i.e.,

$$x \cdot v = \gamma^{-1}(x)v,\ x \in L,\ v \in U.$$

For an ℓ-modular representation (ρ, V) of $\mathcal{G}$, the Frobenius twist of ρ, denoted by $\rho^{(\ell)}$, is defined to be $\rho^{(\mathrm{Fr})}$, where Fr is the automorphism $x \mapsto x^{\ell}$.

(5) For an integral ℓ-adic representation (π, V) of a pro-finite group $\mathcal{G}$, we denote by $r_\ell(\pi)$ the ℓ-modular reduction (we again discuss the definitions in subsection 3.2).
(6) For a closed subgroup J' of a locally profinite group J, we will denote by $\mathrm{Ind}_{J'}^{J}$ and $\mathrm{ind}_{J'}^{J}$ the smooth induction functor and the compact induction functor, respectively.
(7) We use $K_0(F)$ for the compact open subgroup $\mathrm{GL}_n(\mathfrak{o}_F)$ of $\mathrm{GL}_n(F)$, and the notation $K_1(F)$ is used for the first principal congruence subgroup of $K_0(F)$.
(8) Let P be an F-rational parabolic subgroup of G and the unipotent radical of P is denoted by N. Let M be a Levi subgroup of P defined over F. For a smooth representation (τ, W) of $M(F)$, we denote by $i_{P(F)}^{G(F)}(\tau)$ the normalised parabolic induction of τ to $G(F)$. We use the notation $\pi_{N(F)}$ for the Jacquet module of a smooth representation (π, V) of $G(F)$. Given smooth representations τ_i of $\mathrm{GL}_{n_i}(F)$, for $1 \le i \le k$, we denote by

$$\tau_1 \times \tau_2 \times \cdots \times \tau_k$$

the normalised induced representation $i_{P(F)}^{\mathrm{GL}_m(F)}(\otimes_{i=1}^{k}\tau_i)$, where $m = \sum_{i=1}^{k} n_i$ and P is an F-rational parabolic subgroup with a Levi subgroup isomorphic to $\prod_{i=1}^{k}\mathrm{GL}_{n_i}$. We use similar notation when F is a finite field for the induced representation. Let (ρ, V) be a smooth representation of $M(F)$, where V is an L-vector space, and we assume that $q_F^{1/2}$ is fixed in L and the modulus character of $P(F)$ is defined using this fixed element $q_F^{1/2}$. We then note that

$$\big(i_{P(F)}^{G(F)}\rho\big)^{(\gamma)} \simeq i_{P(F)}^{G(F)}\rho^{(\gamma)},$$

for all $\gamma \in \mathrm{Aut}(L)$ with $\gamma(q_F^{1/2}) = q_F^{1/2}$. We note that

$$\big(\mathrm{Ind}_{P(F)}^{G(F)}\rho\big)^{(\gamma)} \simeq \mathrm{Ind}_{P(F)}^{G(F)}\rho^{(\gamma)}.$$

In the case where $L = \overline{\mathbb{F}}_\ell$, and $\gamma = \mathrm{Fr}$, we have

$$\big(i_{P(F)}^{G(F)}\rho\big)^{(\ell)} \simeq i_{P(F)}^{G(F)}(\rho^{(\ell)} \otimes \kappa_P) \tag{1.3}$$

where $\kappa_P : P(F) \to \{\pm 1\}$ is the character given by

$$g \mapsto \Big(\frac{q_F}{\ell}\Big)^{\mathrm{val}_F(\det(\mathrm{Ad}_g|\mathfrak{n}))}, g \in P(F),$$

where $\mathfrak{n}$ is the Lie algebra of $N(F)$ and $\big(\frac{q_F}{\ell}\big)$ is the Legendre symbol.
(9) In this article, a smooth representation (π, V) of $G(F)$ is called cuspidal if $\pi_{N(F)} = 0$, where N is the unipotent radical of any proper F-rational parabolic subgroup P of G.

2. Tate cohomology of smooth representations

In this section, the field F is a non-Archimedean local field with residue field of characteristic p. Let Π be a smooth finite length representation of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$. We prove that the Tate cohomology groups $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ are finitely generated representations of $G^\sigma(F)$. We use Schneider–Stuhler resolutions and their mod-ℓ version by Vignéras for proving this finiteness statement.

2.1. Let $\mathcal{K}$ be the maximal unramified extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, and let Λ be the ring of integers of $\mathcal{K}$. All $\Lambda[G(F)]$-modules are assumed to be smooth, i.e., every element is fixed by a compact open subgroup of $G(F)$. Let M be a $\Lambda[G(F) \rtimes \Gamma]$-module. Let Nr be the norm element $\sum_{k=0}^{\ell-1} \sigma^k \in \Lambda[\Gamma]$. Consider the chain complex of $\Lambda[\Gamma]$-modules

$$\cdots \longrightarrow M \xrightarrow{\mathrm{id}-\sigma} M \xrightarrow{\mathrm{Nr}} M \xrightarrow{\mathrm{id}-\sigma} M \xrightarrow{\mathrm{Nr}} M \longrightarrow \cdots$$

The Tate cohomology groups of M, with respect to the action of Γ, are 2-periodic and we have

$$\widehat{\mathrm{H}}^0(\Gamma, M) := \frac{\mathrm{Ker}(\mathrm{id}-\sigma)}{\mathrm{Im}(\mathrm{Nr})}, \ \ \widehat{\mathrm{H}}^1(\Gamma, M) := \frac{\mathrm{Ker}(\mathrm{Nr})}{\mathrm{Im}(\mathrm{id}-\sigma)}.$$

The action of $G(F)$ on M induces an action of $G^\sigma(F)$ on the Tate cohomology spaces $\widehat{\mathrm{H}}^i(\Gamma, M)$, and we get two ℓ-modular smooth representations $\widehat{\mathrm{H}}^i(\Gamma, M)$, for $i \in \{0,1\}$, of the fixed point subgroup $G^\sigma(F)$. Similar cohomology groups can be defined when F is a finite field, and we will assume the easy modifications in the definitions. For a short exact sequence of $\Lambda[G(F) \rtimes \Gamma]$-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

we have the following hexagonal exact sequence of Tate cohomology groups ([Ser, Chapter VIII, Section 4]):

$$\begin{tikzcd}
\widehat{\mathrm{H}}^0(\Gamma, M') \arrow[r] & \widehat{\mathrm{H}}^0(\Gamma, M) \arrow[r] & \widehat{\mathrm{H}}^0(\Gamma, M'') \arrow[d] \\
\widehat{\mathrm{H}}^1(\Gamma, M'') \arrow[u] & \widehat{\mathrm{H}}^1(\Gamma, M) \arrow[l] & \widehat{\mathrm{H}}^1(\Gamma, M') \arrow[l]
\end{tikzcd} \tag{2.1}$$

2.2. Let X be a Hausdorff, locally compact, and totally disconnected topological space (an ℓ-space in the sense of [BZ76, Section 1.1]). Let G be a locally profinite group with an order ℓ automorphism σ, generating a group Γ of automorphisms of G. Assume that $G \rtimes \Gamma$ acts continuously on X. Let $\mathcal{F}$ be a $G \rtimes \Gamma$-equivariant sheaf on X. The natural restriction map

$$\widehat{\mathrm{H}}^i(\Gamma, H^0(X, \mathcal{F})) \to H^0(X^\sigma, \widehat{\mathrm{H}}^i(\Gamma, \mathcal{F}|_{X^\sigma}))$$

is an isomorphism, for $i \in \{0,1\}$ (see [TV16, Proposition 3.3]). Here, $H^0(X, \mathcal{F})$ is the space of global sections of a sheaf $\mathcal{F}$ on X. Let K be a closed subgroup of G such that $\sigma(K) = K$. Let V be a $\Lambda[K \rtimes \Gamma]$-module. This data defines a $G \rtimes \Gamma$-equivariant sheaf $\mathcal{F}_V$ on G/K whose sections on an open set U of G/K are the sections of the vector bundle $G \times_K V \to G/K$ restricted to U. The global sections of the sheaf $\mathcal{F}_V$ is the induced representation $\mathrm{ind}_K^G V$, and the action of Γ is given by

$$(\sigma \cdot f)(g) = \sigma\big(f(\sigma^{-1}(g))\big),$$

for all $g \in G$ and $f \in \mathrm{ind}_K^G V$. Assume that $(G/K)^\sigma = G^\sigma / K^\sigma$. Then, the restriction to G^σ map induces a G^σ-equivariant isomorphism

$$\widehat{\mathrm{H}}^i(\Gamma, \mathrm{ind}_K^G V) \to \mathrm{ind}_{K^\sigma}^{G^\sigma}(\widehat{\mathrm{H}}^i(\Gamma, V)), \ \ f \mapsto \mathrm{res}_{G^\sigma}(f) \tag{2.2}$$

2.3. Let (Π, V) be a smooth finite length representation of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$. We show that $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is a finitely generated $G^\sigma(F)$-representation. We use the mod-ℓ version of Schneider and Stuhler resolutions ([SS97, Theorem II.3.1]) of smooth irreducible representations as discussed in Vignéras ([Vig97, Proposition 2.6]). Let $\mathfrak{X}$ be the extended Bruhat–Tits building of a connected reductive algebraic group G defined over a non-Archimedean local field F. Let d be the F-rank of G. For $0 \leq k \leq d$, let $\mathfrak{X}_k$ be the set of k-dimensional facets of $\mathfrak{X}$, and let $\mathfrak{X}_{(k)}$ be the set of oriented k-facets which are tuples of the form $(\mathcal{F}, c_\mathcal{F})$, where $\mathcal{F} \in \mathfrak{X}_k$ and $c_\mathcal{F}$ is an orientation of $\mathcal{F}$. Let $e \geq 0$. For a facet $\mathcal{F}$ of $\mathfrak{X}$, Schneider and Stuhler associate a compact open subgroup $U_\mathcal{F}^{(e)}$, and let $\underline{V}$ be the coefficient system defined by associating every facet $\mathcal{F}$ to $V^{U_\mathcal{F}^e}$. Let $C(\mathfrak{X}_{(k)}, \underline{V})$ be the

$\overline{\mathbb{F}}_\ell$-vector space of oriented cellular k-chains on $\mathfrak{X}_{(k)}$. Then we have a G-equivariant resolution of the representation

$$0 \to C(\mathfrak{X}_{(d)}, \underline{V}) \xrightarrow{\partial_{d-1}} C(\mathfrak{X}_{(d-1)}, \underline{V}) \xrightarrow{\partial_{d-2}} \cdots \to C(\mathfrak{X}_{(1)}, \underline{V}) \xrightarrow{\partial_0} C(\mathfrak{X}_{(0)}, \underline{V}) \xrightarrow{\epsilon} V \to 0 \tag{2.3}$$

for some $e \geq 0$. We refer to [SS97, Section II.2] for details. Note that the automorphism σ acts on $\mathfrak{X}$ simplicially, i.e., it preserves the sets $\mathfrak{X}_d$ and $\mathfrak{X}_{(d)}$.

Lemma 2.3.1. *For each $0 \leq k \leq d$, the Tate cohomology groups $\widehat{\mathrm{H}}^i(\Gamma, C(\mathfrak{X}_{(k)}, \underline{V}))$ are finitely generated $\overline{\mathbb{F}}_\ell[G^\sigma(F)]$-modules.*

Proof. Consider the set $\mathfrak{X}_{(k)}$ with discrete topology, and the action of the group $G(F)$ on $\mathfrak{X}_{(k)}$ is continuous. Also note that the sheaf $\underline{V}$ is G and Γ-equivariant. Since V is an admissible representation of $\overline{\mathbb{F}}_\ell[G(F)]$, the stalks of the sheaf $\underline{V}$ are all finite dimensional vector spaces over $\overline{\mathbb{F}}_\ell$. Using [TV16, Proposition 3.3], we get that the restriction map

$$\widehat{\mathrm{H}}^i(\Gamma, C(\mathfrak{X}_{(k)}, \underline{V})) \to C(\mathfrak{X}^\sigma_{(k)}, \widehat{\mathrm{H}}^i(\Gamma, \underline{V}))$$

is an isomorphism. Recall H denotes the group $(G^\sigma)^0$. Let $\mathcal{Y}$ be the Bruhat–Tits building of H over F, then $\mathfrak{X}^\sigma_k$ is a subset of facets of $\mathcal{Y}$ (see [PY02]). Thus, there are only finitely many orbits for the action of $H(F)$ on $\mathfrak{X}^\sigma_{(k)}$. Since the module $\widehat{\mathrm{H}}^i(\Gamma, V^{U^e_{\mathcal{F}}})$ is finite, we get that the space $C(\mathfrak{X}^\sigma_{(k)}, \widehat{\mathrm{H}}^i(\Gamma, \underline{V}))$ is a finitely generated ℓ-modular representation of $H(F)$, and hence, these spaces are finitely generated as ℓ-modular representations of $G^\sigma(F)$. $\square$

In what follows, we need the fact that the submodules of finitely generated smooth $\overline{\mathbb{F}}_\ell[\mathcal{G}(F)]$ representations for a connected reductive group $\mathcal{G}$ over F, are again finitely generated. This is a fundamental result in the theory of smooth complex representations due to Bernstein and in the mod-ℓ case it follows from the deep results of the paper [DHKM24, Corollary 1.3] seen with the work of Dat ([Dat09, Corollary 4.5]).

Lemma 2.3.2. *Let (Π, V) be an irreducible smooth representation of $\overline{\mathbb{F}}_\ell[G(F)]$. Assume that (Π, V) extends as a representation of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$. Then the Tate cohomology groups $\widehat{\mathrm{H}}^i(\Gamma, V)$ are finitely generated representations of $\overline{\mathbb{F}}_\ell[G^\sigma(F)]$.*

Proof. Note that the resolution (2.3) is G and Γ-equivariant, and the Tate cohomology groups

$$\widehat{\mathrm{H}}^i(\Gamma, C(\mathfrak{X}_{(i)}, \underline{V})), \quad 0 \leq i \leq d,$$

are finitely generated representations of $\overline{\mathbb{F}}_\ell[G^\sigma(F)]$ by Lemma 2.3.1. For each $2 \leq k \leq d$, consider the short exact sequence of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$-modules

$$0 \to \mathrm{Im}(\partial_{k-1}) \to C(\mathfrak{X}_{(k-1)}, \underline{V}) \to \mathrm{Im}(\partial_{k-2}) \to 0.$$

Then we have the following hexagonal exact sequence using (2.1):

$$\begin{tikzcd}
\widehat{\mathrm{H}}^0(\Gamma, \mathrm{Im}(\partial_{k-1})) \arrow[r] & \widehat{\mathrm{H}}^0(\Gamma, C(\mathfrak{X}_{(k-1)}, \underline{V})) \arrow[r] & \widehat{\mathrm{H}}^0(\Gamma, \mathrm{Im}(\partial_{k-2})) \arrow[d] \\
\widehat{\mathrm{H}}^1(\Gamma, \mathrm{Im}(\partial_{k-2})) \arrow[u] & \widehat{\mathrm{H}}^1(\Gamma, C(\mathfrak{X}_{(k-1)}, \underline{V})) \arrow[l] & \widehat{\mathrm{H}}^1(\Gamma, \mathrm{Im}(\partial_{k-1})) \arrow[l]
\end{tikzcd}$$

which we denote by $S(k)$. Note that $\mathrm{Im}(\partial_{d-1})$ is identified with $C(\mathfrak{X}_{(d)}, \underline{V})$. Since the spaces $\widehat{\mathrm{H}}^i(\Gamma, C(\mathfrak{X}_{(d)}, \underline{V}))$ and $\widehat{\mathrm{H}}^i(\Gamma, C(\mathfrak{X}_{(d-1)}, \underline{V}))$ are finitely generated ℓ-modular representations of $G^\sigma(F)$, it follows from the diagram $S(d)$ that $\widehat{\mathrm{H}}^i(\Gamma, \mathrm{Im}(\partial_{d-2}))$ is also finitely generated. A similar hexagonal exact sequence of Tate cohomology corresponding to the following short exact sequence of $G(F) \rtimes \Gamma$-modules

$$0 \to \mathrm{Im}(\partial_{d-2}) \to C(\mathfrak{X}_{(d-2)}, \underline{V}) \to \mathrm{Im}(\partial_{d-3}) \to 0$$

shows that $\widehat{\mathrm{H}}^i(\Gamma, \mathrm{Im}(\partial_{d-3}))$ is finitely generated. Since $\widehat{\mathrm{H}}^i(\Gamma, C(\mathfrak{X}_{(d-2)}, \underline{V}))$ is finitely generated, it follows from the hexagonal diagram $S(d-1)$ that $\widehat{\mathrm{H}}^i(\Gamma, \mathrm{Im}(\partial_{d-3}))$ is finitely generated. Thus, following a finite number of inductive steps, the hexagonal diagram $S(2)$ shows that the space $\widehat{\mathrm{H}}^i(\Gamma, \mathrm{Im}(\partial_0))$ is finitely generated. Finally, using the hexagonal exact sequence (2.1) of Tate cohomology groups corresponding to the following short exact sequence of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$-modules

$$0 \to \mathrm{Im}(\partial_0) \to C(\mathfrak{X}_{(0)}, \underline{V}) \xrightarrow{\epsilon} V \to 0,$$

we get that $\widehat{\mathrm{H}}^i(\Gamma, V)$ is finitely generated module over $\overline{\mathbb{F}}_\ell[G^\sigma(F)]$. □

Theorem 2.3.3. *Let (Π, V) be a finite length representation of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$. Then each Tate cohomology group $\widehat{\mathrm{H}}^i(\Gamma, V)$ is a finitely generated representation of $\overline{\mathbb{F}}_\ell[G^\sigma(F)]$.*

Proof. We use induction on the length of (Π, V) as an $\overline{\mathbb{F}}_\ell[G(F)]$-representation. If (Π, V) is an irreducible representation of $G(F)$, then the theorem follows from Lemma 2.3.2. Let W be an irreducible $G(F)$-subrepresentation of V. First consider the case when W is Γ-stable. The long exact sequence of Tate cohomology associated with the following short exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

gives two exact sequences

$$\widehat{\mathrm{H}}^0(\Gamma, W) \longrightarrow \widehat{\mathrm{H}}^0(\Gamma, V) \longrightarrow \widehat{\mathrm{H}}^0(\Gamma, V/W)$$

and

$$\widehat{\mathrm{H}}^1(\Gamma, W) \longrightarrow \widehat{\mathrm{H}}^1(\Gamma, V) \longrightarrow \widehat{\mathrm{H}}^1(\Gamma, V/W).$$

Since W is irreducible, each Tate cohomology $\widehat{\mathrm{H}}^i(\Gamma, W)$ is a finitely generated representation of $\overline{\mathbb{F}}_\ell[G^\sigma(F)]$ by Lemma 2.3.2. Also note that the length of the $G(F)$-representation V/W is strictly less than the length of V, inductively we may assume that the Tate cohomology groups $\widehat{\mathrm{H}}^i(\Gamma, V/W)$ are finitely generated representations of $G^\sigma(F)$. Now, it follows from the above exact sequences that each $\widehat{\mathrm{H}}^i(\Gamma, V)$ is a finitely generated representation of $\overline{\mathbb{F}}_\ell[G^\sigma(F)]$. Let us now consider the case where W is not Γ-stable. Then

$$W' = \bigoplus_{i=0}^{\ell-1} W^{\sigma^i}$$

is a $G(F) \rtimes \Gamma$-stable subspace of V, where the action of $G(F)$ on the space W^{σ^i} is twisted by σ^i for $0 \leq i < \ell$. Since W' is a free $\overline{\mathbb{F}}_\ell[\Gamma]$-module, we get that $\widehat{\mathrm{H}}^i(\Gamma, W')$ is trivial, for $i \in \{0, 1\}$. Now, the long exact sequence of Tate cohomology groups applied to the following short exact sequence of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$-modules

$$0 \longrightarrow W' \longrightarrow V \longrightarrow V/W' \longrightarrow 0$$

gives the following isomorphism of $G^\sigma(F)$-representations

$$\widehat{\mathrm{H}}^i(\Gamma, V) \to \widehat{\mathrm{H}}^i(\Gamma, V/W'),\ i \in \{0, 1\}.$$

Since the length of V/W' is strictly less than that of V, the $G^\sigma(F)$-representation $\widehat{\mathrm{H}}^i(\Gamma, V/W')$ is finitely generated by induction. Thus, we get that the Tate cohomology group $\widehat{\mathrm{H}}^i(\Gamma, V)$ is a finitely generated representation of $G^\sigma(F)$. □

3. Finiteness of Tate cohomology

In this section, we prove some results on the compatibility of Jacquet modules and twisted Jacquet modules with Tate cohomology groups, and we use it later to prove the main admissibility result of Tate cohomology spaces. We assume that F is any non-Archimedean local field of residue characteristic p. Some of these results were proved in [DN24, Section 6] for the group $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$, where E/F is a finite Galois extension of prime degree ℓ, and σ is an automorphism induced by a generator of the Galois group $\mathrm{Gal}(E/F)$. The results on (twisted) Jacquet modules, although proved for groups defined over non-Archimedean local fields, are valid for finite groups of Lie type. For convenience, in this section, we drop the notation Γ from the Tate cohomology groups $\widehat{\mathrm{H}}^i(\Gamma, M)$ and denote them by $\widehat{\mathrm{H}}^i(M)$.

3.1. **Twisted Jacquet modules.** Let (π, V) be a smooth $\Lambda[G(F)]$-module, where Λ denotes the ring of integers of the maximal unramified extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$. Let $P = MN$ be an F-rational parabolic subgroup of G, where M is an F-rational Levi subgroup of P and N is the unipotent radical of P. Let $\psi : N(F) \to \Lambda^\times$ be a smooth character. Let $V(N(F), \psi)$ be the Λ-module generated by the following set of vectors

$$\{\pi(n)v - \psi(n)v : v \in V, \, n \in N(F)\}.$$

The quotient space $V/V(N(F), \psi)$ is called a twisted Jacquet module of V, which is denoted by $V_{N(F),\psi}$. This space is characterized by the set of elements $v \in V$ for which there exists some compact open subgroup $\mathcal{N}_v$ of $N(F)$ such that

$$\int_{\mathcal{N}_v} \psi^{-1}(n)\, \pi(n)\, dn = 0,$$

where dn is a Haar measure on $N(F)$. The space $V_{N(F),\psi}$ is a smooth representation of the $M(F)$-stabiliser of the character ψ, to be denoted by $M(F)_\psi$. In particular, when $\psi = 1$, the trivial character of $N(F)$, we write $V_{N(F)}$ for $V_{N(F),1}$. The smooth $\Lambda[M(F)]$-module $V_{N(F)}$ is called the Jacquet module of V.

3.2. **Integral representation and mod-ℓ reduction.** A smooth ℓ-adic representation (π, V) of $G(F)$, i.e., V is a vector space over $\overline{\mathbb{Q}}_\ell$, is called integral if V is a finite length $G(F)$-representation and if there exists a $G(F)$-stable $\overline{\mathbb{Z}}_\ell$-lattice $\mathcal{L}$ in V. Suppose (π, V) is an admissible, ℓ-adic, integral smooth representation defined over $\mathcal{K}$, i.e., there exists a $G(F)$-stable $\mathcal{K}$-vector subspace V_0 in V such that $V_0 \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell = V$, then there exists a $\Lambda[G(F)]$-stable lattice $\mathcal{L}_0$ in V_0 and $\mathcal{L}_0 \otimes_\Lambda \overline{\mathbb{Z}}_\ell$ is a $G(F)$-stable $\overline{\mathbb{Z}}_\ell$-lattice in V. If P is an F-rational parabolic subgroup of G with an F-rational Levi subgroup M and unipotent radical N, then it follows from [Dat05, Proposition 1.4] that $\mathcal{L}_{N(F)}$ is an $M(F)$-stable $\overline{\mathbb{Z}}_\ell$-lattice in $V_{N(F)}$. The natural action of $G(F)$ on the $\overline{\mathbb{F}}_\ell$-vector space $\mathcal{L} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$ gives an ℓ-modular representation of $G(F)$, which depends on the choice of the lattice $\mathcal{L}$. Using [Vig96, Chapter II, 5.11.a and 5.11.b], the representation $(\pi, \mathcal{L} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell)$ has finite length and its semi-simplification is independent of the choice of the $G(F)$ stable $\overline{\mathbb{Z}}_\ell$-lattice $\mathcal{L}$ in V. The semi-simplification of $(\pi, \mathcal{L} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell)$, denoted by $r_\ell(\pi)$, is called the mod-ℓ reduction of π.

3.3. Consider the tuple $(G, \Gamma, \sigma, G^\sigma, H)$ as in (1.2). Let P be a σ-stable F-parabolic subgroup of G, and let N be the unipotent radical of P. Note that N is σ-stable and let M be a σ-stable Levi subgroup P (for the existence of M, see [KP23, Lemma 12.5.4]). Given any F-parabolic subgroup Q of H, there exists a σ-stable F-parabolic subgroup P of G such that $Q = (P^\sigma)^0$ (see [KP23, Lemma 12.5.4] for a reference). Note that $U = N^\sigma$ is the unipotent radical of Q and $(M^\sigma)^0$ is a Levi subgroup of Q. Let $\psi : N(F) \to \Lambda^\times$ be a smooth character such that $\psi^\sigma = \psi$. For a smooth $\Lambda[G(F) \rtimes \Gamma]$-module $\mathcal{L}$, let $\mathcal{L}^\sigma$ denote the space of σ-fixed points of $\mathcal{L}$. Note that $\mathcal{L}(N(F), \psi)$ is stable under the action of Γ.

3.3.1. We need to define a σ-stable filtration of compact open subgroups on the unipotent radical N of a σ-stable parabolic subgroup P of G. For this, we need some details in the proof of [KP23, Lemma 12.5.4]. Let P be a σ-stable parabolic subgroup of G, and let $Q = P \cap H$. Note that Q is a parabolic subgroup of H, and let S be a maximal F-split torus in Q. Let $\lambda : \mathbb{G}_m \to S$ be a one-parameter subgroup defining the parabolic subgroup Q of H, and the group P is defined as the parabolic subgroup defined by λ. Let $\mathfrak{n}_0$ be an $\mathfrak{o}_F$-lattice in the Lie algebra of N such that $[\mathfrak{n}_0, \mathfrak{n}_0] \subseteq \mathfrak{n}_0$. Let N_0 be the subgroup of N generated by $\exp(X)$, for $X \in \mathfrak{n}_0$. We set $N_k = \lambda(\varpi_F^{-k}) N_0 \lambda(\varpi_F^k)$. The subgroups N_k, for $k \in \mathbb{Z}$ provide a σ-stable filtration of compact open subgroups of N.

Lemma 3.3.1. *Let $\mathcal{N}$ be a σ-stable compact open subgroup of N. Then, we have* $\mathrm{H}^1(\Gamma, \mathcal{N}) = 0$.

Proof. The group $\mathrm{H}^1(\Gamma, A)$ is an ℓ-torsion group for any abelian subgroup A with an action of Γ. If A is pro-p abelian group, then $\mathrm{H}^1(\Gamma, A)$ is trivial. We set $\mathcal{D}_0(N) = N$ and $\mathcal{D}_k(N) = [\mathcal{D}_{k-1}(N), \mathcal{D}_{k-1}(N)]$, for $k \geq 1$. Let $\mathcal{N}_k$ be the image of $\mathcal{N} \cap \mathcal{D}_{k-1}(N)$ under the natural map $\mathcal{D}_{k-1}(N) \to \mathcal{D}_{k-1}(N)/\mathcal{D}_k(N)$, for $k \geq 1$. The group $\mathcal{N}_k$ is a compact open subgroup of the abelian unipotent vector group $\mathcal{D}_{k-1}(N)/\mathcal{D}_k(N)$. Thus, $\mathrm{H}^1(\Gamma, \mathcal{N}_k)$ is trivial. This implies that the map

$$\mathrm{H}^1(\Gamma, \mathcal{D}_k(N) \cap \mathcal{N}) \to \mathrm{H}^1(\Gamma, \mathcal{D}_{k-1}(N) \cap \mathcal{N})$$

is surjective, for $k \geq 1$. Hence, we get that $\mathrm{H}^1(\Gamma, \mathcal{N})$ is trivial since $\mathcal{D}_k(N)$ is trivial for sufficiently large k. □

3.4. We begin with our basic results on the Jacquet modules of Tate cohomology.

Lemma 3.4.1. *Let $(\pi, \mathcal{L})$ be a smooth $\Lambda[G(F) \rtimes \Gamma]$-module. For each $i \in \{0, 1\}$, the image of the natural map $\widehat{\mathrm{H}}^i(\mathcal{L}(N(F), \psi)) \longrightarrow \widehat{\mathrm{H}}^i(\mathcal{L})$ is equal to $\widehat{\mathrm{H}}^i(\mathcal{L})(N^\sigma(F), \overline{\psi})$, where $\overline{\psi}$ is the mod-ℓ reduction of ψ.*

Proof. For $i \in \{0, 1\}$, let φ_i be the natural map

$$\widehat{\mathrm{H}}^i(\mathcal{L}(N(F), \psi)) \longrightarrow \widehat{\mathrm{H}}^i(\mathcal{L}).$$

We first consider the case $i = 0$. Let v be in the image of φ_0, and let $\tilde{v}$ be a lift of v in $\mathcal{L}(N(F), \psi)^\sigma$. There exists a compact open subgroup $\mathcal{N} \subseteq N(F)$ such that

$$\int_{\mathcal{N}} \psi^{-1}(n)\, \pi(n)\tilde{v}\, dn = 0. \tag{3.1}$$

Since $(\pi, \mathcal{L})$ is smooth, there exists a compact open subgroup $\mathcal{N}' \subseteq \mathcal{N}$ of finite index such that $\mathcal{N}'$ fixes the vector $\tilde{v}$ and ψ is trivial on $\mathcal{N}'$. Then

$$\int_{\mathcal{N}} \psi^{-1}(n)\, \pi(n)\tilde{v}\, dn = \mathrm{Vol}(\mathcal{N}') \sum_{n \in \mathcal{N}/\mathcal{N}'} \psi^{-1}(n)\, \pi(n)\tilde{v}\, dn,$$

where $\mathrm{Vol}(\mathcal{N}')$ is the volume of $\mathcal{N}'$. Since $N(F)$ admits a filtration of Γ stable compact open subgroups, we assume that both $\mathcal{N}$ and $\mathcal{N}'$ are Γ-stable. If X denotes the coset space $\mathcal{N}/\mathcal{N}'$, then we have

$$\sum_{x \in X} \psi^{-1}(x)\, \pi(x)\tilde{v} = \sum_{y \in X^\sigma} \psi^{-1}(y)\, \pi(y)\tilde{v} \; + \sum_{z \in X \setminus X^\sigma} \psi^{-1}(z)\, \pi(z)\tilde{v}. \tag{3.2}$$

Since the action of Γ on $X \setminus X^\sigma$ is free, there exists a subset $\mathcal{U} \subseteq X \setminus X^\sigma$ such that $X \setminus X^\sigma$ is the disjoint union of $\sigma^i \mathcal{U}$ for $1 \leq i \leq \ell$. As $\tilde{v}$ is a σ-fixed vector, we get

$$\sum_{z \in X \setminus X^\sigma} \psi^{-1}(z)\, \pi(z)\tilde{v} = \sum_{i=1}^{\ell} \sum_{u \in \mathcal{U}} \psi^{-1}(\sigma^i u)\, \pi(\sigma^i u)\tilde{v} = \mathrm{Nr}\big(\sum_{u \in \mathcal{U}} \psi^{-1}(u)\, \pi(u)\tilde{v} \big),$$

where $\mathrm{Nr} = 1 + \sigma + \cdots + \sigma^{\ell-1} \in \Lambda[\Gamma]$. This shows that

$$\sum_{z \in X \setminus X^{\sigma}} \overline{\psi}^{-1}(z)\, \pi(z) v = 0$$

in $\widehat{\mathrm{H}}^0(\mathcal{L})$. Then it follows from (3.1) and (3.2) that

$$\sum_{y \in X^{\sigma}} \overline{\psi}^{-1}(y)\, \pi(y) v = 0.$$

This implies that the vector v belongs to $\widehat{\mathrm{H}}^0(\mathcal{L})(N^{\sigma}(F), \overline{\psi})$. Conversely, let w be an element of $\widehat{\mathrm{H}}^0(\mathcal{L})(N^{\sigma}(F), \overline{\psi})$. Let $\mathcal{N}_{\sigma}$ be the compact open subgroup of $N^{\sigma}(F)$ with

$$\int_{\mathcal{N}_{\sigma}} \overline{\psi}^{-1}(n)\, \pi(n) w \, dn = 0.$$

Since π is smooth, there exists a compact open subgroup $\mathcal{N}_{\sigma}' \subseteq \mathcal{N}_{\sigma}$ of finite index such that

$$\int_{\mathcal{N}_{\sigma}} \overline{\psi}^{-1}(n)\, \pi(n) w \, dn = \mathrm{Vol}(\mathcal{N}_{\sigma}') \sum_{n \in \mathcal{N}_{\sigma}/\mathcal{N}_{\sigma}'} \overline{\psi}^{-1}(n)\, \pi(n) w.$$

Let $\tilde{w}$ be a lift of w in $\mathcal{L}^{\sigma}$. Consider the element

$$\tilde{w}_1 = \tilde{w} - \frac{\mathrm{Vol}(\mathcal{N}_{\sigma}')}{|\mathcal{N}_{\sigma}/\mathcal{N}_{\sigma}'|} \sum_{n \in \mathcal{N}_{\sigma}/\mathcal{N}_{\sigma}'} \psi^{-1}(n)\, \pi(n) \tilde{w}.$$

Then $\tilde{w}_1$ belongs to $\mathcal{L}(N(F), \psi)^{\sigma}$ and $\varphi_0(\tilde{w}_1) = w$. Thus, we get the lemma for $i = 0$.

Now, consider the case $i = 1$. Let v be an element in the image of φ_1. Let $\tilde{v}$ be a lift of v in $\mathrm{Ker}(\mathrm{Nr}_{|\mathcal{L}(N(F),\psi)})$. Then there exists Γ-stable compact open subgroups $\mathcal{N}' \subset \mathcal{N}$ such that $\mathcal{N}'$ fixes the vector $\tilde{v}$, the character ψ is trivial on $\mathcal{N}'$, and

$$\sum_{x \in X} \psi^{-1}(x)\, \pi(x) \tilde{v} = 0,$$

where X denotes the quotient space $\mathcal{N}/\mathcal{N}'$. Since the Γ action on $X \setminus X^{\sigma}$ is free, there exists a subset $\mathcal{V} \subseteq X \setminus X^{\sigma}$ such that $X \setminus X^{\sigma} = \bigsqcup_{i=0}^{\ell-1} \sigma^i \mathcal{V}$. Then we have the identity

$$\sum_{x \in X} \psi^{-1}(x)\, \pi(x) \tilde{v} = \sum_{x \in \mathcal{V}} \psi^{-1}(x)\, \pi(x) \mathrm{Nr}(\tilde{v}) + \sum_{x \in X^{\sigma}} \psi^{-1}(x)\, \pi(x) \tilde{v}.$$

Thus, we get that $\sum_{x \in X^{\sigma}} \overline{\psi}^{-1}(x)\, \pi(x) v$ is trivial in $\widehat{\mathrm{H}}^1(\mathcal{L})$, and this implies that

$$v \in \widehat{\mathrm{H}}^1(\mathcal{L})(N^{\sigma}(F), \overline{\psi}).$$

Conversely, let v be an element in $\widehat{\mathrm{H}}^1(\mathcal{L})(N^{\sigma}(F), \overline{\psi})$. Let $\tilde{v} \in \mathrm{Ker}(\mathrm{Nr})$ be a lift of v. Consider the element

$$\tilde{v}_1 = \tilde{v} - \frac{1}{|\mathcal{N}_{\sigma}/\mathcal{N}_{\sigma}'|} \sum_{n \in \mathcal{N}_{\sigma}/\mathcal{N}_{\sigma}'} \psi^{-1}(n)\, \pi(n) \tilde{v}.$$

Note that $\tilde{v}_1 \in \mathcal{L}(N(F), \psi)$ and $\mathrm{Nr}(\tilde{v}_1) = 0$ as Nr commutes with the action of $N^{\sigma}(F)$. This completes the proof. $\square$

Using the above lemma, we get the following crucial result on Jacquet modules of the Tate cohomology groups.

Theorem 3.4.2. *Let P be a Γ-stable F-parabolic subgroup of G, and let N be the unipotent radical of P. Let $\mathcal{L}$ be a smooth $\Lambda[G(F) \rtimes \Gamma]$-module. Then the natural map $\widehat{\mathrm{H}}^i(\mathcal{L})_{N^{\sigma}(F), \overline{\psi}} \to \widehat{\mathrm{H}}^i(\mathcal{L}_{N(F),\psi})$ is injective.*

Proof. Consider the short exact sequence of $\Lambda[\Gamma]$-modules

$$0 \longrightarrow \mathcal{L}(N(F), \psi) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{N(F),\psi} \longrightarrow 0.$$

The long exact sequence (2.1) of Tate cohomology groups corresponding to the above short exact sequence gives the following exact sequences:

$$\widehat{\mathrm{H}}^0(\mathcal{L}(N(F), \psi)) \xrightarrow{\varphi_0} \widehat{\mathrm{H}}^0(\mathcal{L}) \to \widehat{\mathrm{H}}^0(\mathcal{L}_{N(F),\psi})$$

and

$$\widehat{\mathrm{H}}^1(\mathcal{L}(N(F), \psi)) \xrightarrow{\varphi_1} \widehat{\mathrm{H}}^1(\mathcal{L}) \to \widehat{\mathrm{H}}^1(\mathcal{L}_{N(F),\psi}).$$

Using Lemma 3.4.1, the image of φ_i is equal to $\widehat{\mathrm{H}}^i(\mathcal{L})(N^\sigma(F), \overline{\psi})$. Hence, we get the theorem. ◻

As a corollary, we get the following result.

Corollary 3.4.3. *Let G be a connected reductive algebraic group defined over F, and let σ be an order ℓ automorphism of G, where ℓ and p are distinct primes. Let Π be an ℓ-modular cuspidal representation of $G(F)$ such that Π extends to a representation of $G(F) \rtimes \Gamma$. Then each Tate cohomology group $\widehat{\mathrm{H}}^i(\Pi)$ is a cuspidal representation of $G^\sigma(F)$.*

Proof. Let Q be a proper F-parabolic subgroup of G^σ. By [KP23, Lemma 12.5.4], there exists a proper σ-stable F-parabolic subgroup P of G such that $(P^\sigma)^0 = Q$. Let N be the unipotent radical of P. Then N^σ is equal to the unipotent radical of Q. Now, using Theorem 3.4.2 for $\psi = 1$, the trivial character of $N(F)$, we get an injection

$$(\widehat{\mathrm{H}}^i(\Pi))_{N^\sigma(F)} \hookrightarrow \widehat{\mathrm{H}}^i(\Pi_{N(F)}).$$

Since Π is cuspidal, the Jacquet module $\Pi_{N(F)}$ is trivial, and so is $(\widehat{\mathrm{H}}^i(\Pi))_{N^\sigma(F)}$. Therefore, the Tate cohomology group $\widehat{\mathrm{H}}^i(\Pi)$ is cuspidal. ◻

3.5. In the paper [TV16, Conjecture 6.1], the authors conjectured that if Π is an irreducible ℓ-modular representation of $G(F)$ such that Π extends to a representation of $G(F) \rtimes \Gamma$, then the Tate cohomology groups $\widehat{\mathrm{H}}^i(\Pi)$, $i \in \{0, 1\}$, are finite length representations of $G^\sigma(F)$. In our previous work [DN24, Proposition 5.5], this conjecture is proved for $G = \mathrm{GL}_n$ where σ is induced by a Galois automorphism. In this subsection, we generalise this result for arbitrary connected reductive groups. We prove

Theorem 3.5.1. *Let G be a connected reductive group defined over a non-Archimedean local field F of residue characteristic p, and let σ be an automorphism of G, defined over F, and has prime order $\ell \neq p$. Set $\Gamma = \langle \sigma \rangle$, and let Π be a finite length ℓ-modular representation of $G(F) \rtimes \Gamma$. Then $\widehat{\mathrm{H}}^i(\Pi)$ is a finite length ℓ-modular representation of $G^\sigma(F)$, for $i \in \{0, 1\}$.*

3.5.1. Let $\mathcal{G}$ be a connected reductive algebraic group defined over F. Fix a minimal F-parabolic subgroup Q_0 of $\mathcal{G}$. Let Ω be the set of all proper F-parabolic subgroups of $\mathcal{G}$ containing Q_0. For $Q \in \Omega$, we denote by U the unipotent radical of Q. Let π be a smooth R-representation of $\mathcal{G}(F)$, where the coefficient field R is either $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$. We have the natural map obtained from Frobenius reciprocity corresponding to identity maps:

$$\pi \longrightarrow \prod_{Q \in \Omega} \mathrm{ind}_{Q(F)}^{\mathcal{G}(F)}(\pi_{U(F)}), \tag{3.3}$$

and let π_c be the kernel of the map (3.3). If the Jacquet module $(\pi_c)_{U(F)}$ is non-zero for some $P \in \Omega$, then Frobenius reciprocity gives the following non-zero map

$$\pi_c \longrightarrow \prod_{Q \in \Omega} \mathrm{ind}_{Q(F)}^{\mathcal{G}(F)}((\pi_c)_{U(F)}). \tag{3.4}$$

Since the induction functor $\mathrm{ind}_{Q(F)}^{\mathcal{G}(F)}$ preserves injective maps and the Jacquet functor $\pi \mapsto \pi_{U(F)}$ is exact, we get a contradiction to the fact that π_c is the kernel of the map (3.3). Thus, π_c is a cuspidal representation.

3.6. **First reduction.** The hurdle to prove admissibility is with the case where the rank of a maximal F-split torus in $Z(G^\sigma)$ is larger than the rank of a maximal F-split torus of $Z(G)$. The following proposition (Proposition 3.6.1) reduces Theorem 3.5.1 to the case where the maximal F-split torus of $Z(G^\sigma)$ is contained in $Z(G)$.

Let C be the maximal F-split torus of $Z(G^\sigma)$. Let S be a maximal torus of H, defined over F, where H is the connected component of G^σ containing the identity. Then the centralizer $Z_G(S)$ is a σ-stable maximal torus of G defined over F, and we denote it by T (see [KP23, Proposition 12.8.1(2)]). Let $\Phi(G,T)$ be the set of roots of T in G. Note that $\Phi(G,T)$ is the absolute root system of G. Let M be the centralizer of C in G (denoted by $Z_G(C)$). The group M is a σ-stable Levi subgroup of an F-rational parabolic subgroup P of G (see [Bor91, Proposition 20.4] and [BT65, Theorem 4.15]). We want to claim that M is a Levi subgroup of a σ-stable parabolic subgroup P of G. For this, we follow the arguments in the proof of [Bor91, Proposition 20.4]. Let $\lambda \in X_*(C)$ be a non-zero element such that for all $\alpha \in \Phi(G,T)$ which restrict to a non-zero element in $X^*(C)$, we have $\langle \alpha, \lambda \rangle \neq 0$. We set

$$\Psi = \{\alpha \in \Phi(G,T) : \langle \alpha, \lambda \rangle > 0\}.$$

Note that

$$\Phi(G,T) = \Phi(Z_G(C),T) \bigsqcup \Psi \bigsqcup -\Psi.$$

The unipotent group U_Ψ generated by the root groups U_β, for $\beta \in \Psi$, is a σ-stable subgroup defined over F. The group $P = Z_G(C)U_\Psi$ is a σ-stable parabolic subgroup defined over F and contains $M = Z_G(C)$ as its Levi subgroup.

For maintaining uniform notation, the unipotent radical of P, i.e., the group U_Ψ, will be denoted by N. The group N is σ-stable and note that N^σ is trivial. By definition, G^σ is contained in M, and thus, we get that $M^\sigma = G^\sigma$. Observe that the maximal F-split torus in $Z(M^\sigma) = Z(G^\sigma)$ is contained in $Z(M)$. We denote by $\Pi_{N(F)}$, the Jacquet module of Π with respect to $N(F)$, and let $\Pi(N(F))$ be the kernel of the natural map $\Pi \to \Pi_{N(F)}$. Note that the space $\Pi(N(F))$ is σ-stable. Using Theorem 3.4.2, we get an injective map

$$\widehat{\mathrm{H}}^i(\Pi) \hookrightarrow \widehat{\mathrm{H}}^i(\Pi_{N(F)}). \tag{3.5}$$

This suffices for the proof of Theorem 3.5.1. However, we prove that the injective map (3.5) is an isomorphism. This might be of interest for explicit computation of Tate cohomology in some cases.

Proposition 3.6.1. *Let Π be a smooth ℓ-modular representation of $G(F)$ such that Π extends as a representation of $G(F) \rtimes \Gamma$. Then the Tate cohomology space $\widehat{\mathrm{H}}^i(\Pi(N(F)))$ is trivial, for $i \in \{0,1\}$.*

Proof. The above proposition can be proved using Glauberman's correspondence. However, we give a direct proof. Let us first consider the case $i = 0$. Let $w \in \Pi(N(F))^\sigma$ and let N_0 be a σ-stable compact open subgroup of $N(F)$ such that

$$\int_{N_0} \Pi(n)w\, dn = 0, \tag{3.6}$$

where dn is a Haar measure on N_0. Let N_1 be a σ-stable open normal subgroup of N_0 such that $w \in \Pi^{N_1}$. Let $\mathcal{N}$ be the quotient group N_0/N_1. Since N^σ is trivial, the long exact sequence for Γ-group cohomology corresponding to the exact sequence

$$1 \to N_1 \to N_0 \to \mathcal{N} \to 1$$

gives the exact sequence

$$1 \to \mathcal{N}^\sigma \to \mathrm{H}^1(\Gamma, N_0).$$

The pointed set $\mathrm{H}^1(\Gamma, N_0)$ is trivial from Lemma 3.3.1. Hence, we get that $\mathcal{N}^\sigma$ is trivial.

Then we have

$$\int_{N_0} \Pi(n)w\, dn = \sum_{n\in\mathcal{N}} \Pi(n)w = \sum_{n\in\mathcal{N}^\sigma} \Pi(n)w + \sum_{n\in\mathcal{N}\backslash\mathcal{N}^\sigma} \Pi(n)w.$$

Since the Γ-action on $\mathcal{N}\backslash\mathcal{N}^\sigma$ is free, there exists a subset $X \subset \mathcal{N}\backslash\mathcal{N}^\sigma$ such that $\mathcal{N}\backslash\mathcal{N}^\sigma = \coprod_{i=0}^{\ell-1} \sigma^i X$, and we get the following equality

$$\sum_{n\in\mathcal{N}} \Pi(n)w = \sum_{n\in\mathcal{N}^\sigma} \Pi(n)w + \mathrm{Nr}\big(\sum_{x\in X} \Pi(x)w\big).$$

Thus, we get

$$\sum_{h\in\mathcal{N}} \Pi(n)w = w$$

in $\widehat{\mathrm{H}}^0(\Pi(N(F)))$. Now, using the equation (3.6), we get that $w = 0$ in $\widehat{\mathrm{H}}^0(\Pi(N(F)))$.

We now consider the case where $i = 1$. Let $v \in \mathrm{Ker}\big(\mathrm{Nr} : \Pi(N(F)) \to \Pi(N(F))\big)$. There exists a σ-stable compact open subgroup N_0 of $N(F)$ such that

$$\int_{N_0} \Pi(n)v\, dn = 0,$$

where dn is a Haar measure on N_0. Let N_1 be a σ-stable open normal subgroup of N_0 such that $v \in \Pi^{N_1}$. If $\mathcal{N}$ denotes the quotient group N_0/N_1, then we have

$$\int_{N_0} \Pi(n)v\, dn = \sum_{n\in\mathcal{N}} \Pi(n)v = \sum_{n\in\mathcal{N}^\sigma} \Pi(n)v + \sum_{n\in\mathcal{N}\backslash\mathcal{N}^\sigma} \Pi(n)v.$$

Note that $\mathcal{N}^\sigma = \{e\}$, and there exists a subset $X \subseteq \mathcal{N} \setminus \mathcal{N}^\sigma$ such that $\mathcal{N} \setminus \mathcal{N}^\sigma = \coprod_{i=0}^{\ell-1} \sigma^i X$. Then the above identity becomes

$$v \;+\; \sum_{i=0}^{\ell-1} \sum_{x\in X} \Pi(\sigma^i(x))v = 0. \tag{3.7}$$

Set $w = \sum_{i=0}^{\ell-1} \sum_{x\in X} \Pi(\sigma^i(x))v$. We will show that $w \in \mathrm{Im}(1-\sigma)$. Using the fact that $\mathrm{Nr}(v) = 0$, we have

$$\begin{aligned}
w &= \sum_{x\in X} \Pi(x)v \;+\; \sum_{i=1}^{\ell-1} \sum_{x\in X} \Pi(\sigma^i(x))v \\
&= -\sum_{i=1}^{\ell-1} \sum_{x\in X} \Pi(x)\sigma^i v \;+\; \sum_{i=1}^{\ell-1} \sum_{x\in X} \Pi(\sigma^i(x))v \\
&= -\sum_{i=1}^{\ell-1} \sigma^i \sum_{x\in X} \Pi(\sigma^{\ell-i}(x))v \;+\; \sum_{i=1}^{\ell-1} \sum_{x\in X} \Pi(\sigma^{\ell-i}(x))v \\
&= \sum_{i=1}^{\ell-1} (1-\sigma^i) \sum_{x\in X} \Pi^{\sigma^{\ell-i}}(x)v.
\end{aligned}$$

Thus, we get that $w = (1-\sigma)w_1$ for some $w_1 \in \Pi(N(F))$, and hence it follows from (3.7) that $v = 0$ in $\widehat{\mathrm{H}}^1(\Pi(N(F))$. $\square$

Corollary 3.6.2. *Let G be a connected reductive algebraic group and let P be a σ-stable parabolic subgroup with unipotent radical N and a σ-stable Levi subgroup M. Assume that $N \cap G^\sigma$ is trivial and $G^\sigma \subset M$. Then, the natural map*

$$\widehat{\mathrm{H}}^i(\Pi) \to \widehat{\mathrm{H}}^i(\Pi_{N(F)})$$

is an isomorphism.

Corollary 3.6.3. *Let G be a connected reductive algebraic group defined over a p-adic field F and let s be an element of order $\ell \neq p$ such that the centralizer $C_G(s)$ is contained in a proper parabolic subgroup of G. Let Π be an ℓ-modular cuspidal representation of $G(F)$. Then the Tate cohomology space $\widehat{\mathrm{H}}^i(\langle s\rangle, \Pi)$ is trivial, for $i \in \{0,1\}$.*

3.7. **Proof of Theorem 3.5.1.** We use induction on the length of Π and the dimension of G to prove the theorem. First assume that Π is an irreducible as $G(F)$-representation. We stress that in the proof of this theorem, we use a deep result that the subrepresentations of finitely generated smooth ℓ-modular representations of $G(F)$ are again finitely generated. This fundamental result that follows from the works of [DHKM24, Corollary 1.3] and [Dat09, Corollary 4.5]. Using the Proposition 3.6.1, we can assume that the maximal F-split torus C of $Z(G^\sigma)$ is contained in $Z(G)$. Let Q be a proper parabolic subgroup of $H = (G^\sigma)^0$ and let U be the unipotent radical of Q. Let L be a Levi-subgroup of Q. For a smooth representation π of $H(F)$, let Φ be the following map obtained by Frobenius reciprocity corresponding to identity maps:

$$\Phi : \pi \to \prod_{Q=LU, Q\neq H} \mathrm{ind}_{Q(F)}^{H(F)} \pi_{U(F)}$$

and let π_c be the kernel of Φ. Note that the Jacquet module $(\pi_c)_{U(F)}$ is trivial, for any unipotent radical of a proper parabolic Q of H. Applying Φ to the Tate cohomology space $\widehat{\mathrm{H}}^i(\Pi)$, we get the following exact sequence

$$0 \to \widehat{\mathrm{H}}^i(\Pi)_c \to \widehat{\mathrm{H}}^i(\Pi) \xrightarrow{\Phi} \prod_{Q=LU, Q\neq H} \mathrm{ind}_{Q(F)}^{H(F)}\big(\widehat{\mathrm{H}}^i(\Pi)_{U(F)}\big) \tag{3.8}$$

Note that $\widehat{\mathrm{H}}^i(\Pi)_c$ is a cuspidal representation, and it is an $H(F)$-subrepresentation of the space $\widehat{\mathrm{H}}^i(\Pi)$–which is a finitely generated $\overline{\mathbb{F}}_\ell[H(F)]$-representation (see Theorem 2.3.3). This implies that $\widehat{\mathrm{H}}^i(\Pi)_c$ is an ℓ-modular finitely generated cuspidal representation of $H(F)$, and using the fact that the split component C of $Z(G^\sigma)$ is contained in $Z(G)$, we get that $\widehat{\mathrm{H}}^i(\Pi)_c$ is a finite length representation of $H(F)$ (this follows from the fact that finitely generated cuspidal representations with central character are of finite length [Vig96, Chapter I, Proposition 7.11(2), Chapter II 2.7]). Let P be a σ-stable parabolic subgroup of G such that $(P^\sigma)^0 = Q$, and let N be the unipotent radical of P. Let M be a σ-stable Levi subgroup of P such that the connected component of M^σ is a Levi subgroup L of Q. Then, using Theorem 3.4.2, we have the injection

$$\widehat{\mathrm{H}}^i(\Pi)_{U(F)} \hookrightarrow \widehat{\mathrm{H}}^i(\Pi_{N(F)}).$$

Using induction, we get that $\widehat{\mathrm{H}}^i(\Pi_{N(F)})$ is a finite length representation of $L(F)$. Since parabolic induction preserves finite length, it follows from exact sequence (3.8) that $\widehat{\mathrm{H}}^i(\Pi)$ is a finite length representation of $H(F)$. Thus, the space $\widehat{\mathrm{H}}^i(\Pi)$ is a finite length representation of $G^\sigma(F)$, for $i \in \{0,1\}$.

Now, consider the general case where Π is not necessarily an irreducible representation of $G(F)$. Let σ be a generator of Γ. Let Π_1 be an irreducible subrepresentation of Π and set Π_1' to be the subrepresentation of Π spanned by $\Pi_1^{\sigma^i}$, for $0 \leq i < \ell$. Note that Π_1' is σ-stable, and we consider the short exact sequence of $\overline{\mathbb{F}}_\ell[G(F) \rtimes \Gamma]$ modules

$$0 \longrightarrow \Pi_1' \longrightarrow \Pi \longrightarrow \Pi/\Pi_1' \longrightarrow 0.$$

From the long exact sequence of Tate cohomology, we get the following exact sequence:

$$\widehat{\mathrm{H}}^i(\Pi_1') \to \widehat{\mathrm{H}}^i(\Pi) \to \widehat{\mathrm{H}}^i(\Pi/\Pi_1').$$

If Π_1 is not σ-stable, then $\widehat{\mathrm{H}}^i(\Pi_1')$ is trivial for $i \in \{0,1\}$, and if Π_1 is σ-stable, then $\widehat{\mathrm{H}}^i(\Pi_1')$ is a finite length representation of $G^\sigma(F)$ from the previous case. Note that the length of Π/Π_1' is now strictly less than the length of Π. Then, using induction hypothesis, the Tate cohomology space $\widehat{\mathrm{H}}^i(\Pi/\Pi_1')$ is a finite length representation of $G^\sigma(F)$. Thus, we get that $\widehat{\mathrm{H}}^i(\Pi)$ is a finite length representation of $G^\sigma(F)$. As a corollary, we get the following result.

Corollary 3.7.1. *Let G be a connected reductive group defined over a non-Archimedean local field F of residue characteristic p. Let σ be an automorphism of G, defined over F and has prime order $\ell \neq p$. Let (Π, V) be a finite length integral $\mathcal{K}$-representation of $G(F) \rtimes \Gamma$. Then, for any $\Lambda[G(F) \rtimes \Gamma]$-stable lattice $\mathcal{L}$ in V, the space $\widehat{\mathrm{H}}^i(\mathcal{L})$ is a finite length ℓ-modular representations of $G^\sigma(F)$, for $i \in \{0,1\}$.*

Proof. Recall that $\mathcal{L}$ is a free Λ-module and that the ℓ-modular representation $\mathcal{L}/\ell\mathcal{L}$ is a finite length representation of $G(F)$ (see [Vig96, II.5.11.a] for finiteness). Consider the short exact sequence of $G(F) \rtimes \Gamma$-modules

$$0 \longrightarrow \mathcal{L} \xrightarrow{\text{mult.}\ell} \mathcal{L} \longrightarrow \mathcal{L}/\ell\mathcal{L} \longrightarrow 0. \tag{3.9}$$

By Theorem 3.5.1, the Tate cohomology groups $\widehat{\mathrm{H}}^i(\mathcal{L}/\ell\mathcal{L})$ are of finite length as representations of $G^\sigma(F)$. From the long exact sequence of Tate cohomology corresponding to the above exact sequence, we have

$$0 \to \widehat{\mathrm{H}}^0(\mathcal{L}) \to \widehat{\mathrm{H}}^0(\mathcal{L}/\ell\mathcal{L}) \to \widehat{\mathrm{H}}^1(\mathcal{L}) \to 0.$$

and

$$0 \to \widehat{\mathrm{H}}^1(\mathcal{L}) \to \widehat{\mathrm{H}}^1(\mathcal{L}/\ell\mathcal{L}) \to \widehat{\mathrm{H}}^0(\mathcal{L}) \to 0.$$

Thus, we get that each $\widehat{\mathrm{H}}^i(\mathcal{L})$ is of finite length. □

4. Some genericity and irreducibility results

For a generic irreducible ℓ-modular representation (Π, V) of $\mathrm{GL}_n(E)$, which is stable under the action of $\mathrm{Gal}(E/F)$, we showed in [DN24, Proposition 6.3] that the Tate cohomology space $\widehat{\mathrm{H}}^i(\mathrm{Gal}(E/F), \Pi)$ admits a unique irreducible generic subquotient; we used Kirillov model together with a version of Lemma 3.4.1. We prove this result in some generality here using the work of Rodier and Moeglin–Waldspurger as a substitute for the Kirillov model for GL_n. We will assume that F is a p-adic field in this section.

4.1. Let $(G, \Gamma, \sigma, G^\sigma, H)$ be a tuple as in (1.2). Let $\mathfrak{g}$ be the Lie algebra of G, and there is an induced action of Γ on $\mathfrak{g}$. The σ-fixed point Lie subalgebra $\mathfrak{g}^\sigma$ is the Lie algebra of H. Let $B_\mathfrak{g}$ be the Killing form of G. We assume that H is a quasi-split reductive algebraic group and that there exists an element $X \in \mathfrak{g}^\sigma$ such that X is regular nilpotent in $\mathfrak{g}$. This implies that X is regular in $\mathfrak{g}^\sigma$. Let $\varphi : \mathbb{G}_m \to G$ be a co-character such that $X \in \mathfrak{g}_{-2}$, where $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is the grading induced by φ. Let N be the closed subgroup of G with Lie algebra $\bigoplus_{i>0} \mathfrak{g}_i$, and let B be the Borel subgroup normalising N. Let B^- be the Borel subgroup of G with Lie algebra $\bigoplus_{i \leq 0} \mathfrak{g}_i$. Note that the group B is stable under Γ, and $(B^\sigma)^0$ is a Borel subgroup of H. Also note that N is the unipotent radical of B and N^σ is the unipotent radical of $(B^\sigma)^0$.

Let $\mathfrak{o}_F$ be the ring of integers of F, and let $\mathfrak{p}_F$ be the maximal ideal of F. Fix a uniformizer ϖ of F. Let $\psi : F \to \overline{\mathbb{Z}}_\ell^\times$ be an additive character which is trivial on $\mathfrak{p}_F$ and non-trivial on $\mathfrak{o}_F$. We denote by $\overline{\psi} : F \to \overline{\mathbb{F}}_\ell^\times$ the mod-ℓ reduction of ψ. Let $\psi_X : N(F) \to \overline{\mathbb{Z}}_\ell^\times$ be the character

$$\psi_X(\exp(Y)) = \psi(\mathrm{B}_\mathfrak{g}(X, Y)).$$

Note that $B_{\mathfrak{g}}(\sigma(X),\sigma(Y)) = B_{\mathfrak{g}}(X,Y)$, and $\exp(\sigma(Y)) = \sigma(\exp(Y))$, for all $Y \in \mathrm{Lie}(N)$. For $X \in \mathfrak{g}^\sigma$, the character ψ_X is invariant under the action of σ on elements of $N(F)$. The mod-ℓ reduction of ψ_X is denoted by $\overline{\psi}_X$. Recall that for an irreducible smooth ℓ-modular representation (Π, V) of $G(F)$, the multiplicity one statement

$$\dim_{\overline{\mathbb{F}}_\ell}(V_{N(F),\overline{\psi}_X}) \leq 1$$

follows from [MT24, Corollary 7.2], and the representation (Π, V) is called $(N(F), \overline{\psi}_X)$-generic if $\dim_{\overline{\mathbb{F}}_\ell}(V_{N(F),\overline{\psi}_X}) = 1$. Note that Γ acts trivially on $V_{N(F),\overline{\psi}_X}$. We now prove the following theorem.

Theorem 4.1.1. *Let (Π, V) be an irreducible smooth ℓ-modular representation of $G(F)$ such that Π extends as a representation of $G(F) \rtimes \Gamma$. Let $\overline{\psi}_X : N(F) \to \overline{\mathbb{F}}_\ell^\times$ be a non-degenerate character of $N(F)$ associated with a σ-invariant element $X \in \mathfrak{g}$ such that $\overline{\psi}_X$ restricts to a non-degenerate character of $N^\sigma(F)$. If (Π, V) is $(N(F), \overline{\psi}_X)$-generic, then $\widehat{\mathrm{H}}^i(\Gamma, \Pi)$ is non-zero and it has a unique irreducible $(N^\sigma(F), \overline{\psi}_X)$-generic subquotient, for $i \in \{0, 1\}$.*

Proof. We will use the fact that the natural map $V \to V_{N(F),\overline{\psi}_X}$ is an isomorphism on twisted-coinvariants with respect to some compact open subgroup K of $G(F)$, this important result is due to Rodier in the case of smooth complex representations, and it is later generalised by Moeglin and Waldspurger ([MgW87]) for degenerate Whittaker models. In [Vig04, Section III], Vignéras used these ideas to prove $G(F)$-stability of certain natural lattices in the Whittaker model of a smooth irreducible representation of $G(F)$ on a $\overline{\mathbb{Q}}_\ell$-vector space. We will recall the constructions in Moeglin–Waldspurger and Vigneras's paper and verify that these constructions are compatible with the action of σ.

Let $\widetilde{L}$ be a lattice in $\mathfrak{g}$ such that $[\widetilde{L}, \widetilde{L}] \subset \widetilde{L}$. The lattice

$$L = \widetilde{L} + \sigma(\widetilde{L}) + \cdots + \sigma^{\ell-1}(\widetilde{L}),$$

is a σ-stable lattice in $\mathfrak{g}$ such that $[L, L] \subset L$. Let $\mathfrak{g}^X$ be the centralizer in $\mathfrak{g}$ of the element X. Let $\mathfrak{m}$ be a φ-stable complement of $\mathfrak{g}^X$. Note that we can choose $\mathfrak{m}$ such that $\sigma(\mathfrak{m}) = \mathfrak{m}$. The nil-radical of the alternating form $B_X(Y, Z) = B_{\mathfrak{g}}(X, [Y, Z])$ is $\mathfrak{g}^X$. Choose a σ-stable lattice M_1 in $\mathfrak{m}$ such that $B_X(M_1, M_1) \subset \mathfrak{o}_F$ (such an M_1 can be constructed by averaging the Γ-translates of a lattice $\widetilde{M}_1$ in $\mathfrak{m}$ with $B_X(\widetilde{M}_1, \widetilde{M}_1) \subset \mathfrak{o}_F$). The lattice

$$L_1 = M_1 \oplus \sum_{i \in \mathbb{Z}} (L \cap \mathfrak{g}^X \cap \mathfrak{g}_i)$$

defines the groups $G_n = \exp(\varpi^n L_1)$, for all $n \geq B$, where B is a constant defined in [MgW87, Lemma I.3]. Let χ_n be the character of G_n given by

$$\exp(Z) \mapsto \psi(B_{\mathfrak{g}}(\varpi^{-2n} X, Z)),$$

for all $Z \in \varpi^n L_1$. Let $t = \varphi(\varpi)$ and let

$$K_n = t^{-n} G_n t^n, \ \chi'_n(k) = \chi_n(t^n k t^{-n}).$$

Note that $\chi'_n(x) = \psi_X(x)$, for $x \in N(F) \cap K_n$, and $\chi'_n(x) = 1$, for $x \in B^-(F) \cap K_n$. The mod-ℓ reduction of χ'_n is denoted by $\overline{\chi}'_n$. Note that K_n and $\overline{\chi}'_n$ are Γ-invariant. As in the complex case, we claim that the natural map

$$V^{K_n, \overline{\chi}'_n} \to V_{N(F),\overline{\psi}_X} \tag{4.1}$$

is an isomorphism for all n large enough.

To prove the isomorphism (4.1), given a $(N(F), \overline{\psi}_X)$-generic ℓ-modular representation (Π, V), there exists an ℓ-adic integral representation $\widetilde{\Pi}$ of $G(F)$ such that Π is the unique $(N(F), \psi_X)$-generic constituent of the mod-ℓ reduction of $\widetilde{\Pi}$ (see [MT24, Corollary 7.2]). We may assume that $\widetilde{\Pi}$ is defined over a finite extension R_1 of $\mathbb{Q}_\ell$ (see [Vig96, II.4.7]), and let R be the extension of R_1

obtained by attaching all the p-power roots of unity. Note that the ring of integers $\mathfrak{o}_R$ of R is a DVR. Let $\mathcal{L}$ be a $(G(F) \rtimes \Gamma)$-stable $\mathfrak{o}_R$-lattice in $\widetilde{\Pi}$ such that Π is the unique $(N(F), \psi_X)$-generic constituent of $\mathcal{L} \otimes_{\mathfrak{o}_R} \overline{\mathbb{F}}_\ell$.

For a large enough integer n, it follows from [MgW87, I.14] that the natural map

$$\widetilde{\Pi}^{K_n,\chi_n'} \to \widetilde{\Pi}_{N(F),\psi_X}$$

is an isomorphism. For n as above, Vignéras showed that the natural map

$$\mathcal{L}^{K_n,\chi_n'} \to \mathcal{L}_{N(F),\psi_X} \tag{4.2}$$

is an isomorphism (see ([Vig04, Proposition III.4.6]). Since $\mathcal{L}^{K_n,\chi_n'}$ is a $\mathfrak{o}_R$-lattice in $\widetilde{\Pi}^{K_n,\chi_n'}$, we get that the modules in (4.2) are free $\mathfrak{o}_R$-modules of rank one.

Since K_n is a pro-p group, we have

$$V^{K_n,\overline{\chi}_n'} \subset \mathcal{L}^{K_n,\chi_n'} \otimes_{\mathfrak{o}_R} \overline{\mathbb{F}}_\ell,$$

and

$$\mathcal{L}_{N(F),\psi_X} \otimes_{\mathfrak{o}_R} \overline{\mathbb{F}}_\ell = V_{N(F),\overline{\psi}_X}.$$

It remains to show that $V^{K_n,\overline{\chi}_n'}$ is a non-zero space for n large, and to show this we follow the arguments in [MgW87, Lemme I.10]. Consider the quotient map

$$V \to V_{N(F),\overline{\psi}_X}$$

and let $v \in V$ be an element which maps to a non-zero element in $V_{N(F),\overline{\psi}_X}$. There exists a large n such that $v \in V^{K_n \cap B^-(F)}$. Note that the element

$$I_n'(v) = \int_{N(F)\cap K_n} (\overline{\chi}_n')^{-1}(x)\Pi(x)v\,dx$$

has the same image as v in $V_{N(F),\overline{\psi}_X}$ and we have $I_n'(v) \in V^{K_n,\overline{\chi}_n'} \backslash \{0\}$. This shows that the natural map (4.1) is an isomorphism.

When (Π, V) extends as a representation of $G(F) \rtimes \Gamma$, we observe that the map (4.1) is a σ-equivariant map. Note that the group K_n is a pro-p group. Let $V(K_n, \overline{\chi}_n')$ be the kernel of the projection map

$$V \to V^{K_n,\overline{\chi}_n'}.$$

Thus, we have a σ-equivariant splitting

$$V = V(K_n, \overline{\chi}_n') \bigoplus V^{K_n,\overline{\chi}_n'}.$$

Since $\dim_{\overline{\mathbb{F}}_\ell} V^{K_n,\overline{\chi}_n'} = 1$, the group Γ acts trivially. Thus the space $\widehat{\mathrm{H}}^i(\Gamma, V)$ is non-zero and the natural map

$$\widehat{\mathrm{H}}^i(\Gamma, V) \to \widehat{\mathrm{H}}^i(\Gamma, V^{K_n,\overline{\chi}_n'}) \xrightarrow{\simeq} \widehat{\mathrm{H}}^i(\Gamma, V_{N(F),\overline{\psi}_X})$$

is non-zero for $i \in \{0, 1\}$. It follows from Theorem 3.4.2 that the natural map

$$\widehat{\mathrm{H}}^i(\Gamma, V) \to \widehat{\mathrm{H}}^i(\Gamma, V_{N(F),\overline{\psi}_X})$$

factorises through an injective map

$$\widehat{\mathrm{H}}^i(\Gamma, V)_{N^\sigma(F),\overline{\psi}_X} \to \widehat{\mathrm{H}}^i(\Gamma, V_{N(F),\overline{\psi}_X}),$$

and it is non-zero as noted above. Hence, the above map is an isomorphism from multiplicity one. Thus, from our result on finiteness of Tate cohomology (Theorem 3.5.1), and the uniqueness of Whittaker model in the ℓ-modular case [MT24, Corollary 7.2], we get that there exists a unique irreducible generic subquotient of

$$\widehat{\mathrm{H}}^i(\Gamma, \Pi).$$

This proves the theorem. □

Remark 4.1.2. The above theorem can be applied in many interesting cases we list a few.

(1) In the case where $G = \mathrm{Res}_{E/F}(\mathcal{G}_E)$, where $\mathcal{G}$ is a quasi-split connected reductive group defined over F, and E/F is a cyclic Galois extension of degree ℓ.
(2) The case where $G = \mathrm{GL}_{2n+1}$ and $G^\sigma = \mathrm{O}_{2n+1}$,
(3) when $G = \mathrm{GL}_{2n}$ and $G^\sigma = \mathrm{Sp}_{2n}$.
(4) It also deals with the case where G is a semisimple group of type D_4 equipped with a triality automorphism σ of order 3.

The above theorem does not deal with the case where $G = \mathrm{GL}_{2n}$ and $G^\sigma = \mathrm{O}_{2n}$.

We end this section with a result on extending (uniquely) a σ-stable, generic, absolutely irreducible $\mathcal{K}$-representation (Π, V) to a representation of $G(F) \rtimes \Gamma$. The mod-ℓ version of the following lemma, for all σ-stable irreducible smooth $\overline{\mathbb{F}}_\ell$-representations, can be proved using a simple Schur's lemma argument (see ([TV16, Section 6.1]). It is important to note that any ℓ-adic irreducible smooth representation ξ of $G(F)$ such that $\xi^\sigma \simeq \xi$ extends as a representation of $G(F) \rtimes \Gamma$–although not uniquely. However, since we cannot extract ℓ-th roots of unity in $\mathcal{K}$, it is not completely clear if this holds for absolutely irreducible smooth representations defined over $\mathcal{K}$. The following lemma shows that this holds for generic representations.

Lemma 4.1.3. *Let $(N(F), \psi_X)$ be a pair as defined in Theorem 4.1.1. Let (Π, V) be an absolutely irreducible $(N(F), \psi_X)$-generic $\mathcal{K}$-representation of $G(F)$ such that $\Pi^\sigma \simeq \Pi$. Then Π extends uniquely to a representation of $G(F) \rtimes \Gamma$.*

Proof. Let $W : V \to \mathcal{K}$ be a non-zero $(N(F), \psi_X)$-Whittaker linear functional, i.e.,

$$W(\Pi(n)v) = \psi_X(n)W(v),\ v \in V,\ n \in N(F).$$

For $v \in V$, we set $W_v(g) = W(\Pi(g)v)$. Let $\mathbb{W}(\Pi, \psi_X)$ be the Whittaker model of Π defined as the subspace $\{W_v : v \in V\}$ of $\mathrm{Ind}_{N(F)}^{G(F)}\psi_X$. As ψ_X is a σ-invariant character of $N(F)$, we get that the image of the operator

$$T_\sigma : \mathbb{W}(\Pi, \psi_X) \to \mathrm{Ind}_{N(F)}^{G(F)}\psi_X$$

given by $T_\sigma(f)(g) = f(\sigma^{-1}(g))$, is $\mathbb{W}(\Pi^\sigma, \psi_X)$. Since $\Pi^\sigma \simeq \Pi$, using the multiplicity one property of the space of Whittaker linear functionals, the image of T_σ is equal to $\mathbb{W}(\Pi, \psi_X)$. Thus, we get the required extension in the lemma. Uniqueness of the extension is a consequence of the fact that $\mathcal{K}$ is the maximal unramified extension of $\mathbb{Q}_\ell$. Let T_σ and T'_σ be the action of σ on V for two different extensions of Π. The operators T_σ and T'_σ are intertwining operators from Π to Π^σ, and from Schur's lemma we have $T_\sigma = \lambda T'_\sigma$, for some $\lambda \in \mathcal{K}$. So, $\lambda^\ell = 1$ and $\lambda \in \mathcal{K}$. We conclude that $\lambda = 1$. □

4.2. **Irreducibility of Tate cohomology for** GL_n**.** In this section, we assume that G is the group $\mathrm{Res}_{E/F}\mathrm{GL}_n$, where E/F is a finite Galois extension of prime degree ℓ, and we prove the irreducibility of the $\mathrm{Gal}(E/F)$-Tate cohomology of $\Lambda[\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)]$-stable lattices in certain integral generic $\mathcal{K}$-representations of $\mathrm{GL}_n(E)$. Assume that $n = \ell m$ for some integer $m \geq 1$. Let (Π, V) be the representation

$$\tau \times \tau^\sigma \times \cdots \times \tau^{\sigma^{\ell-1}}, \tag{4.3}$$

where τ is an absolutely irreducible cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_m(E)$ such that τ is not isomorphic to twisted representations τ^{σ^i}, for $1 \leq i \leq \ell - 1$. This implies that Π is absolutely irreducible; to see this if $\tau^{\sigma^i}\nu \simeq \tau$, for some $1 \leq i \leq \ell - 1$, and ν is the modulus character, then we get that $\tau\nu^j \simeq \tau$, for some j, and this is not possible by considering the central characters on both sides (see irreducibility criterion [BZ77, Theorem 4]). Note that $\Pi \simeq \Pi^\sigma$.

Hypothesis 4.1. *In this subsection, we assume that $r_\ell(\tau) \not\simeq r_\ell(\tau)^\sigma$.*

Since τ is integral, Π is an integral $\mathcal{K}$-representation of $\mathrm{GL}_n(E)$. Let $\mathcal{L}$ be a $\Lambda[\mathrm{GL}_n(E)]$-stable lattice in V, then $\sum_{i=0}^{\ell-1}\sigma^i(\mathcal{L})$ is a $\Lambda[\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)]$-stable lattice in V. In later sections, we will recall the notion of base change lifting and the representation Π arises as a base change lifting of a smooth cuspidal integral $\mathcal{K}$-representation π of $\mathrm{GL}_n(F)$–with the additional property that the mod-ℓ reduction of π is supercuspidal (see Lemma 6.1.1). We now prove the following result.

Theorem 4.2.1. *Let τ be an absolutely irreducible, integral, cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_m(E)$ such that τ is not σ-stable, and $r_\ell(\tau)$ is also not σ-stable. Let Π be the representation (4.3). Then, for any $\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)$-stable Λ-lattice $\mathcal{L} \subseteq V$, we have $\widehat{\mathrm{H}}^0(\mathrm{Gal}(E/F), \mathcal{L})$ is irreducible and cuspidal. Moreover, the space $\widehat{\mathrm{H}}^1(\mathrm{Gal}(E/F), \mathcal{L})$ is trivial.*

Proof. We set $\Gamma = \mathrm{Gal}(E/F)$ and σ a generator of Γ. We prove the cuspidality of the ℓ-modular representation $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{L})$ of the group $\mathrm{GL}_n(F)$, for $i \in \{0,1\}$. Consider the Jacquet module $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{L})_{N(F)}$, where N is the unipotent radical of an F-rational proper parabolic subgroup P of GL_n. Let M be an F-rational Levi subgroup of P. Using Proposition 3.4.2, we have $M(F)$-equivariant injection

$$\widehat{\mathrm{H}}^i(\Gamma, \mathcal{L})_{N(F)} \hookrightarrow \widehat{\mathrm{H}}^i(\Gamma, \mathcal{L}_{N(E)}).$$

We show that the Tate cohomology group $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{L}_{N(E)})$ is trivial for $i \in \{0,1\}$.

Say W is a finite length integral $\mathcal{K}[M(E) \rtimes \Gamma]$-representation such that

1. the Jordan-Holder factors of the $\mathcal{K}$-representation are multiplicity free,
2. the $M(E)$ representation $r_\ell(W)$ is multiplicity free, and
3. the irreducible $M(E)$-subquotients of W and the irreducible $M(E)$-subquotients of $r_\ell(W)$ are not stable under the action of σ.

It follows from the geometric lemma that the Jordan–Holder factors of $V_{N(E)}$, as a representation of $M(E)$, are absolutely irreducible and $V_{N(E)}$ is multiplicity free. Moreover, the irreducible subquotients of $V_{N(E)}$ are not Γ-stable. The Jacquet module $V_{N(E)}$ is integral as $\mathcal{L}_{N(E)}$ is a $M(E) \rtimes \Gamma$-stable Λ-lattice in $V_{N(E)}$ by [Dat05, Proposition 1.4]. Using Hypothesis 4.1, we get that the mod-ℓ reduction of $V_{N(E)}$ is multiplicity free and the irreducible subquotients of the mod-ℓ reduction of $V_{N(E)}$ are not Γ-stable. We use induction on the length of W to show that $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{W})$ is trivial, for all $\Lambda[M(E) \rtimes \Gamma]$-stable lattice $\mathcal{W}$ in W.

Let W_1 be an irreducible $M(E)$-subrepresentation of W and set $W_2 = \bigoplus_{i=0}^{\ell-1}\sigma^i(W_1)$. Let $\mathcal{M}$ be the lattice $W_2 \cap \mathcal{W}$. Note that $\mathcal{M}/\ell\mathcal{M}$ is isomorphic to $\bigoplus_{i=0}^{\ell-1} r_\ell(W_1)^{\sigma^i}$. Thus,

$$\widehat{\mathrm{H}}^i(\Gamma, \mathcal{M}/\ell\mathcal{M}) = \{0\},\ i \in \{0,1\}.$$

Using the hexagonal exact sequence (2.1) associated with the exact sequence

$$0 \to \mathcal{M} \xrightarrow{\ell} \mathcal{M} \to \mathcal{M}/\ell\mathcal{M} \to 0,$$

we get that $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{M})$ is trivial for $i \in \{0,1\}$. Now, using the hexagonal exact sequence (2.1) associated with the following exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{W} \longrightarrow \mathcal{W}/\mathcal{M} \longrightarrow 0,$$

we get that

$$\widehat{\mathrm{H}}^i(\Gamma, \mathcal{W}) \simeq \widehat{\mathrm{H}}^i(\Gamma, \mathcal{W}/\mathcal{M}).$$

Since the quotient $\mathcal{W}/\mathcal{M}$ is a $M(E) \rtimes \Gamma$-stable Λ-lattice in W/W_2, and the length of W/W_2 is strictly less than that of W, the Tate cohomology $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{W}/\mathcal{M})$ is trivial (by induction), and hence $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{W})$ is trivial, for $i \in \{0,1\}$. Thus, we get that $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{L}_{N(E)})$ is trivial, for $i \in \{0,1\}$. Hence, the representation $\widehat{\mathrm{H}}^i(\Gamma, \mathcal{L})$ is cuspidal, for $i \in \{0,1\}$.

The $G(F)$-representation $\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L})$ is non-zero and it admits a unique irreducible generic subquotient (see the proof of Theorem 4.1.1 or [DN24, Proposition 6.3]). Since, the finite length representation $\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L})$ is also cuspidal, it is irreducible as every irreducible cuspidal is generic. Let N_n be the unipotent radical of the Borel subgroup of upper unipotent matrices in GL_n. Let Θ_E be a non-degenerate σ-invariant character of $N_n(E)$ such that Θ_E restricts to a non-degenerate character Θ_F of $N_n(F)$. Note that $\widehat{\mathrm{H}}^1(\Gamma,\mathcal{L}_{N_n(E),\Theta_E})$ is trivial as Γ acts trivially on a free Λ-module of rank 1. From the inclusion

$$\widehat{\mathrm{H}}^1(\Gamma,\mathcal{L})_{N_n(F),\Theta_F}\hookrightarrow \widehat{\mathrm{H}}^1(\Gamma,\mathcal{L}_{N_n(E),\Theta_E}),$$

we get that $\widehat{\mathrm{H}}^1(\Gamma,\mathcal{L})_{N_n(F),\Theta_F}$ is trivial. Hence, the space $\widehat{\mathrm{H}}^1(\Gamma,\mathcal{L})$ is zero, as every irreducible ℓ-modular cuspidal representation of $\mathrm{GL}_n(F)$ is generic. □

Corollary 4.2.2. *Let $\mathcal{L}$ be a $\Lambda[\mathrm{GL}_n(E)\rtimes\Gamma]$-stable lattice in Π–as defined in Theorem 4.2.1. Then,*

$$\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L})\simeq\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}/\ell\mathcal{L}).$$

Remark 4.2.3. In section 6, we will discuss base change lifting from $\mathrm{GL}_n(F)$ to $\mathrm{GL}_n(E)$. If the base change lifting to $\mathrm{GL}_n(E)$ of an absolutely irreducible cuspidal $\mathcal{K}$-representation π of $\mathrm{GL}_n(F)$–with supercuspidal ℓ-modular reduction–is not cuspidal, then it is isomorphic to Π as defined in (4.3) for some τ which satisfies Hypothesis 4.1, and we prove this in Lemma 6.1.1. In the case where the base change lifting is cuspidal, the analogue of Theorem 4.2.1 follows from the existence of integral Kirillov model. This can be seen in the proof of [Ron16, Theorem 6]. We also refer to [DN24, Subsection 6.5.1], for more details.

5. Tate cohomology and Shintani correspondence

In this section, we study $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$-Tate cohomology of Galois invariant lattices in Shintani liftings of cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$ to $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$. The primary novelty of this section is that it treats the case where the Shintani lifting of a cuspidal representation–with supercuspidal ℓ-modular reduction–is not cuspidal.

5.1. Character formula for finite groups of Lie type.

5.1.1. We begin with the character theory of the finite group $\mathrm{GL}_n(\mathbb{F}_q)$. Although, irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ are obtained by J. A. Green ([Gre55]), for the present purpose, we find it convenient to use Deligne–Lusztig theory. Let us consider a connected reductive algebraic group G defined over $\mathbb{F}_q$, and let $F: G\to G$ be the corresponding Frobenius endomorphism. Let (S_0,B_0) be a pair consisting of an F-stable Borel subgroup B_0 and an F-stable maximal torus S_0 of B_0. Let $W(G,S_0)$ be the Weyl group $N_G(S_0)/S_0$, where $N_G(S_0)$ is the normalizer of S_0 in G. The automorphism induced by the Frobenius endomorphism F on $W(G,S_0)$ is denoted by F^*. For an F-stable maximal torus T of G, there exists $w\in W(G,S_0)$ such that (T,F) is G^F-conjugate to $(S_0,\mathrm{Ad}(w)\circ F)$. Two F-stable maximal tori T_1 and T_2 of G are conjugate in G^F if and only if the corresponding Weyl group elements w_1 and w_2 are F^*-conjugate in $W(G,S_0)$, i.e., if $w_1 = ww_2F^*(w)^{-1}$, for some $w\in W(G,S_0)$. For $w\in W(G,S_0)$, we denote by T_w an F-stable maximal torus of G such that (T_w,F) is isomorphic to $(S_0,\mathrm{Ad}(w)\circ F)$. For the above results, we refer to [DL76, Corollary 1.14].

5.1.2. For an F-stable maximal torus T and a character $\theta: T^F\to\overline{\mathbb{Q}}_\ell^\times$, Deligne and Lusztig associate a virtual character $R^G_{T,\theta}$. Every irreducible ℓ-adic representation of G^F occurs in a virtual character $R^G_{T,\theta}$, for some torus T and $\theta\in\widehat{T^F}$. For all unipotent elements $u\in G^F$, we denote by $Q_{G,T,F}(u)$, the Green's function, which is defined as the value $R^G_{T,1}(u)$. Let $g\in G^F$ be an element with Jordan decomposition $g=su$. Let $C_G(s)^0$ be the connected component of the centralizer of $s \in G^F$. Let $T_1, T_2, \dots, T_k$ be representatives of the $(C_G(s)^0)^F$-conjugacy classes of F-stable maximal tori in $C_G(s)^0$, which are conjugate to T in G^F. For simplicity, we denote by G_s the group $C_G(s)^0$. The Deligne–Lusztig character formula (see [DL76, Theorem 4.2] or [GM20, Lemma 2.2.23, Chapter 2]) gives

$$R^G_{T,\theta}(su) = \sum_{i=1}^{k} Q_{G_s,T_i,F}(u) \sum_{w \in W(G,T_i)^{F^*}/W(G_s,T_i)^{F^*}} \theta(wsw^{-1}). \tag{5.1}$$

Let T be an F-stable maximal torus of G such that T is contained in an F-stable parabolic subgroup P of G. Let M be an F-stable Levi subgroup of P. Let $\theta \in \widehat{T^F}$. Then we have

$$R^G_{T,\theta} = \mathrm{Ind}^G_{P,M}(R^M_{T,\theta}). \tag{5.2}$$

The notation $\mathrm{Ind}^G_{P,M}$ means the parabolic induction of a virtual character of M to G.

5.1.3. When G is GL_n, let F_n be the Frobenius endomorphism sending $(g_{ij}) \mapsto (g^q_{ij})$, and let D_n and B_n be the diagonal torus and the upper triangular Borel subgroup of GL_n, respectively. Clearly D_n and B_n are stable under F_n. The induced action of F_n on $W(G, D_n) \simeq S_n$ is trivial and the F_n-stable maximal tori of G are parametrized by the conjugacy classes of S_n, where S_n is the symmetric group on n-symbols. The character of every cuspidal representation of $G^{F_n} = \mathrm{GL}_n(\mathbb{F}_q)$ is equal to $(-1)^{n-1}R^G_{T,\theta}$, where T is an F_n-stable torus corresponding to the conjugacy class of $W(G, S_0)$ containing n-cycles, and $\theta \in \widehat{T^{F_n}}$ is in general position, i.e., $\theta^w \neq \theta$ for all $w \in W(G,T)^{F^*_n}$. For a unipotent element $u \in G^{F_n}$ with associated partition μ of n, and a maximal torus T corresponding to a partition λ of n, the Green's function $Q_{G,T,\mathrm{Fr}}(u)$ are Hall polynomials corresponding to the pair (μ, λ). We will not make use of this relation with Hall polynomials, however, we will use the connection of Green's polynomials with the representation theory of S_n due to T. Springer. In the fundamental work ([Spr76, Theorem 7.8]), T. Springer showed that

$$Q_{G,T_w,F_n}(u) = \sum_{i \geq 0} a_i(w,u) q^i$$

where the map

$$w \mapsto a_i(w,u)$$

is the character of an irreducible representation of S_n. These results have their analogues for all finite reductive groups.

5.1.4. Let $w \in S_n$ be an n-cycle, and let T_w be an F_n-stable maximal torus of GL_n corresponding to the element w. Let $s \in T^{F_n}_w$ be an element such that the characteristic polynomial of s is $f(X)^r$ where $f(X) \in \mathbb{F}_q[X]$ is an irreducible polynomial of degree d. We have $n = dr$. Let G_s be the connected group $C_{\mathrm{GL}_n}(s)$. Note that $G_s(\overline{\mathbb{F}}_q)$ is isomorphic to $\mathrm{GL}_r(\overline{\mathbb{F}}_q)^d$ and the Frobenius endomorphism $F_n : G_s \to G_s$, under this identification, is given by

$$(g_1, g_2, \dots, g_d) \longmapsto (F_r(g_d), F_r(g_1), \dots, F_r(g_{d-1})).$$

We get that $G^{F_n}_s$ is isomorphic to $\mathrm{GL}_r(\mathbb{F}_{q^d})$. Note that D^d_r and B^d_r give an F_n-stable maximal torus and an F_n-stable Borel subgroup to be denoted by S_{G_s} and B_{G_s}, respectively. The Frobenius endomorphism F_n on G_s induces a map F^*_n on $W(G_s, S_{G_s})$. If we identify $W(G_s, S_{G_s})$ with S^d_r, then

$$F^*_n(\sigma_1, \sigma_2, \dots, \sigma_d) = (\sigma_d, \sigma_1, \dots, \sigma_{d-1}).$$

The next lemma is about the F^ℓ_n-stable maximal tori in G_s which are conjugate to T_w by an element in $\mathrm{GL}^{F^\ell_n}_n = \mathrm{GL}_n(\mathbb{F}_{q^\ell})$. Along the way, we prove some standard observations which we could not find a convenient reference.

Lemma 5.1.1. *Let w be an n-cycle in S_n, and let T_w be an F_n-stable maximal torus of GL_n as above. Let $s \in T_w^{F_n}$ be an element with characteristic polynomial $f(X)^r$, where $f(X) \in \mathbb{F}_q[X]$ is an irreducible polynomial of degree d.*

(1) *Let T_1 and T_2 be two F_n-stable maximal tori of G_s such that $gT_1g^{-1} = T_2$ for some $g \in \mathrm{GL}_n(\mathbb{F}_q)$. Then there exists an element $h \in G_s^{F_n}$ such that $hT_1h^{-1} = T_2$.*
(2) *Assume that $(\ell, d) = 1$. Let T_1 and T_2 be two F_n^ℓ-stable maximal tori of G_s such that $gT_1g^{-1} = T_2$ for some $g \in \mathrm{GL}_n(\mathbb{F}_{q^\ell})$. Then there exists an element $h \in G_s^{F_n^\ell}$ such that $hT_1h^{-1} = T_2$.*
(3) *Let S be an F_n-stable maximal torus of G_s such that $gSg^{-1} = T_w$ for some $g \in \mathrm{GL}_n(\mathbb{F}_{q^\ell})$. There exists an element $h \in G_s^{F_n^\ell}$ such that $hSh^{-1} = T_w$.*

Proof. Let w_1 and w_2 be two elements of $W(G_s, S_{G_s})$ corresponding to the tori T_1 and T_2, respectively. We identify the elements w_1 and w_2 with the d-touples $(s_1, \ldots, s_d)$ and $(t_1, \ldots, t_d)$ respectively, via the isomorphism

$$W(G_s, S_{G_s}) \simeq S_r \times \cdots \times S_r,$$

where $s_i, t_j \in S_r$ for $1 \leq i, j \leq d$. Assume that there exists an element $k \in S_r$ such that

$$ks_1s_2 \cdots s_dk^{-1} = t_1t_2 \cdots t_d.$$

We define k_r by setting

$$k_r = (t_r \cdots t_2t_1)\, k\, (s_r \cdots s_2s_1)^{-1},\ 1 \leq r \leq d-1.$$

Let $w = (k_1, \ldots, k_{d-1}, k)$ be an element of $W(G_s, S_{G_s})$, and we have

$$ww_1F_n^*(w)^{-1} = w_2.$$

Thus, w_1 is F_n^*-conjugate to w_2. This shows that the tuples $(s_1, \ldots, s_d)$ and $(t_1, \ldots, t_d)$ are F_n^* conjugate in $W(G_s, S_{G_s})$ if and only if $s_1s_2 \cdots s_d$ and $t_1t_2 \cdots t_d$ are conjugate in S_r. We deduce that any F_n^*-conjugacy class in $W(G_s, S_{G_s})$ has a representative $(s_1, \ldots, s_d)$ such that $s_1 = s_2 = \cdots = s_{d-1} = \mathrm{id}$, and we choose such a representatives for the tuples $(s_1, s_2, \ldots, s_d)$ and $(t_1, t_2, \ldots, t_d)$. Assume that s_d corresponds to the partition $(r_1, r_2, \ldots, r_k)$ of r. Then T_1 is an F_n-stable maximal torus of GL_n corresponding to the partition $(dr_1, dr_2, \ldots, dr_k)$. This shows the lemma in this case.

The part (2) of the lemma is similar to part (1). First assume that $(\ell, d) = 1$. Note that the Frobenius endomorphism F_n on G_s is the composition $\omega \circ \mathrm{F}_r$, where ω is the d-cycle $(1, 2, \ldots, d)$. Then we have the identity $F_n^\ell = \omega^\ell \circ F_r^\ell$. Since ℓ does not divide d, ω^ℓ is also a d-cycle. Using a similar argument as in part (1), we get that T_1 and T_2 are conjugate in $G_s^{F_n^\ell}$.

For part (3) of the lemma, using the above discussion for every $(F_n^*)^\ell$-conjugacy class in $W(G_s, S_{G_s})$, we can associate ℓ-partitions $(r_{i1}, r_{i2}, \ldots, r_{ik_i})$ of the positive integer d/ℓ, for $1 \leq i \leq \ell$; and the torus associated with this $(F_n^*)^\ell$-conjugacy class of $W(G_s, S_{G_s})$ is $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$-conjugate to a torus of GL_n corresponding to the partition

$$(dr_{11}/\ell, dr_{12}/\ell, \ldots, dr_{1k_1}/\ell, dr_{21}/\ell, dr_{22}/\ell, \ldots, dr_{2k_2}/\ell, \ldots, dr_{\ell 1}/\ell, dr_{\ell 2}/\ell, \ldots, dr_{lk_\ell}/\ell).$$

Now the lemma follows since $k_i = 1$, for $1 \leq i \leq \ell$ and $r_{i1} = r$. This completes the proof. □

5.2. Let $G \rtimes \Gamma$ be a finite group such that $\Gamma \simeq \mathbb{Z}/\ell\mathbb{Z}$, and we fix a generator σ of Γ. The following lemma, in certain cases, gives the trace of G^σ action on the Tate cohomology with respect to the action of Γ. This will be useful when we consider the Tate cohomology of a lattice in Shintani lifts to be discussed in the next subsection. For $x \in \Lambda$, we denote by $\overline{x}$ its image in $\Lambda/\ell\Lambda$.

Lemma 5.2.1. *Let M be a $\Lambda[G \rtimes \Gamma]$-module such that M is a free Λ-module of finite rank. Assume that $\widehat{\mathrm{H}}^1(\Gamma, M) = 0$. Then*

$$\mathrm{Tr}(g|_{\widehat{\mathrm{H}}^0(\Gamma,M)}) = \overline{\mathrm{Tr}(g|_M)},$$

for all $g \in G^\sigma$.

Proof. This lemma is a modification of a similar result proved in [BR96, Proposition 2.2] for Brauer characters (see [Ron16, Theorem 12] for a version of the above result). Since we crucially use it, we give a proof here. Note that the only indecomposable $\Lambda[\Gamma]$-modules are the trivial module Λ, $\Lambda[\Gamma]$ and the argumentation ideal I of $\Lambda[G]$. Since $\widehat{\mathrm{H}}^1(\Gamma, M)$ is trivial, we get that M is a direct sum of the $\Lambda[\Gamma]$-modules Λ and $\Lambda[\Gamma]$, and we get

$$\dim_{\overline{\mathbb{F}}_\ell} \widehat{\mathrm{H}}^0(\Gamma, M) \equiv \mathrm{rank}_\Lambda(M) \pmod{\ell}.$$

The above congruence implies the lemma when the order of g is a power of ℓ. Assume that $g = g_1 g_2 = g_2 g_1$, where the order of g_1 is prime to ℓ and the order of g_2 is a power of ℓ. Note that $g_1, g_2 \in G^\sigma$. We have the decomposition $M = \oplus_{\xi \in \Lambda^\times} M_\xi$, where M_ξ is the ξ-eigenspace for the action of g_1. The Λ-module M_ξ is stable under the action of g_2 and Γ. The lemma follows from the following congruences:

$$\mathrm{Tr}(g|_{\widehat{\mathrm{H}}^0(\Gamma,M)}) \equiv \bigoplus_{\xi \in \Lambda^\times} \mathrm{Tr}(g_2|_{\widehat{\mathrm{H}}^0(\Gamma,M_\xi)})\xi \equiv \bigoplus_{\xi \in \Lambda^\times} \mathrm{rank}_\Lambda(M_\xi)\xi \equiv \mathrm{Tr}(g|_M) \pmod{\ell}.$$

□

5.3. **Shintani correspondence.** Let σ be the Frobenius element in $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$. For any representation τ of $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$, we denote by τ^σ the composition of the representation τ with the automorphism of the group $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ induced by σ (which is the automorphism F_n). Also, for convenience, we denote by g^{F_n} the element $F_n(g)$. Let Π be an ℓ-adic irreducible representation of $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ such that Π is $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$ equivariant, i.e., $\Pi \simeq \Pi^\sigma$. In the article [Shi76, Theorem 1], Shintani proved that for a suitable choice of isomorphism $I_\sigma : \Pi \to \Pi^\sigma$, there exists an ℓ-adic irreducible representation π of $\mathrm{GL}_n(\mathbb{F}_q)$ such that

$$\mathrm{Tr}(I_\sigma \circ \Pi(g)) = \mathrm{Tr}\big(\pi(\mathrm{Norm}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(g))\big), \ g \in \mathrm{GL}_n(\mathbb{F}_{q^\ell}), \tag{5.3}$$

where $\mathrm{Norm}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(g)$ is the unique conjugacy class of $\mathrm{GL}_n(\mathbb{F}_q)$ containing an element conjugate to

$$g g^{F_n} \cdots g^{F_n^{\ell-1}}.$$

The association $\Pi \mapsto \pi$ gives a bijection from the set of isomorphism classes of $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$-stable irreducible ℓ-adic representation of $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ onto the set of isomorphism classes of irreducible ℓ-adic representations of $\mathrm{GL}_n(\mathbb{F}_q)$. The inverse of this bijection is called the Shintani base change lifting, denoted by $\mathrm{bc}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}$. Shintani correspondence can be described using Deligne–Lusztig theory. The character of an irreducible cuspidal ℓ-adic representation of $\mathrm{GL}_n(\mathbb{F}_q)$ is given by $(-1)^{n-1}R_{T,\theta}^{\mathrm{GL}_n}$, where T is the maximal F_n-stable torus of GL_n corresponding to an n-cycle in S_n. Let $\mathrm{Nr} : T^{F_n^\ell} \to T^{F_n}$ be the norm map. Then the character of Π is equal to the class function $(-1)^{n-\ell}R_{T,\theta\circ\mathrm{Nr}}^{\mathrm{GL}_n}$ on the group $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$. If π is a generic representation of $\mathrm{GL}_n(\mathbb{F}_q)$, then its Shintani lift Π is also generic (see [Zel23, Section 4.4] for a review on genericity and Shintani lifting).

Notation. *To simplify notations we denote by $R_{T,\theta}$ the virtual character $R_{T,\theta}^{\mathrm{GL}_n}$. We use $R_{T,\theta\circ\mathrm{Nr},\ell}$ for the virtual character $R_{T,\theta\circ\mathrm{Nr}}^{\mathrm{GL}_n}$ – this notation is chosen to signify the fact that the latter is a virtual character on the group $\mathrm{GL}_n^{F_n^\ell} = \mathrm{GL}_n(\mathbb{F}_{q^\ell})$.*

The following observation is well known, but we state it for later use.

Lemma 5.3.1. *Let π be an irreducible cuspidal ℓ-adic representation of $\mathrm{GL}_n(\mathbb{F}_q)$ defined over a finite extension L of $\mathbb{Q}_\ell$. Let Π be the Shintani lift of π. Then Π is defined over L.*

Proof. Let $\mathbb{Q}(\chi_\pi)$ be the field obtained by adjoining the values of the character χ_π to $\mathbb{Q}$. Since π is defined over L, we get that $\mathbb{Q}(\chi_\pi) \subset L$. Note that the Schur index of any absolutely irreducible L-representation of the group $\mathrm{GL}_n(\mathbb{F}_{q^k})$ is 1 (see [Ohm96, Main Theorem] or [Zel81, Proposition I2.6]). Note that the character of π is equal to $(-1)^{n-1}R_{T,\theta}$. Since the Green functions in the Deligne–Lusztig character formula take values in $\mathbb{Z}$, we get that $\mathbb{Q}(\chi_\pi)$ is equal to the field generated by the values of the character θ of T^{F_n}. Thus, the character values of the representation $\Pi = (-1)^{n-\ell}R_{T,\theta\circ\mathrm{Nr},\ell}$ are contained in $\mathbb{Q}(\chi_\pi)$. So, we get that the representation Π is defined over L. □

For the next two lemmas, we use the parametrisation of cuspidal representations in terms of semisimple conjugacy classes as in [Vig96, Chapter III Section 2], and also see the exposition in [Hel16, Section 5]. These references also describe the mod-ℓ reductions of cuspidal representations. In particular, we use the fact that the mod-ℓ-reduction of an irreducible ℓ-adic cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ is irreducible. This fact follows from the existence of Kirillov model (see [Vig96, Chapter III Théorème 1.1]). We will also need the fact that the ℓ-modular cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$ are parametrised by ℓ-regular parts of regular semisimple elements in $\mathrm{GL}_n(\mathbb{F}_q)$, and these aspects are discussed in [Vig96, Chapter III section 2.1].

Lemma 5.3.2. *Let $\overline{\pi}$ be an irreducible ℓ-modular supercuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$. Then $\overline{\pi}$ lifts to an integral cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_n(\mathbb{F}_q)$.*

Proof. The supercuspidal representation $\overline{\pi}$ corresponds to an ℓ-regular semisimple element $s \in \mathrm{GL}_n(\mathbb{F}_q)$ such that the characteristic polynomial of s is irreducible. Let π be an ℓ-adic cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ corresponding to the element s. Note that the character of π takes values in $\mathcal{K}$, and thus the representation π is defined over $\mathcal{K}$ (see [Ohm96, Main Theorem] or [Zel81, Proposition I2.6]). By definition, the mod-ℓ reduction of π is isomorphic to $\overline{\pi}$. □

Lemma 5.3.3. *Let π be an irreducible ℓ-adic cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ with supercuspidal mod-ℓ reduction. Let Π be the Shintani lifting of π to $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$. Then the cuspidal support of Π has supercuspidal mod-ℓ reduction. If the Shintani lifting is non-cuspidal, then*

$$\Pi \simeq \tau \times \tau^\sigma \times \cdots \times \tau^{\sigma^{\ell-1}},$$

where τ is cuspidal with $r_\ell(\tau)$ supercuspidal and $r_\ell(\tau)^\sigma \not\simeq r_\ell(\tau)$.

Proof. Let $s \in \mathrm{GL}_n(\mathbb{F}_q)$ be a semisimple element with irreducible characteristic polynomial (over $\mathbb{F}_q$) which corresponds to the representation π. Since the ℓ-modular reduction of π is supercuspidal, the ℓ-regular part s, denoted by s', has irreducible (over $\mathbb{F}_q$) characteristic polynomial. The cuspidal support of Π corresponds to the element $s \in \mathrm{GL}_n(\mathbb{F}_{q^\ell})$. The characteristic polynomial of $s \in \mathrm{GL}_n(\mathbb{F}_q)$, denoted by $f(t)$, is irreducible over $\mathbb{F}_q$. If $f(t)$ splits over $\mathbb{F}_{q^\ell}$, then it splits into ℓ-distinct irreducible factors, say

$$f(t) = f_0(t)f_1(t)\cdots f_{\ell-1}(t),$$

where $f_i(t) \in \mathbb{F}_{q^\ell}[t]$ is irreducible. Then the cuspidal support of Π is

$$(\prod_{i=0}^{\ell-1} \mathrm{GL}_{n/\ell}(\mathbb{F}_{q^\ell}), \bigotimes_{i=0}^{\ell-1} \pi_{s_i}),$$

where $s_i \in \mathrm{GL}_{n/\ell}(\mathbb{F}_{q^\ell})$ is a semisimple element with characteristic polynomial $f_i(t)$, and π_{s_i} is the cuspidal representation associated to s_i. Let $g(t)$ be the characteristic polynomial of s', and note that $g(t) \in \mathbb{F}_q[t]$ is irreducible as π has supercuspidal mod-ℓ reduction. Since s' commutes with s, the polynomial $g(t)$ splits over $\mathbb{F}_{q^\ell}$ and say

$$g(t) = g_0(t)g_1(t)\cdots g_{\ell-1}(t).$$

From the uniqueness of ℓ-regular parts, permuting s_i if necessary, we get that the ℓ-regular part of s_i, denoted by s_i', has characteristic polynomial $g_i(t)$. Thus, the mod-ℓ reduction of π_{s_i} is supercuspidal. Note that the characteristic polynomial of s' splits in $\mathbb{F}_{q^\ell}$ if and only if the characteristic polynomial of s splits. Hence, in the case where $f(t)$ remains irreducible over $\mathbb{F}_{q^\ell}$, i.e., when Π is cuspidal, we get that $r_\ell(\Pi)$ is supercuspidal. The last statement is clear from the proof. ☐

Proposition 5.3.4. *Let π and Π be as defined in the above lemma (5.3.1) with $L = \mathcal{K}$. Let $\Pi = \Pi_{\mathcal{K}} \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$, for some $\mathcal{K}$-representation $\Pi_{\mathcal{K}}$ of $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$. Then we have*

(1) $\Pi_{\mathcal{K}}^\sigma \simeq \Pi_{\mathcal{K}}$,
(2) $\Pi_{\mathcal{K}}$ extends uniquely as a representation of $\mathrm{GL}_n(\mathbb{F}_{q^\ell}) \rtimes \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$.
(3) If the mod-ℓ reduction of π is supercuspidal. Then, for any $\Lambda[\mathrm{GL}_n(\mathbb{F}_{q^\ell}) \rtimes \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)]$-stable lattice $\mathcal{L}$ in $\Pi_{\mathcal{K}}$, the space $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is irreducible and cuspidal ℓ-modular representation of $\mathrm{GL}_n(\mathbb{F}_q)$. Moreover, we have $\widehat{\mathrm{H}}^1(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is trivial.

Proof. The first part of the proposition is clear from the fact that Π is irreducible and $\Pi \simeq \Pi^\sigma$. Note that Π is generic and so is $\Pi_{\mathcal{K}}$. Then the second part is similar to Lemma 4.1.3. Let us prove the third part. Let $\mathcal{L}$ be a $\Lambda[\mathrm{GL}_n(\mathbb{F}_{q^\ell}) \rtimes \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)]$-stable lattice in $\Pi_{\mathcal{K}}$. Using similar arguments on Jacquet modules as in the proof of Theorem 4.2.1 along with Lemma 5.3.3, we get that $\widehat{\mathrm{H}}^i(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ are ℓ-modular cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$. Moreover, using Theorem 3.4.2 for finite fields, we get the following injective maps

$$\widehat{\mathrm{H}}^i(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})_{N_n(\mathbb{F}_q),\overline{\psi}} \hookrightarrow \widehat{\mathrm{H}}^i(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L}_{N_n(\mathbb{F}_{q^\ell}),\psi}) \tag{5.4}$$

where N_n is the unipotent radical of the upper triangular Borel subgroup of GL_n and ψ is a non-degenerate character of $N_n(\mathbb{F}_{q^\ell})$ such that the restriction of ψ to $N_n(\mathbb{F}_q)$ is non-degenerate and $\psi^\sigma = \psi$, for a generator σ of $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$. Since $\Pi_{\mathcal{K}}$ is generic, the twisted Jacquet module $\mathcal{L}_{N_n(\mathbb{F}_{q^\ell}),\psi}$ is a free Λ-module of rank 1. This implies that $\widehat{\mathrm{H}}^1(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L}_{N_n(\mathbb{F}_{q^\ell}),\psi})$ is trivial, and so is $\widehat{\mathrm{H}}^1(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})_{N_n(\mathbb{F}_q),\overline{\psi}}$ (by (5.4)). Since every cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ is generic, we get that $\widehat{\mathrm{H}}^1(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is trivial.

From (5.4), we get that

$$\dim_{\overline{\mathbb{F}}_\ell} \widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})_{N_n(\mathbb{F}_q),\overline{\psi}} \leq 1.$$

Therefore, for proving the irreducibility of $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$, it remains to show that the space $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is non-zero. To get a contradiction, assume that $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is trivial, then it follows from Lemma 5.2.1 that the character $\chi_{\Pi_{\mathcal{K}}}$ of $\Pi_{\mathcal{K}}$ is divisible by ℓ, for all $g \in \mathrm{GL}_n(\mathbb{F}_q)$. Note that we have the equality of the class functions on $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$

$$\chi_{\Pi_{\mathcal{K}}} = (-1)^{n-\ell} R_{T,\theta\circ\mathrm{Nr},\ell}.$$

We now obtain a contradiction since

$$R_{T,\theta\circ\mathrm{Nr},\ell}(u) = R_{T,1,\ell}(u) = 1$$

where u is a regular unipotent element in $\mathrm{GL}_n(\mathbb{F}_q)$ (see [DL76, Theorem 9.16]). ☐

5.4. **Congruence of characters.** In this subsection, we prove certain congruences between the character of a cuspidal representation and the character of its Shintani lifting to $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$, when evaluated at an element of $\mathrm{GL}_n(\mathbb{F}_q)$. We begin with the following preliminary lemma. For any representation ρ, we denote by χ_ρ its character.

Lemma 5.4.1. *Let τ be a representation of $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ over a $\mathcal{K}$-vector space. Let*

$$\begin{aligned}\eta =&\tau\otimes\tau^{\sigma}\otimes\cdots\otimes\tau^{\sigma^{\ell-1}},\\ \iota(g) =&(g, g^{F_n},\ldots, g^{F_n^{\ell-1}}).\end{aligned}$$

Then, we have

$$\sum_{w\in S_\ell}\chi_\eta^{(w)}(\iota(g))\equiv\sum_{i=0}^{\ell-1}\chi_\tau(g^{F_n^i})^\ell \ mod\ (\ell),$$

for all $g\in\mathrm{GL}_n(\mathbb{F}_{q^\ell})$, where $\chi_\eta^{(w)}$ is the character of η–twisted by a permutation w acting on the factors of the tensor product, for all $w\in S_\ell$ (S_ℓ is the symmetric group on ℓ-symbols).

Proof. For any $w\in S_\ell$, we have the following character identity

$$\chi_\eta^{(w)}(g_0,\ldots,g_{\ell-1})=\chi_\eta\left(g_{w^{-1}(0)},\ldots,g_{w^{-1}(\ell-1)}\right)=\prod_{i=0}^{\ell-1}\chi_\tau^{F_n^i}(g_{w^{-1}(i)}),$$

for all $g_i\in\mathrm{GL}_n(\mathbb{F}_{q^\ell})$. Then

$$\sum_{w\in S_\ell}\chi_\eta^{(w)}(\iota(g))=\sum_{w\in S_\ell}\Big(\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{i+w(i)}})\Big). \tag{5.5}$$

Let X denote the set of functions

$$\{\mathrm{id}+w:\mathbb{Z}/\ell\mathbb{Z}\to\mathbb{Z}/\ell\mathbb{Z}\mid w\in S_\ell\},$$

and let w_0 be the ℓ-cycle $(0,1,\ldots,\ell-1)$. Note that the obvious action of w_0 on the space of functions $\{f:\mathbb{Z}/\ell\mathbb{Z}\to\mathbb{Z}/\ell\mathbb{Z}\}$ given by

$$(w_0f)(x)=f(w_0(x))$$

preserves the set X; this can be seen from the following identity:

$$(w_0(\mathrm{id}+w))(i)=i+w'(i),$$

where w' is the permutation of $\mathbb{Z}/\ell\mathbb{Z}$ defined by $w'=1+ww_0$. Now, for $w\in S_\ell$, the permutation $\mathrm{id}+w$ belongs to the fixed point set X^{w_0} if and only if there exists an unique integer $k\in\mathbb{Z}/\ell\mathbb{Z}$ such that

$$i+w(i)=k,\ 0\leq i\leq\ell-1.$$

Therefore, from the identity (5.5), we get

$$\sum_{w\in S_\ell}\chi_\eta^{(w)}(\iota(g))=\sum_{i=0}^{\ell-1}\chi_\tau(g^{F_n^i})^\ell+\sum_{\mathrm{id}+w\in X\setminus X^{w_0}}\Big(\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{i+w(i)}})\Big). \tag{5.6}$$

Since $\langle w_0\rangle$ acts freely on $X\setminus X^{w_0}$, there exists a subset $U\subseteq X\setminus X^{w_0}$ such that

$$X\setminus X^{w_0}=\coprod_{r=0}^{\ell-1}w_0^rU.$$

Then we have

$$\sum_{\mathrm{id}+w\in X\setminus X^{w_0}}\Big(\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{(\mathrm{id}+w)(i)}})\Big)=\sum_{r=0}^{\ell-1}\sum_{\mathrm{id}+w\in U}\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{[w_0^r(\mathrm{id}+w)](i)}}). \tag{5.7}$$

Recall that $[w_0^r(\mathrm{id}+w)](i)=(i+r)+w(i+r)$, $0\leq r\leq \ell-1$, and we have

$$\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{[w_0^r(\mathrm{id}+w)](i)}})=\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{i+w(i)}}).$$

Using this identity to the equation (5.7), we get

$$\sum_{\mathrm{id}+w\in X\setminus X^{w_0}}\Big(\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{i+w(i)}})\Big)=\ell\sum_{\mathrm{id}+w\in U}\prod_{i=0}^{\ell-1}\chi_\tau(g^{F_n^{i+w(i)}}).$$

Then it follow from the equation (5.6), we get

$$\sum_{w\in S_\ell}\chi_\eta^{(w)}(\iota(g))\equiv\sum_{i=0}^{\ell-1}\chi_\tau(g^{F_n^i})^\ell \bmod (\ell).$$

Hence the lemma. □

Proposition 5.4.2. *Let π be an absolutely irreducible cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_n(\mathbb{F}_q)$ such that the mod-ℓ reduction of π is supercuspidal. Let Π be the absolutely irreducible $\mathcal{K}$-representation of $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ such that $\Pi\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ is the Shintani lift of $\pi\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ (such Π exists by Lemma 5.3.1). Let $\mathcal{L}$ be a $\Lambda[\mathrm{GL}_n(\mathbb{F}_{q^\ell})\rtimes\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)]$-stable lattice in Π. Then we have*

$$\mathrm{Tr}(g|_{\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q),\mathcal{L})})=\overline{\chi_\pi(g)}^{\ell},$$

for all $g\in\mathrm{GL}_n(\mathbb{F}_q)$.

Proof. Since $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q),\mathcal{L})$ is irreducible and cuspidal (from part (3) of Proposition 5.3.4), it is sufficient to show that

$$\mathrm{Tr}(g|_{\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q),\mathcal{L})})=\overline{\chi_\pi(g)}^{\ell}, \tag{5.8}$$

for all g with Jordan decomposition $g=su$, where the characteristic polynomial of s is of the form $f(X)^r$ for some irreducible polynomial $f(X)\in\mathbb{F}_q[X]$; let $\mathcal{S}$ be the set of all such elements $g\in\mathrm{GL}_n(\mathbb{F}_q)$. Since the space $\widehat{\mathrm{H}}^1(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q),\mathcal{L})$ is trivial (from part (3) of Proposition 5.3.4), it follows from Lemma 5.2.1 that the equality (5.8) is equivalent to the following identity

$$\overline{\chi_\Pi(g)}=\overline{\chi_\pi(g)}^{\ell},\ g\in\mathcal{S}.$$

For simplicity the Frobenius morphism F_n on GL_n will be denoted by F in this proof. We use Deligne–Lusztig character formulas for χ_Π and χ_π. Let S_0 be an F-stable split maximal torus of GL_n. Let T be an F-stable maximal torus of GL_n such that (T,F) is $\mathrm{GL}_n(\mathbb{F}_q)$ conjugate to $(S_0,\mathrm{Ad}(w_0)\circ F)$, where w_0 is an n-cycle in $W(\mathrm{GL}_n,S_0)$. Let $\theta:T^F\to\mathcal{K}^\times$ be a character in general position such that $\pi=(-1)^{n-1}R_{T,\theta}$. Let $\mathrm{Nr}:T^{F^\ell}\to T^F$ be the norm map

$$t\mapsto tt^Ft^{F^2}\dots t^{F^{\ell-1}}.$$

Let η be the composite

$$T^{F^\ell}\xrightarrow{\mathrm{Nr}}T^F\xrightarrow{\theta}\overline{\mathbb{Q}}_\ell^\times.$$

The character of the Shintani lift Π of π to $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ is equal to $\chi_\Pi=(-1)^{n-\ell}R_{T,\eta,\ell}$ (see [Kaw87] for a reference). Here $R_{T,\eta,\ell}$ is the Deligne–Lusztig induction of the character η of T^{F^ℓ} to G^{F^ℓ}.

First, we assume that $n = \ell m$ for some integer m. We identify T^{F^ℓ} with $(\mathbb{F}_{q^n}^\times)^\ell$ and T^F is identified with $\mathbb{F}_{q^n}^\times$ such that Nr is given by

$$(t_1, t_2, \ldots, t_\ell) \mapsto t_1 t_2^F \ldots t_\ell^{F^{\ell-1}},\ t_i \in \mathbb{F}_{q^n}^\times.$$

Let P be an F^ℓ-stable parabolic subgroup containing T, and let M be an F^ℓ-stable Levi subgroup of P containing T. We can identify the group M^{F^ℓ} with $\mathrm{GL}_{n/\ell}(\mathbb{F}_{q^\ell})^\ell$ and $\Pi = (-1)^{n-\ell} R_{T,\eta,\ell}$ with the parabolic induction

$$\tau \times \tau^\sigma \times \cdots \times \tau^{\sigma^{\ell-1}},$$

where τ is the irreducible cuspidal representation of $\mathrm{GL}_{n/\ell}(\mathbb{F}_{q^\ell})$, obtained as the Deligne–Lusztig induction of the character θ of $\mathbb{F}_{q^n}^\times$ (which is considered as F^ℓ-fixed points of a F^ℓ-stable maximal torus of $\mathrm{GL}_{n/\ell}$ corresponding to an n/ℓ-cycle).

Let $s \in T^F$ be an element with characteristic polynomial $f(X)^r$, where $f(X) \in \mathbb{F}_q[X]$ is an irreducible polynomial of degree d. We have $n = dr$. Let G_s be the centralizer $C_{\mathrm{GL}_n}(s)$. Let $u \in G_s^F$ be an unipotent element. From the Deligne–Lusztig character formula (5.1) and Lemma 5.1.1, we get that

$$\chi_\Pi(su) = Q_{G_s,T,F^\ell}(u) \sum_{w \in W(G,T)^{F^\ell}/W(G_s,T)^{F^\ell}} \eta(wsw^{-1}).$$

We begin with the case where $\ell \mid d$. Let $f = f_1 f_2 \cdots f_\ell$ be the splitting of f, where $f_i(X) \in \mathbb{F}_{q^\ell}[X]$ is an irreducible polynomial. With the above identification, the element $s \in T^F \subset M^{F^\ell}$ is of the form

$$s = (s_0, s_1, \ldots, s_{\ell-1}), s_i \in \mathbb{F}_{q^{d/\ell}},\ 0 \leq i \leq \ell - 1,$$

where s_i, considered as an element of $\mathrm{GL}_{n/\ell}(\mathbb{F}_{q^\ell})$, has characteristic polynomial $f_i(X)^r$. Thus, we have

$$F_{n/\ell}(s_0) = s_1,\ F_{n/\ell}(s_1) = s_2, \ldots, F_{n/\ell}(s_{\ell-1}) = s_0.$$

Note that $G_s^{F^\ell}$ can be identified with $\mathrm{GL}_r(\mathbb{F}_{q^d})^\ell$ and the element $u \in G_s^F \subset G_s^{F^\ell}$ with

$$(u_1, u_2, \ldots, u_\ell)$$

where u_i is the unipotent element of $\mathrm{GL}_r(\mathbb{F}_{q^d})$ conjugate to u. Thus we have

$$Q_{G_s,T,F^\ell}(u) = Q_{G_s,T,F}(u)^\ell.$$

Let γ be the Frobenius element of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_{q^\ell})$, then the set of conjugates

$$\begin{aligned}&\{wsw^{-1} : w \in W(G,T)^{F^\ell}/W(G_s,T)^{F^\ell}\}\\ &= \{(\gamma^{k_0}(s_{\alpha(0)}), \gamma^{k_1}(s_{\alpha(1)}), \ldots, \gamma^{k_{\ell-1}}(s_{\alpha(\ell-1)}) : 0 \leq k_i \leq d/\ell,\ \alpha \in S_\ell\}.\end{aligned}$$

We observe that

$$\begin{aligned}&\eta((\gamma^{k_0}(s_{\alpha(0)}), \gamma^{k_1}(s_{\alpha(1)}), \ldots, \gamma^{k_{\ell-1}}(s_{\alpha(\ell-1)})))\\ &= \eta((\gamma^{k_1}(F(s_{\alpha(1)})), \gamma^{k_2}(F(s_{\alpha(2)})), \ldots, \gamma^{k_{\ell-1}}(F(s_{\alpha(\ell-1)})), \gamma^{k_0}(F(s_{\alpha(0)})))).\end{aligned}$$

Let Y be the set

$$\{(\gamma^k(s_{\alpha(0)}), \gamma^k(s_{\alpha(1)}), \ldots, \gamma^k(s_{\alpha(\ell-1)}) : 0 \leq k \leq d/\ell,\ \alpha \in S_\ell\}.$$

We deduce that

$$\sum_{w \in W(G,T)^{F^\ell}/W(G_s,T)^{F^\ell}} \eta(wsw^{-1}) \equiv \sum_{wsw^{-1} \in Y} \eta(wsw^{-1}) \bmod (\ell).$$

Using Lemma 5.4.1, we get that

$$Q_{G_s,T,F}(u)^\ell \sum_{wsw^{-1}\in\in Y} \eta(wsw^{-1}) \equiv Q_{G_s,T,F}(u)^\ell \sum_{0\leq i\leq d} \theta(s^{q^i})^\ell \text{ mod } (\ell).$$

This shows the proposition in this case.

We now assume that $(d,\ell)=1$. In this case, we have

$$\chi_\pi(su) = Q_{G_s,T,F^\ell}(u) \sum_{w\in W(G,T)^{F^\ell}/W(G_s,T)^{F^\ell}} \eta(wsw^{-1}).$$

Note that

$$Y = \{wsw^{-1} : w\in W(G,T)^{F^\ell}/W(G_s,T)^{F^\ell}\} = \{(s^{F^{\alpha(0)}}, s^{F^{\alpha(1)}},\dots, s^{F^{\alpha(\ell-1)}}) : \alpha\in S_\ell\}.$$

Using Lemma 5.4.1, we get that

$$\sum_{y\in Y}\eta(y) = \sum_{0\leq i\leq d}\theta(s^{q^i})^\ell \text{ mod } (\ell).$$

We must now analyse the Green function, for which we first recall the following identifications.

$$G_s^F = \mathrm{GL}_r(\mathbb{F}_{q^d}),\ G_s^{F^\ell} = \mathrm{GL}_r(\mathbb{F}_{q^{d\ell}}), T^F = \mathbb{F}_{q^n}^*, T^{F^\ell} = (\mathbb{F}_{q^n}^*)^\ell.$$

Assume that $\mu=(\mu_1,\mu_2,\dots,\mu_k)$ is the partition associated to the conjugacy class of u in G_s^F. Then μ is the partition of the $G_s^{F^\ell}$ conjugacy class of u. The partition associated with T^F is (n) and $(n/\ell, n/\ell,\dots,n/\ell)$ is the partition of T^{F^ℓ}. Let $w\in S_n$ be an n-cycle. It follows from Springer that

$$Q_{G_s,T,F}(u) = \sum_{i\geq 0} a_i(w,u)q^i,$$

and

$$Q_{G_s,T,F^\ell}(u) = \sum_{i\geq 0} a_i(w^\ell,u)q^{i\ell},$$

where $g\mapsto a_i(g,u)$, for $g\in S_n$, is a character of a representation of S_n. Clearly, we have

$$a_i(w,u)^\ell \equiv a_i(w^\ell,u) \text{ mod } (\ell).$$

From which we get that

$$Q_{G_s,T,F^\ell}(u) \equiv Q_{G_s,T,F}(u)^\ell \text{ mod } (\ell).$$

We thus conclude in all cases

$$\chi_\Pi(su) \equiv \chi_\pi(su)^\ell \text{ mod } (\ell).$$

In the case where $(\ell,n)=1$, the above proposition is proved by Ronchetti ([Ron16, Theorem 13]). For the uniformity of the proof, we treat this case using our method. For $g\in\mathcal{S}$, we have

$$R_{T,\eta,\ell}(g) = Q_{G_s,T,F^\ell}(u) \sum_{w\in W(G,T)^{F^\ell}/W(G_s,T)^{F^\ell}} \theta(wsw^{-1})$$

and in this case, we have

$$G_s^F = \mathrm{GL}_r(\mathbb{F}_q),\ G_s^{F^\ell} = \mathrm{GL}_r(\mathbb{F}_{q^{d\ell}}),\ T^F = \mathbb{F}_{q^\ell},\ T^{F^\ell} = \mathbb{F}_{q^{n\ell}}$$

and we have

$$Q_{G_s,T,F^\ell}(u) = \sum_{i\geq 0} a_i(w,u)q^{i\ell} \equiv (\sum_{i\geq 0} a_i(w,u)q^i)^\ell \text{ mod } (\ell).$$

Then we have

$$\sum_{w\in W(G,T)^{F^\ell}/W(G_s,T)^{F^\ell}} \theta(wsw^{-1}) = \sum_{i=1}^d \theta(s^{q^{i\ell}}) \equiv \left(\sum_{i=1}^d \theta(s^{q^i})\right)^\ell \text{ mod } (\ell)$$

□

The above character computations give the following theorem.

Theorem 5.4.3. *Let π be an absolutely irreducible cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_n(\mathbb{F}_q)$ such that the mod-ℓ reduction of π is supercuspidal. Let Π be the $\mathcal{K}$-representation of $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$ such that $\Pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ is the Shintani lift of $\pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$. Then for any $\Lambda[\mathrm{GL}_n(\mathbb{F}_{q^\ell}) \rtimes \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)]$-stable lattice $\mathcal{L}$ in Π, the Tate cohomology $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is isomorphic to the Frobenius twist $r_\ell(\pi)^{(\ell)}$.*

Proof. To prove the theorem, we use the fact that two irreducible ℓ-modular representations of a finite group are isomorphic if and only if they have the same character ([Lam01, Corollary 7.21]). Using Proposition 5.4.2, we get the character identity

$$\mathrm{Tr}(g|_{\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q),\mathcal{L})}) = \overline{\chi_\pi(g)}^{\ell},$$

for all $g \in \mathrm{GL}_n(\mathbb{F}_q)$. Note that the Tate cohomology group $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is irreducible from Proposition 5.3.4, and the Frobenius twist $r_\ell(\pi)^{(\ell)}$ is irreducible since the mod-ℓ reduction of an absolutely irreducible cuspidal $\mathcal{K}$-representation is irreducible (see [Vig96, Chapter III, Théorème 1.1(d)]). Thus, the Tate cohomology $\widehat{\mathrm{H}}^0(\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q), \mathcal{L})$ is isomorphic to $r_\ell(\pi)^{(\ell)}$ as $\mathrm{GL}_n(\mathbb{F}_q)$-representations. Hence the theorem. □

Remark 5.4.4. *In this section we only determine the Tate cohomology of a lattice in Shintani lifting of a cuspidal with supercuspidal mod-ℓ reduction. In the next section, we deduce a theorem on Tate cohomology of some lattices in base change lifts of generic depth-zero representations. The above theorem will be the first step in the inductive argument.*

6. Lifting to p-adic groups

In this section, we give an application of Theorem 5.4.3 for computing the Tate cohomology of depth-zero generic smooth irreducible representations of $\mathrm{GL}_n(F)$, where F is a p-adic field.

6.1. Local Langlands correspondence and base change.

We begin with a quick recap of local Langlands correspondence and base change lifting to fix some notations.

6.1.1. For a finite extension K of $\mathbb{Q}_p$, we denote by $\mathcal{W}_K$ the Weil group of K and let ν_K be the modulus character of $\mathcal{W}_K$ (see [BH06, Chapter 7] for the notations and details on Weil-groups and inertial and wild inertial subgroups). Let $\mathcal{A}(n, K)$ be the set of isomorphism classes of irreducible ℓ-adic smooth representations of $\mathrm{GL}_n(K)$. Let $\mathcal{G}(n, K)$ be the set of isomorphism classes of pairs (ρ, N), where ρ is a semisimple n-dimensional smooth ℓ-adic representation of $\mathcal{W}_K$ and N is an element in $\mathrm{Hom}_{\mathcal{W}_K}(\rho, \rho\nu_K)$. The pair (ρ, N) is called a Frobenius semisimple Weil–Deligne representation. The ℓ-adic local Langlands correspondence is a bijection

$$\mathrm{LLC}_K : \mathcal{A}(n, K) \to \mathcal{G}(n, K),$$

which is uniquely characterized by certain identities of local constants that one attaches to the elements of $\mathcal{A}(n, K)$ and $\mathcal{G}(n, K)$ (see [HT01], [Hen00], [Sch13]). Moreover, the set of irreducible ℓ-adic cuspidal representations of $\mathrm{GL}_n(K)$ are mapped onto the set of pairs $(\rho, 0)$, where ρ is an n-dimensional irreducible ℓ-adic representations of $\mathcal{W}_K$ via LLC_K.

6.1.2. Note that the classical local Langlands correspondence is a bijection between the set of isomorphism classes of irreducible smooth complex representations of $\mathrm{GL}_n(K)$ and the set of isomorphism classes of n-dimensional complex semisimple Weil–Deligne representations. To get a correspondence over $\overline{\mathbb{Q}}_\ell$, one twists the original correspondence by the character $\nu^{(1-n)/2}$, where ν is the composition of the determinant character of $\mathrm{GL}_n(K)$ and the absolute value of K. This twisting makes the following equivariance property, which is very useful for rationality questions:

$$\mathrm{LLC}_K(\pi^\gamma) = \mathrm{LLC}_K(\pi)^\gamma,$$

for all $\gamma \in \mathrm{Aut}(\overline{\mathbb{Q}}_\ell)$. For details, see [Clo90, Conjecture 4.4, Section 4.2] and [Hen01, Section 7].

6.1.3. Let π be an irreducible ℓ-adic representation of $\mathrm{GL}_n(F)$. Let P be an F-rational parabolic subgroup of GL_n with Levi subgroup M and let τ be an irreducible cuspidal ℓ-adic representation of $M(F)$ such that π occurs as a subrepresentation of $i_{P(F)}^{\mathrm{GL}_n(F)}\tau$. Note that $(M(F), \tau)$ is unique up to $\mathrm{GL}_n(F)$-conjugation and the equivalence class of $(M(F), \tau)$ for $\mathrm{GL}_n(F)$-conjugation is called the cuspidal support of π. Let (ρ, N) be the Weil–Deligne representation $\mathrm{LLC}_F(\pi)$. The cuspidal support of π corresponds to the representation ρ, i.e., say

$$\rho \simeq \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k,$$

where ρ_i is an irreducible ℓ-adic representation of $\mathcal{W}_F$, then $M(F)$ is isomorphic to $\prod_{i=1}^k \mathrm{GL}_{n_i}(F)$ and $\tau \simeq \otimes_{i=1}^k \tau_i$, where $\mathrm{LLC}_F(\tau_i) = \rho_i$.

6.1.4. Let F be a finite extension of $\mathbb{Q}_p$, and let E/F be a finite Galois extension such that $[E : F] = \ell$, where ℓ is a prime and $\ell \neq p$. Let π be an irreducible smooth ℓ-adic representation of $\mathrm{GL}_n(F)$. Fixing an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, the Arthur–Clozel's base change lifting gives a map from irreducible smooth ℓ-adic representations of $\mathrm{GL}_n(F)$ to those of $\mathrm{GL}_n(E)$:

$$\mathrm{bc}_{E/F} : \mathcal{A}(n, F) \to \mathcal{A}(n, E),$$

the representation $\Pi = \mathrm{bc}_{E/F}(\pi)$ is characterised by the Shintani character identity (see [AC89, Section 6, Theorem 6.2(a)]). In terms of local Langlands correspondence, Π is the unique irreducible ℓ-adic representation of $\mathrm{GL}_n(E)$ such that

$$\mathrm{res}_{\mathcal{W}_E}\big(\mathrm{LLC}_F(\pi)\big) \simeq \mathrm{LLC}_E(\Pi).$$

We refer to [Hen01, Section 7] for a discussion on these properties. If $\Pi = \mathrm{bc}_{E/F}(\pi)$, then we get that $\Pi \simeq \Pi^\sigma$ for all $\sigma \in \mathrm{Gal}(E/F)$. Moreover, any irreducible smooth ℓ-adic representation Π such that $\Pi^\sigma \simeq \Pi$, is equal to $\mathrm{bc}_{E/F}(\pi)$, for some $\pi \in \mathcal{A}(n, F)$. (see [AC89, Theorem 6.2 and Section 6.4]).

Lemma 6.1.1. *Let π be an irreducible cuspidal integral ℓ-adic representation of $\mathrm{GL}_n(F)$ such that $r_\ell(\pi)$ is supercuspidal. Let Π be the base change lifting of π to $\mathrm{GL}_n(E)$. Assume that Π is non-cuspidal, i.e.,*

$$\Pi \simeq \tau \times \tau^\sigma \times \cdots \times \tau^{\sigma^{\ell-1}},$$

where τ is an irreducible integral cuspidal representation with $\tau^\sigma \not\simeq \tau$. Then, the representation $r_\ell(\tau)$ is supercuspidal and $r_\ell(\tau)^\sigma \not\simeq r_\ell(\tau)$.

Proof. Let ρ be the irreducible smooth ℓ-adic representation of $\mathcal{W}_F$ associated to π via the local Langlands correspondence. Note that ρ is integral. When the restriction $\mathrm{Res}_{\mathcal{W}_E}\rho$ is reducible, we have

$$\mathrm{Res}_{\mathcal{W}_E}\rho \simeq \rho_1 \oplus \rho_1^\sigma \oplus \cdots \rho_1^{\sigma^{\ell-1}},$$

for some irreducible representation ρ_1 of $\mathcal{W}_E$. In other words, ρ is isomorphic to $\mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}\rho_1$. The cuspidal support of Π is

$$(\mathrm{GL}_{n/\ell}(E)^{\ell}, \bigotimes_{i=0}^{\ell-1} \tau^{\sigma^i}),$$

where $\mathrm{LLC}(\tau) = \rho_1$. Since $r_\ell(\pi)$ is supercuspidal, we get that $r_\ell(\rho)$ is irreducible [Vig01, Théorème 1.6]. Thus, we get that $r_\ell(\rho_1)$ is irreducible and $r_\ell(\tau)$ is supercuspidal. Note that $\mathrm{Res}_{\mathcal{W}_E} r_\ell(\rho)$ is semisimple, and hence multiplicity free as $\mathcal{W}_F/\mathcal{W}_E$ is cyclic. Thus, we get that $r_\ell(\rho_1)^\sigma \not\simeq r_\ell(\rho_1)$, and from mod-$\ell$ local Langlands correspondence we get that $r_\ell(\tau)^\sigma \not\simeq r_\ell(\tau)$. □

6.1.5. In this paragraph, we will show that the base change lift of a depth-zero representation is a depth-zero representation. Let us recall some notations. Let $K_0(F)$ and $K_1(F)$ be the compact open subgroups $\mathrm{GL}_n(\mathfrak{o}_F)$ and $I_n + \mathfrak{p}_F M_n(\mathfrak{o}_F)$ respectively, of the group $\mathrm{GL}_n(F)$, where I_n is the $n \times n$ identity matrix. A smooth irreducible representation (π, V) of $\mathrm{GL}_n(F)$, where V is a vector space either over $\overline{\mathbb{Q}}_\ell$, $\mathcal{K}$ or $\overline{\mathbb{F}}_\ell$, is said to be of depth-zero if $V^{K_1(F)} \neq 0$.

Let π be an ℓ-adic depth-zero representation. Let (ρ, N) be the Weil–Deligne representation $\mathrm{LLC}_F(\pi)$ and we use the notations as in paragraph 6.1.3. The cuspidal support (M, τ) of π is of depth-zero, i.e., if $\pi^{K_1(F)}$ is non-trivial, then $\tau^{K_1(F)\cap M}$ is non-trivial (see [MP96, Theorem 5.2]). Hence, the representation τ_i is a depth-zero cuspidal representation of $\mathrm{GL}_{n_i}(F)$. From the compatibility of local ϵ-factors under local Langlands correspondence for irreducible ℓ-adic cuspidal representations, we get that ρ_i is trivial on the wild inertial subgroup $\mathcal{P}_F$ of the Weil group $\mathcal{W}_F$ (see [BH14, Section 1] for a convenient reference). Since $[E : F] = \ell$, we get that the representation $\mathrm{Res}_{\mathcal{W}_E}\rho_i$ is trivial on the $\mathcal{P}_E$ (which is equal to $\mathcal{P}_F$) the wild inertial subgroup of $\mathcal{W}_E$. Thus, the cuspidal support of the representation $\Pi = \mathrm{bc}_{E/F}(\pi)$ is of depth-zero and applying [MP96, Theorem 5.2] we get that Π is a depth-zero representation.

6.2. **Rational structures.** We have to address some rationality questions before we proceed further.

6.2.1. The following lemma is well known and we state it for convenience of reference.

Lemma 6.2.1. *Let (π, V) be a smooth irreducible generic ℓ-adic representation of $\mathrm{GL}_n(F)$ such that $\pi^{(\gamma)} \simeq \pi$, for all $\gamma \in \mathrm{Aut}(\overline{\mathbb{Q}}_\ell/\mathcal{K})$, then π is defined over $\mathcal{K}$. In particular, the base change lifting of π to $\mathrm{GL}_n(E)$ is defined over $\mathcal{K}$.*

Proof. Let $\gamma \in \mathrm{Aut}(\overline{\mathbb{Q}}_\ell/\mathcal{K})$. If $W : V \to \overline{\mathbb{Q}}_\ell$ is a non-zero Whittaker functional, then $W^{(\gamma)}$, defined by setting

$$W^{(\gamma)}(v) = \gamma(W(v)),\ v \in V$$

is a Whittaker functional on $\pi^{(\gamma)}$. Let $\mathbb{W}(\pi, \Theta_F)$ be the Whittaker model of π, where Θ_F is a non-degenerate character of $N_n(F)$–the group of unipotent upper triangular matrices in $\mathrm{GL}_n(F)$. For $v \in V$, we denote by $W_v(g)$, the function $g \mapsto W(\pi(g)v)$, and let $W_v^{(\gamma)}$ be the function $g \mapsto \gamma(W(\pi(g)v))$. By multiplicity one theorem for Whittaker functionals, we have $W_v^{(\gamma)} \in \mathbb{W}(\pi, \Theta_F)$, and the map

$$W_v \mapsto W_v^{(\gamma)},\ v \in V$$

defines a γ-semilinear map on $\mathbb{W}(\pi, \Theta_F)$. As the representation π is admissible, from Hilbert 90, the representation π is defined over $\mathcal{K}$. Note that $\mathrm{LLC}_F(\pi)$ is fixed by twisting with $\gamma \in \mathrm{Aut}(\overline{\mathbb{Q}}_\ell/\mathcal{K})$. Therefore, $\mathrm{LLC}_E(\Pi)$ is fixed by $\mathrm{Aut}(\overline{\mathbb{Q}}_\ell/\mathcal{K})$, and the lemma follows from the fact that Π is generic. □

6.2.2. We will need $\mathcal{K}$-rational structures of the cuspidal support of π and that of $\Pi = \mathrm{bc}_{E/F}(\pi)$. Let π be an irreducible ℓ-adic representation of $\mathrm{GL}_n(F)$ which is defined over $\mathcal{K}$. Let (ρ, N) be the Weil–Deligne representation associated to π via the local Langlands correspondence. Since π is defined over $\mathcal{K}$, the representation ρ is fixed by $\mathrm{Aut}(\overline{\mathbb{Q}}_\ell/\mathcal{K})$. Thus, the representation τ is defined over $\mathcal{K}$ by Lemma 6.2.1. From the above relation between cuspidal support and the associated Weil–Deligne representation, the base change lifting Π of the representation π to $\mathrm{GL}_n(E)$ occurs as a subrepresentation of $i_{P(E)}^{\mathrm{GL}_n(E)}\xi$, where ξ is the base change lifting of τ. Now, applying Lemma 6.2.1, the representation ξ is defined over $\mathcal{K}$. A $\mathcal{K}$-rational structure of ξ gives a $\mathcal{K}$-rational structure on Π as well.

6.2.3. We will show that supercuspidal ℓ-modular representations can be lifted to an integral $\mathcal{K}$-representation. This will be a key input for the proof of the concluding theorem of this section Theorem 6.4.3.

Lemma 6.2.2. *Let $\overline{\pi}$ be an ℓ-modular supercuspidal representation of $\mathrm{GL}_n(F)$. Then $\overline{\pi}$ lifts to an integral $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$.*

Proof. We use Bushnell–Kutzko's type theory to prove this lemma. Let $\mathfrak{r} = [\mathfrak{A}, n, 0, \beta]$ be a simple stratum contained in the representation $\overline{\pi}$, and set $L = F[\beta]$. Let (J, λ) be a maximal simple type constructed from a simple character θ associated with the strata $\mathfrak{r}$. Note that J contains a normal pro-p subgroup J^1 such that $J/J^1 = \mathrm{GL}_m(k_L)$, where $m = n/[L : F]$. There exists an irreducible representation κ of J constructed from the simple character θ, and a cuspidal representation τ of J/J^1 such that $\lambda = \kappa \otimes \tau$; under the assumption that $\overline{\pi}$ is supercuspidal, the representation τ is a supercuspidal representation [Vig96, III 4.24 and III.5.14(3)]. The group $L^\times$ normalises J and $\kappa \otimes \tau$–and it extends to a representation ξ of the group $L^\times J$ such that

$$\overline{\pi} \simeq \mathrm{ind}_{L^\times J}^{\mathrm{GL}_n(F)}\xi.$$

It follows from [Hel16, Lemma 4.7] that the representation κ of J can be lifted to a $\mathcal{K}$-representation. Since τ is a supercuspidal representation, the representation τ can be lifted to a $\mathcal{K}$-representation of the group J (see Lemma 5.3.2). We denote the $\mathcal{K}$-liftings of κ and τ by $\tilde{\kappa}$ and $\tilde{\tau}$ respectively. We will show that the representation $\widetilde{\kappa} \otimes \widetilde{\tau}$ of the group J extends to a $\mathcal{K}$-representation $\widetilde{\xi}$ of $L^\times J$ such that the mod-ℓ reduction of $\widetilde{\xi}$ is equal to ξ. The uniformiser ϖ_L of $L^\times$ generates the quotient $L^\times J/F^\times J$. Since ϖ_L normalises $\kappa \otimes \tau$, it acts as a scalar say x. Let $\widetilde{x} \in \Lambda$ be a lift of the element $x \in \overline{\mathbb{F}}_\ell$. We set the action of ϖ_L on $\widetilde{\kappa} \otimes \widetilde{\tau}$ to be the scalar $\widetilde{x}$. Thus, we obtain the required extension $\widetilde{\xi}$. Note that the representation

$$\mathrm{ind}_{L^\times J}^{\mathrm{GL}_n(F)}\widetilde{\xi}$$

is an absolutely irreducible, integral $\mathcal{K}$-representation such that its mod-ℓ reduction isomorphic to $\overline{\pi}$. ∎

Notation 6.2.1. *From now, E is an unramified extension of F and $[E : F] = \ell$, and we set $\Gamma = \mathrm{Gal}(E/F)$ and σ is a generator of Γ. Let N_n be the unipotent radical of the Borel subgroup consisting of upper triangular matrices in GL_n. Let Θ_E be a non-degenerate σ-invariant character of $N_n(E)$ such that the restriction of Θ_E to $N_n(F)$, denoted by Θ_F, is a non-degenerate character.*

6.3. **Depth-zero base change and Shintani lifting.** In this subsection, we discuss the compatibility of the local base change lifting of depth-zero representations in $\mathcal{A}(n, F)$ and Shintani lift. Let $\mathcal{A}(n, F)_0$ denote the set of isomorphism classes of irreducible smooth depth-zero ℓ-adic representations of $\mathrm{GL}_n(F)$, and let $\mathcal{A}(n, \mathbb{F}_q)$ be the set of isomorphism classes of irreducible ℓ-adic representations of $\mathrm{GL}_n(\mathbb{F}_q)$. In the article [SZ08, Section A.1.3], Silberger–Zink considers a map:

$$\varphi_0^F : \mathcal{A}(n, F)_0 \longrightarrow \mathcal{A}(n, \mathbb{F}_q)$$

which is earlier defined by Schneider–Zink using Bushnell–Kutzko theory for all representations in $\mathcal{A}(n,F)$, and showed in [SZ08, Appendix A, equation (A.7)], the following identity

$$\varphi_0^E \circ \mathrm{bc}_{E/F} = \mathrm{bc}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q} \circ \varphi_0^F, \tag{6.1}$$

where $\mathrm{bc}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}$ is the Shintani lifting map. We will only need the precise description of φ_0^F for cuspidal depth-zero representations; which we now recall.

Let π be an absolutely irreducible, cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$. Using Lemma 6.2.1, the base change lifting of $\pi\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ to $\mathrm{GL}_n(E)$ is defined over $\mathcal{K}$, and let Π be the $\mathcal{K}$-representation of $\mathrm{GL}_n(E)$ such that $\Pi\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ is the base change lift of $\pi\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$. We simply say that Π is the base change lift of π. If π is integral, then Π is integral. By construction, Π is an absolutely irreducible $\mathcal{K}$-representation. Since π is a cuspidal representation, we have

$$\varphi_0^F(\pi) = \pi^{K_1(F)}$$

and since $\Pi^{K_1(E)}$ is irreducible, we have

$$\varphi_0^E(\Pi) = \Pi^{K_1(E)}.$$

The spaces $\pi^{K_1(F)}$ and $\Pi^{K_1(E)}$ are representations of $K_0(F)/K_1(F)\simeq \mathrm{GL}_n(\mathbb{F}_q)$ and $K_0(E)/K_1(E)\simeq \mathrm{GL}_n(\mathbb{F}_{q^\ell})$, respectively. Using the identity (6.1), we get that the irreducible $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$-representation $\Pi^{K_1(E)}\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ is the Shintani lifting of the irreducible ℓ-adic cuspidal representation $\pi^{K_1(F)}\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ of $\mathrm{GL}_n(\mathbb{F}_q)$. If π is absolutely irreducible, depth-zero, integral, cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$ with supercuspidal mod-ℓ reduction, then the mod-ℓ-reduction of the cuspidal support of the $K_0(E)/K_1(E)$-representation $\Pi^{K_1(F)}$ is supercuspidal by Lemma 5.3.3. Using these observations, we now prove the following theorem.

Theorem 6.3.1. *Let F be a finite extension of $\mathbb{Q}_p$, and let E/F be a finite unramified Galois extension with $[E:F]=\ell$, where ℓ and p are distinct primes. Let π be the depth-zero, absolutely irreducible, integral, cuspidal $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$ such that the mod-ℓ reduction of π is supercuspidal. Let Π be the base change lifting to $\mathrm{GL}_n(E)$ of π (described as above). Then*

(1) For any $\mathrm{GL}_n(E)\rtimes\Gamma$-stable Λ-lattice $\mathcal{L}$ of Π, we have $\widehat{\mathrm{H}}^1(\Gamma,\mathcal{L})$ is trivial and

$$\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L})\simeq r_\ell(\pi)^{(\ell)}$$

as $\mathrm{GL}_n(F)$-representations.

(2) If the mod-ℓ reduction $r_\ell(\Pi)$ is irreducible, then

$$\widehat{\mathrm{H}}^0(\Gamma,r_\ell(\Pi))\simeq r_\ell(\pi)^{(\ell)}$$

as $\mathrm{GL}_n(F)$ representations.

Proof. Let $\mathcal{L}$ be a $\Lambda[\mathrm{GL}_n(E)\rtimes\mathrm{Gal}(E/F)]$-lattice in Π. Since $K_1(E)$ is a pro-p group, we have the following decomposition

$$\mathcal{L}=\mathcal{L}^{K_1(E)}\oplus\mathcal{L}(K_1(E)),$$

where $\mathcal{L}(K_1(E))$ is the kernel of the projection

$$\mathcal{L}\to\mathcal{L}^{K_1(E)}.$$

Also note that $\mathcal{L}(K_1(E))$ is stable under the action of $\mathrm{Gal}(E/F)$. Using compatibility of Tate cohomology groups with finite direct sums, we get

$$\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L})=\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}^{K_1(E)})\oplus\widehat{\mathrm{H}}^0\big(\Gamma,\mathcal{L}(K_1(E))\big). \tag{6.2}$$

Note that the $\mathrm{GL}_n(\mathbb{F}_{q^\ell})$-representation $\Pi^{K_1(E)}\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ is the Shintani lift of the irreducible ℓ-adic cuspidal representation $\pi^{K_1(F)}\otimes_{\mathcal{K}}\overline{\mathbb{Q}}_\ell$ of $\mathrm{GL}_n(\mathbb{F}_q)$, and from the assumption that the mod-ℓ reduction of π is supercuspidal, we get that the mod-ℓ reduction of the cuspidal support of the $K_0(E)/K_1(E)$

representation $\Pi^{K_1(E)}$ is supercuspidal (see Lemma 5.3.3). This shows that Π is of the form (4.3) (one can also use Lemma 6.1.1). Now, it follows from Theorem 5.4.3 that $\widehat{\mathrm{H}}^0(\Gamma, \mathcal{L}^{K_1(E)})$ is $\mathrm{GL}_n(\mathbb{F}_q)$-isomorphic to $r_\ell(\pi^{K_1(F)})^{(\ell)}$. Since $\widehat{\mathrm{H}}^0(\Gamma, \mathcal{V})$ is irreducible and cuspidal (by Theorem 4.2.1), and the space $\widehat{\mathrm{H}}^0(\Gamma, \mathcal{L}^{K_1(F)})$ contains $(r_\ell(\pi)^{(\ell)})^{K_1(F)}$, we get that $\widehat{\mathrm{H}}^0(\Gamma, \mathcal{L})$ is isomorphic to $r_\ell(\pi)^{(\ell)}$. The vanishing of $\widehat{\mathrm{H}}^1(\Gamma, \mathcal{L})$ is proved in Theorem 4.2.1. Using the multiplication by ℓ-map as in 3.9, we get that

$$\widehat{\mathrm{H}}^0(\Gamma, \mathcal{L}) \simeq \widehat{\mathrm{H}}^0(\Gamma, \mathcal{L}/\ell\mathcal{L}).$$

This proves the second part of the theorem. □

6.4. **Tate cohomology of depth-zero generic representations.** Let π be an absolutely irreducible generic integral $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$. The base change lift of $\pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ to $\mathrm{GL}_n(E)$ is defined over $\mathcal{K}$ (Lemma 6.2.1), and let Π be the $\mathcal{K}$-representation such that $\Pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ is the base change lift of $\pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$. Note that the representation Π is integral (see [DN24, Lemma 4.1]). The representation π is a subrepresentation of $i_{P(F)}^{\mathrm{GL}_n(F)}\tau$, where τ is an absolutely irreducible cuspidal $\mathcal{K}$-representation of $M(F)$, where M is an F-rational Levi subgroup of P. Note that τ is an integral representation. The base change lift Π is a subrepresentation of $i_{P(E)}^{\mathrm{GL}_n(E)}\xi$ such that ξ is the base change lifting of τ, more precisely, we identify

$$M(F) \simeq \prod_{i=1}^{k} \mathrm{GL}_{n_i}(F)$$

and τ with $\otimes_{i=1}^k \tau_i$, and ξ is isomorphic to $\otimes_{i=1}^k \xi_i$, where ξ_i is the base change lift of τ_i to $\mathrm{GL}_{n_i}(E)$. Note that ξ_i are integral generic $\mathcal{K}$-representations. Let $\mathcal{L}_i$ be a $\Lambda[\mathrm{GL}_{n_i}(E) \rtimes \Gamma]$-stable lattice in ξ_i and let $\mathcal{L}$ be the induced lattice

$$i_{P(E)}^{\mathrm{GL}_n(E)}\Big(\bigotimes_{i=1}^{k} \mathcal{L}_i\Big).$$

Set $\mathcal{L}'$ to be the lattice $\mathcal{L} \cap \Pi$. Note that $\mathcal{L}'$ is a $\Lambda[\mathrm{GL}_n(E) \rtimes \Gamma]$-stable lattice in Π. We assume that the mod-ℓ reduction of τ is supercuspidal.

Theorem 6.4.1. *Let π be an absolutely irreducible, depth-zero, integral, generic $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$ such that the cuspidal support of π has supercuspidal mod-ℓ reduction. Let $J_\ell(\pi)$ the unique generic constituent of the mod-ℓ reduction of π. Let Π be the $\mathcal{K}$-representation of $\mathrm{GL}_n(E)$ such that $\Pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ is base change lift of $\pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$, and let $\mathcal{L}'$ be a lattice as defined above. Then $J_\ell(\pi)^{(\ell)}$ is the unique generic constituent of the space $\widehat{\mathrm{H}}^0(\Gamma, \mathcal{L}')$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{H}}^0(\Gamma, \mathcal{L}') \arrow[r, "\varphi_1"] \arrow[d] & \widehat{\mathrm{H}}^0\big(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L}\big) \arrow[d] \\
\widehat{\mathrm{H}}^0(\Gamma, \mathcal{L}')_{N_n(F), \overline{\Theta}_F} \arrow[r, "\varphi_2"] \arrow[d, "f_1"] & \widehat{\mathrm{H}}^0\big(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L}\big)_{N_n(F), \overline{\Theta}_F} \arrow[d, "f_2"] \\
\widehat{\mathrm{H}}^0\big(\Gamma, \mathcal{L}'_{N_n(E), \Theta_E}\big) \arrow[r, "\varphi_3"'] & \widehat{\mathrm{H}}^0\big(\Gamma, (i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L})_{N_n(E), \Theta_E}\big)
\end{tikzcd}$$

where φ_i are induced by the $\mathrm{GL}_n(E)$-equivariant injection $\mathcal{L}' \hookrightarrow i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L}$. Note that the twisted Jacquet modules $\mathcal{L}'_{N_n(E),\Theta_E}$ and $(i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L})_{N_n(E),\Theta_E}$ are free Λ-modules of rank 1. Now, the exactness of twisted Jacquet module for smooth $\mathcal{K}$-representations implies that the map

$$\mathcal{L}'_{N_n(E),\Theta_E} \to i_{P(E)}^{\mathrm{GL}_n(E)}(\otimes_{i=1}^{k}\mathcal{L}_i)_{N_n(E),\Theta_E}$$

induced by the inclusion $\mathcal{L}' \hookrightarrow i_{P(E)}^{\mathrm{GL}_n(E)}(\otimes_{i=1}^{k}\mathcal{L}_i)$ is an equality. Thus the map φ_3 is an isomorphism. Hence, the map φ_1 is non-zero. Also note that the vertical maps f_1 and f_2 are non-zero and hence isomorphisms since they are maps between one dimensional $\overline{\mathbb{F}}_\ell$-vector spaces. This implies that φ_2 is an isomorphism. Therefore, the map φ_1 takes the generic constituents of $\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}')$ to the generic constituents of $\widehat{\mathrm{H}}^0(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L})$.

Recall that $\xi_i \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ is the base change lift of $\tau_i \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$, and $\mathcal{L}_i$ is a $\mathrm{GL}_{n_i}(E) \rtimes \Gamma$-stable Λ-lattice in ξ_i, for each $1 \leq i \leq r$. Then using Theorem 6.3.1, the $\mathrm{GL}_{n_i}(F)$-representation $\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}_i)$ is isomorphic to $r_\ell(\tau_i)^{(\ell)}$ and $\widehat{\mathrm{H}}^1(\Gamma,\mathcal{L}_i)$ is trivial. As $\mathcal{L}_i$ are Λ-torsion free, and the space $\widehat{\mathrm{H}}^1(\Gamma,\mathcal{L}_i)$ vanishes, we get that the cup-product map

$$\bigotimes_{i=1}^{k}\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}_i) \to \widehat{\mathrm{H}}^0(\Gamma,\bigotimes_{i=1}^{k}\mathcal{L}_i). \tag{6.3}$$

is an isomorphism of $M(F)$-representations (see [Bor98, Corollary 2.2]). Using (2.2) and (6.3) we get that

$$\widehat{\mathrm{H}}^0(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L}) \simeq \mathrm{ind}_{P(F)}^{\mathrm{GL}_n(F)}\{\overline{\delta}_{P(E)}\bigotimes_{i=1}^{k}\big(r_\ell(\tau_i)\big)^{(\ell)}\}. \tag{6.4}$$

From $|k_E| = |k_F|^\ell$, we get that $\mathrm{Res}_{P(F)}\overline{\delta}_{P(E)} = \kappa_{P(F)}\overline{\delta}_{P(F)}$. Hence, we obtain

$$\widehat{\mathrm{H}}^0(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)}\mathcal{L}) \simeq i_{P(F)}^{\mathrm{GL}_n(F)}\big\{\kappa_{P(F)}\bigotimes_{i=1}^{k}\big(r_\ell(\tau_i)\big)^{(\ell)}\big\} \simeq \big(i_{P(F)}^{\mathrm{GL}_n(F)}\bigotimes_{i=1}^{k}r_\ell(\tau_i)\big)^{(\ell)},$$

where the second isomorphism follows from (1.3). Since the Frobenius twist $J_\ell(\pi)^{(\ell)}$ is the unique generic constituent of the above representation, we conclude that $J_\ell(\pi)^{(\ell)}$ is the unique generic constituent of $\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}')$. ∎

The following lemma is useful to prove our next theorem, and its proof is similar to some of the arguments in the proof of the previous theorem.

Lemma 6.4.2. *Let M_1 and M_2 be two ℓ-modular reps of $\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)$. Assume that M_1 and M_2 are of finite length with $\dim_{\overline{\mathbb{F}}_\ell}(M_i)_{N_n(E),\Theta_E} = 1$, for $i \in \{1,2\}$, and M_1 and M_2 have a unique generic subquotient isomorphic to M_0. For any non-zero homomorphism*

$$\varphi : M_1 \to M_2$$

such that M_0 is a subquotient of $\mathrm{Im}(\varphi)$, the map

$$\widehat{\mathrm{H}}^0(\varphi) : \widehat{\mathrm{H}}^0(\Gamma, M_1) \to \widehat{\mathrm{H}}^0(\Gamma, M_2)$$

is non-zero and its image contains the unique generic subquotient of $\widehat{\mathrm{H}}^0(\Gamma, M_1)$ and $\widehat{\mathrm{H}}^0(\Gamma, M_2)$.

Proof. Applying Tate cohomology to the commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "\varphi"] \arrow[d] & M_2 \arrow[d] \\ (M_1)_{N_n(E),\Theta_E} \arrow[r, "\sim"] & (M_2)_{N_n(E),\Theta_E} \end{tikzcd}$$

and using the natural isomorphism of non-zero one dimensional $\overline{\mathbb{F}}_\ell$-vector spaces from Theorem 4.1.1:

$$\widehat{\mathrm{H}}^0(\Gamma, (M_i)_{N_n(E),\Theta_E}) \simeq \widehat{\mathrm{H}}^0(\Gamma, M_i)_{N_n(F),\Theta_F},$$

for $i \in \{1,2\}$, we get the following diagram

$$\begin{tikzcd} \widehat{\mathrm{H}}^0(\Gamma, M_1) \arrow[r, "\widehat{\mathrm{H}}^0(\varphi)"] \arrow[d] & \widehat{\mathrm{H}}^0(\Gamma, M_2) \arrow[d] \\ \widehat{\mathrm{H}}^0(\Gamma, M_1)_{N_n(F),\Theta_F} \arrow[r, "\sim"] & \widehat{\mathrm{H}}^0(\Gamma, M_2)_{N_n(F),\Theta_F} \end{tikzcd}$$

Thus, $\mathrm{Im}(\widehat{\mathrm{H}}^0(\varphi))$ contains the unique generic subquotient of $\widehat{\mathrm{H}}^0(\Gamma, M_1)$ and $\widehat{\mathrm{H}}^0(\Gamma, M_2)$. □

Theorem 6.4.3. *Let π be a depth-zero, absolutely irreducible, integral, generic $\mathcal{K}$-representation of $\mathrm{GL}_n(F)$ and let Π be the base change lifting of π to $\mathrm{GL}_n(E)$. Let $J_\ell(\Pi)$ (resp. $J_\ell(\pi)$) be the unique generic subquotient of the mod-ℓ reduction of Π (resp. π). Then, $J_\ell(\pi)^{(\ell)}$ is the unique generic subquotient of $\widehat{\mathrm{H}}^0(\Gamma, J_\ell(\Pi))$.*

Proof. Let M be the Levi-subgroup of an F-rational parabolic subgroup P of GL_n and let $\overline{\tau}$ be a supercuspidal representation of $M(F)$ such that $J_\ell(\pi)$ occurs as a subquotient of $i_{P(F)}^{\mathrm{GL}_n(F)}(\overline{\tau})$. Let τ be a $\mathcal{K}$-lift of the rep $\overline{\tau}$ (see Lemma 6.2.2) and let τ^E be the base change lifting of τ to the group $M(E)$. We fix an isomorphism of M with $\prod_{i=1}^k \mathrm{GL}_{n_i}$, and τ with $\bigotimes_{i=1}^k \tau_i$. We then have $\tau_E = \bigotimes_{i=1}^k \tau_i^E$, where τ_i^E is the base change lifting of τ_i to $\mathrm{GL}_{n_i}(E)$. We will show that $J_\ell(\Pi)$ occurs in the mod-ℓ reduction of $i_{P(E)}^{\mathrm{GL}_n(E)}\tau^E$.

Let (ρ, N) be the Weil–Deligne representation associated with the representation π by the local Langlands correspondence. The supercuspidal support of the representation $J_\ell(\pi)$ corresponds to $r_\ell(\rho)$ (We refer to [Vig01, Théorème 1.6, Section 1.8.4] for details on the ℓ-modular local Langlands correspondence). Assume that

$$r_\ell(\rho) = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k,$$

where ρ_i is an ℓ-modular irreducible representation of $\mathcal{W}_F$. We denote by $\widetilde{\rho}_i = \mathrm{LLC}_F^{-1}(\tau_i)$. The representation $\widetilde{\rho}_i$ can be assumed to be defined over $\mathcal{K}$, and it is integral with $r_\ell(\widetilde{\rho}_i) = \rho_i$ (although it is not used in the proof, the representations ρ and $\oplus_{i=1}^k \widetilde{\rho}_i$ of $\mathcal{W}_F$ may not be isomorphic). We set

$$\widetilde{\rho}_i^E = \mathrm{Res}_{\mathcal{W}_E}\widetilde{\rho}_i,\ 1 \leq i \leq k.$$

The supercuspidal support of $J_\ell(\Pi)$ corresponds via the ℓ-modular local Langlands correspondence to $r_\ell(\mathrm{Res}_{\mathcal{W}_E}\rho)$. By Brauer–Nesbitt principle, we have

$$r_\ell(\mathrm{Res}_{\mathcal{W}_E}\rho) = r_\ell(\widetilde{\rho}_1^E) \oplus \cdots \oplus r_\ell(\widetilde{\rho}_k^E). \tag{6.5}$$

Via the ℓ-modular local Langlands correspodence, the supercuspidal support of $J_\ell(\Pi)$ corresponds to the left hand side of (6.5), and the supercuspidal support of irreducible subquotients of $i_{P(E)}^{\mathrm{GL}_n(E)}\overline{\tau}^E$ correspond to the right hand side of (6.5). Thus, the representation $J_\ell(\Pi)$ occurs as a subquotient of $i_{P(E)}^{\mathrm{GL}_n(E)}\overline{\tau}^E$ (for convenience we choose the notation $\overline{\tau}^E$ for $r_\ell(\tau_E)$).

Let $\mathcal{L}_i^E$ be a $\Lambda[\mathrm{GL}_n(E) \rtimes \Gamma]$-stable lattice in τ_i^E, and let $\mathcal{L}^E$ be the lattice $\otimes_{i=1}^k \mathcal{L}_i^E$. We showed in the proof of Theorem 6.4.1 (see equation (6.4)) that

$$\widehat{\mathrm{H}}^0(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)} \mathcal{L}^E) = (i_{P(F)}^{\mathrm{GL}_n(F)} \overline{\tau})^{(\ell)}.$$

There exists $\mathrm{GL}_n(E) \rtimes \Gamma$-subrepresentations F_1 and F_2 of $i_{P(E)}^{\mathrm{GL}_n(E)}(\overline{\tau}^E)$ such that $F_1/F_2 = J_\ell(\Pi)$. We have the following maps:

$$\begin{aligned} 0 \to \widehat{\mathrm{H}}^0(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)} \mathcal{L}^E) &\xrightarrow{\varphi_1} \widehat{\mathrm{H}}^0(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)} \overline{\tau}^E) \\ \widehat{\mathrm{H}}^0(\Gamma, F_1) &\xrightarrow{\varphi_2} \widehat{\mathrm{H}}^0(\Gamma, F_1/F_2) = \widehat{\mathrm{H}}^0(\Gamma, J_\ell(\Pi)) \\ \widehat{\mathrm{H}}^0(\Gamma, F_1) &\xrightarrow{\varphi_3} \widehat{\mathrm{H}}^0(\Gamma, i_{P(E)}^{\mathrm{GL}_n(E)} \overline{\tau}^E) \end{aligned}$$

Note that each Tate cohomology group above, as a representation of $\mathrm{GL}_n(F)$, admits a unique generic subquotient. From Lemma 6.4.2 the maps φ_2 and φ_3 are non-zero, and the image contains their respective generic subquotients. Since $J_\ell(\pi)^{(\ell)}$ occurs as the unique generic subquotient of $(i_{P(F)}^{\mathrm{GL}_n(F)} \overline{\tau})^{(\ell)}$, we get that the unique generic subquotient of $\widehat{\mathrm{H}}^0(\Gamma, J_\ell(\Pi))$ is isomorphic to $J_\ell(\pi)^{(\ell)}$. $\square$

7. The unipotent cuspidal representation of $\mathrm{Sp}_4(\mathbb{F}_q)$

Finite classical groups can have at most one irreducible unipotent cuspidal representation. Let G be a connected reductive group defined over $\mathbb{F}_q$, and let $F : G \to G$ be the Frobenius endomorphism. Recall that an irreducible representation of a finite reductive group G^F is said to be unipotent if it occurs in a Deligne–Lusztig virtual character $\pm R_{T,1}$, where T is an F-stable torus in G. When G is a classical group, if an irreducible unipotent cuspidal representation exists, then it is invariant under automorphisms of the underlying algebraic group. Additionally, we have

(1) Unipotent cuspidal representations of finite classical groups are defined over $\mathbb{Q}$ ([Lus02]).
(2) The mod-ℓ reduction of an unipotent cuspidal representation is irreducible ([DM18, Theorem 1]).

So, the case of unipotent cuspidal representations for finite classical groups is a natural testing ground for Treumann–Venkatesh conjectures. In this section, we treat the group $\mathrm{Sp}_4(\mathbb{F}_q)$ which has a unique cuspidal unipotent representation for all q. We relate the zeroth Tate cohomology and Frobenius twist of this unipotent cuspidal representation, and show vanishing of the first Tate cohomology of this unipotent cuspidal representation.

We follow the exposition of [Gol78]. Set $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and let J be the block matrix $\begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}$. The symplectic group $\mathrm{Sp}_4(\mathbb{F}_q)$ is defined as $\{g \in \mathrm{GL}_4(\mathbb{F}_q) : {}^t g J g = J\}$, where ${}^t g$ is the transpose of g. We denote by e_1, e_2, e_3, e_4, the standard basis of $\mathbb{F}_q^4$. In the famous article [Sri68], B. Srinivasan discovered an irreducible cuspidal representation of the group $\mathrm{Sp}_4(\mathbb{F}_q)$, which does not have a Whittaker model, and this representation is denoted by θ_{10}. The representation θ_{10} is a unipotent cuspidal representation. In what follows, we assume that $\ell \neq 2$.

7.1. Mirabolic restriction of of θ_{10}.

In the article [Gol78], Gol'fand proved that θ_{10} has irreducible mirabolic restriction like the cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$. To recall these results, we begin by defining some subgroups. Let P be a $\mathbb{F}_q$-rational Siegel parabolic subgroup of Sp_4 with unipotent radical U and a $\mathbb{F}_q$-rational Levi subgroup M whose $\mathbb{F}_q$ points are

$$M(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & 0 \\ 0 & w\,{}^t a^{-1} w \end{pmatrix} : a \in \mathrm{GL}_2(\mathbb{F}_q) \right\}, \quad U(\mathbb{F}_q) = \left\{ \begin{pmatrix} I_2 & b \\ 0 & I_2 \end{pmatrix} : b \in M_2(\mathbb{F}_q),\ w\,{}^t b w = b \right\}.$$

We use the notations $m(a)$ and $u(b)$ to denote the elements of the groups $M(\mathbb{F}_q)$ and $U(\mathbb{F}_q)$, respectively. We fix a non-trivial additive character $\psi : \mathbb{F}_q \to \Lambda^\times$, and an element $\tau \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$. Set

$$z = \begin{pmatrix} 0 & 1 \\ -\tau & 0 \end{pmatrix}.$$

Consider the character $\psi_z : U(\mathbb{F}_q) \to \Lambda^\times$, sending

$$u(b) \mapsto \psi(\mathrm{Tr}(zb)),$$

where $\mathrm{Tr}(A)$ denotes the trace of a matrix $A \in M_2(\mathbb{F}_q)$. Since $M(\mathbb{F}_q)$ normalizes $U(\mathbb{F}_q)$, we have for each $m(a) \in M(\mathbb{F}_q)$, a character $\psi_z^{m(a)}$ of $U(\mathbb{F}_q)$, defined as

$$\psi_z^{m(a)}(u(b)) = \psi_z(m(a)u(b)m(a)^{-1}).$$

Let $S(\mathbb{F}_q)$ be the subgroup consisting of $m(a)$ in $M(\mathbb{F}_q)$ for which $\psi_z^{m(a)} = \psi_z$. The precise form of S as an algebraic group is given by

$$\Big\{ m(a) \in M : azw\,{}^t a = zw \Big\},$$

Note that the determinant of the matrix a for which $m(a) \in S$ can be both ± 1. The connected component of identity S^0 is the torus of norm 1-elements for the quadratic extension $\mathbb{F}_{q^2}/\mathbb{F}_q$, and $S/S^0 \simeq \mathbb{Z}/2\mathbb{Z}$.

Let $\chi_S : S(\mathbb{F}_q) \to \Lambda^\times$ be the character defined by

$$\chi_S(m(a)) = \iota(\det(a)),\ m(a) \in S,$$

where $\iota : \mathbb{F}_q^\times \to \Lambda^\times$ is the Teichmüller lift. Let $Y(\mathbb{F}_q) = S(\mathbb{F}_q)\,U(\mathbb{F}_q)$. The character ψ_z induces a character $\psi_{Y,z} : Y(\mathbb{F}_q) \to \Lambda^\times$, defined as

$$\psi_{Y,z}(m(a)u(b)) = \chi_S(m(a))\,\psi_z(u(b)), \tag{7.1}$$

for $m(a) \in S(\mathbb{F}_q)$ and $u(b) \in U(\mathbb{F}_q)$. Using Mackey decomposition, we get that the induced representation $\mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\psi_{Y,z})$ is irreducible, and the result [Gol78, Proposition 1] gives:

$$\mathrm{res}_{P(\mathbb{F}_q)}(\theta_{10}) \simeq \mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\psi_{Y,z}).$$

Lemma 7.1.1. *The mod-ℓ reduction of $\mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\psi_{Y,z})$ is irreducible. In particular, the mod-ℓ reduction $r_\ell(\theta_{10})$ is irreducible.*

Proof. Let V be a non-zero $P(\mathbb{F}_q)$-stable subspace of $\mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\overline{\psi}_{Y,z})$, where $\overline{\psi}_{Y,z}$ is the mod-ℓ reduction of $\psi_{Y,z}$. Then there exists a non-trivial element $m \in M(\mathbb{F}_q)/S(\mathbb{F}_q)$ such that the conjugate $\overline{\psi}_{Y,z}^m$ of $\overline{\psi}_{Y,z}$ is contained in V. This implies that V contains $\overline{\psi}_{Y,z}^{(m)}$ for all non-trivial elements $m \in M(\mathbb{F}_q)/S(\mathbb{F}_q)$. Since the dimension of $\mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\overline{\psi}_{Y,z})$ is same as the cardinality of the coset space $M(\mathbb{F}_q)/S(\mathbb{F}_q)$, we get that

$$V = \mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\overline{\psi}_{Y,z}).$$

Hence, the mod-ℓ reduction of $\mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\psi_{Y,z})$ is irreducible. Since the restriction $\mathrm{res}_{P(\mathbb{F}_q)}(\theta_{10})$ is isomorphic to the induced representation $\mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\psi_{Y,z})$, the mod-$\ell$ reduction $r_\ell(\theta_{10})$ is also irreducible. $\square$

7.2. Let Γ be the Galois group $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$ and let σ be the Frobenius element of Γ. Let π (resp. Π) be the unipotent cuspidal $\mathcal{K}$-representation of $\mathrm{Sp}_4(\mathbb{F}_q)$ (resp. $\mathrm{Sp}_4(\mathbb{F}_{q^\ell})$). Note that the twisted representation Π^σ is unipotent by [DM91, Proposition 11.3.8] and is also cuspidal. By the uniqueness of Π, we have $\Pi \simeq \Pi^\sigma$. Let $\eta : \mathbb{F}_{q^\ell} \to \Lambda^\times$ be the additive character given by the composition $\psi \circ \mathrm{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}$, where $\mathrm{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}$ is the trace map. Since ℓ is odd, we have $\tau \in \mathbb{F}_{q^\ell}^\times \setminus (\mathbb{F}_{q^\ell}^\times)^2$. Let η_z be the character of $U(\mathbb{F}_{q^\ell})$, defined by

$$\eta_z(u(b)) = \eta(\mathrm{Tr}(zb)), u(b) \in U(\mathbb{F}_{q^\ell}),$$

where Tr denotes the trace of square matrices. Note that $S(\mathbb{F}_{q^\ell})$ is the stabilizer of η_z in $M(\mathbb{F}_{q^\ell})$. Let $Y(\mathbb{F}_{q^\ell}) = S(\mathbb{F}_{q^\ell})\, U(\mathbb{F}_{q^\ell})$. Let $\eta_{Y,z}$ be the character of $Y(\mathbb{F}_{q^\ell})$ defined as (7.1), where ψ_z is replaced by η_z. The induced representation $\mathrm{Ind}_{Y(\mathbb{F}_{q^\ell})}^{P(\mathbb{F}_{q^\ell})}(\eta_{Y,z})$ is irreducible and the restriction $\mathrm{res}_{P(\mathbb{F}_{q^\ell})}(\Pi)$ is isomorphic to $\mathrm{Ind}_{Y(\mathbb{F}_{q^\ell})}^{P(\mathbb{F}_{q^\ell})}(\eta_{Y,z})$. We use the notations ρ and ρ_ℓ for the representations $\mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\psi_{Y,z})$ and $\mathrm{Ind}_{Y(\mathbb{F}_{q^\ell})}^{P(\mathbb{F}_{q^\ell})}(\eta_{Y,z})$, respectively.

7.2.1. Let $Y = S \ltimes U$ be the subgroup of the Siegel parabolic subgroup P as defined above. Note that P is connected and the group Y/Y^0 is $\mathbb{Z}/2\mathbb{Z}$, where Y^0 is the connected component of Y containing identity. Let $X = Y\backslash P$. Using [Ser, Proposition 36] and [DM91, Lemma 4.2.13], we get the following exact sequence of pointed sets:

$$1 \to Y(\mathbb{F}_{q^k}) \to P(\mathbb{F}_{q^k}) \to X(\mathbb{F}_{q^k}) \to \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^k}), \mathbb{Z}/2\mathbb{Z}) \to 1,$$

for $k \geq 1$. We then have the exact sequence of pointed sets

$$1 \to (Y(\mathbb{F}_{q^\ell})\backslash P(\mathbb{F}_{q^\ell})) \to X(\mathbb{F}_{q^\ell}) \to \mathbb{F}_{q^\ell}^\times/(\mathbb{F}_{q^\ell}^\times)^2 \to 1.$$

By taking Γ-invariants and using the fact that ℓ is odd, we get the exact sequence

$$1 \to (Y(\mathbb{F}_{q^\ell})\backslash P(\mathbb{F}_{q^\ell}))^\Gamma \to X(\mathbb{F}_q) \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \to 1.$$

Note that $Y(\mathbb{F}_q)\backslash P(\mathbb{F}_q)$ is contained in $(Y(\mathbb{F}_{q^\ell})\backslash P(\mathbb{F}_{q^\ell}))^\Gamma$, and we also have the exact sequence

$$1 \to (Y(\mathbb{F}_q)\backslash P(\mathbb{F}_q)) \to X(\mathbb{F}_q) \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \to 1.$$

Thus, we get that

$$Y(\mathbb{F}_q)\backslash P(\mathbb{F}_q) = (Y(\mathbb{F}_{q^\ell})\backslash P(\mathbb{F}_{q^\ell}))^\Gamma. \tag{7.2}$$

We denote by X and X_ℓ for the sets $Y(\mathbb{F}_q)\backslash P(\mathbb{F}_q)$ and $Y(\mathbb{F}_{q^\ell})\backslash P(\mathbb{F}_{q^\ell})$ respectively.

7.2.2. Consider the unipotent cuspidal $\mathcal{K}$-representation Π of $\mathrm{Sp}_4(\mathbb{F}_{q^\ell})$. Note that $\eta_{Y,z}$ is a σ-invariant character of $Y(\mathbb{F}_{q^\ell})$. From the multiplicity one result

$$\dim_{\mathcal{K}}\mathrm{Hom}_{Y(\mathbb{F}_{q^\ell})}(\Pi, \eta_{Y,z}) = 1,$$

and from the fact that $\Pi^\sigma \simeq \Pi$, we get that the representation Π extends to a representation of $\mathrm{Sp}_4(\mathbb{F}_{q^\ell}) \rtimes \Gamma$, and such an extension is unique since $\mathcal{K}$ is the maximal unramified extension of $\mathbb{Q}_\ell$. The mod-ℓ-reduction $r_\ell(\Pi)$ also extends uniquely to a representation of $\mathrm{Sp}_4(\mathbb{F}_{q^\ell}) \rtimes \Gamma$. The extension can be realised on the model

$$\mathrm{Ind}_{Y(\mathbb{F}_{q^\ell})}^{P(\mathbb{F}_{q^\ell})}(\eta_{Y,z})$$

by setting $(\sigma f)(p) = f(\sigma(p))$, for all f in the above space of functions and $p \in P(\mathbb{F}_{q^\ell})$.

Theorem 7.2.1. *For any $\Lambda[\mathrm{Sp}_4(\mathbb{F}_{q^\ell}) \rtimes \Gamma]$-stable lattice $\mathcal{L} \subseteq \Pi$, the space $\widehat{\mathrm{H}}^1(\Gamma, \mathcal{L})$ is trivial.*

Proof. Note that the character $\eta_{Y,z}$ is Γ-invariant, i.e., $\eta_{Y,z}(\sigma(y)) = \eta_{Y,z}(y)$, for $y \in Y(\mathbb{F}_q)$. The representation ρ_ℓ is endowed with an action of Γ, given by

$$(\sigma f)(x) = f(\sigma^{-1}(x)), \quad \sigma \in \Gamma,$$

for all $x \in P(\mathbb{F}_{q^\ell})$ and $f \in \rho_\ell$. Since $\mathrm{res}_{P(\mathbb{F}_{q^\ell})}(\Pi)$ is isomorphic to ρ_ℓ, it suffices to show that the Tate cohomology $\widehat{\mathrm{H}}^1(\Gamma, \mathcal{L})$ is trivial for any $\Lambda[P(\mathbb{F}_{q^\ell}) \rtimes \Gamma]$-stable lattice $\mathcal{L}$ in ρ_ℓ. Let $\mathcal{L}$ be the $\Lambda[P(\mathbb{F}_{q^\ell}) \rtimes \Gamma]$-stable lattice in ρ_ℓ consisting of Λ-valued functions in ρ_ℓ. Since $\eta_{Y,z}$ is Γ-invariant, the space $\widehat{\mathrm{H}}^1(\Gamma, \eta_{Y,z})$ is trivial. Then, using equality (7.2) and the isomorphism (2.2), we get that

$$\widehat{\mathrm{H}}^1(\Gamma, \mathcal{L}) = 0.$$

Since the mod-ℓ reduction $r_\ell(\rho_\ell)$ is irreducible, any $\Lambda[P(\mathbb{F}_{q^\ell})]$-stable lattice is homothetic to $\mathcal{L}$. Thus, $\widehat{\mathrm{H}}^1(\Gamma, \mathcal{L})$ is trivial for any $\Lambda[P(\mathbb{F}_{q^\ell}) \rtimes \Gamma]$-stable lattice $\mathcal{L}$ in ρ_ℓ. □

The next lemma shows the irreducibility of the zeroth Tate cohomology group of the mod-ℓ reduction of Π.

Lemma 7.2.2. *The space $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))$ is irreducible and cuspidal as a representation of $\mathrm{Sp}_4(\mathbb{F}_q)$.*

Proof. Note that $r_\ell(\Pi)$ is irreducible by Lemma 7.1.1, and it is cuspidal since Π is cuspidal. By Corollary 3.4.3, we get that $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))$ is cuspidal. To prove the irreducibility of $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))$, we consider the following restriction to $P(\mathbb{F}_q)$ map

$$\big(\mathrm{Ind}_{Y(\mathbb{F}_{q^\ell})}^{P(\mathbb{F}_{q^\ell})}(\overline{\eta}_{Y,z})\big)^\sigma \longrightarrow \mathrm{Ind}_{Y(\mathbb{F}_q)}^{P(\mathbb{F}_q)}(\overline{\psi}_{Y,z}^{\ell})$$

$$f \longmapsto \mathrm{res}_{P(\mathbb{F}_q)}(f)$$

where $\overline{\psi}_{Y,z}$ and $\overline{\eta}_{Y,z}$ denote the mod-ℓ reductions of $\psi_{Y,z}$ and $\eta_{Y,z}$, respectively. The above map factors through $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\rho_\ell))$ and induces the following $P(\mathbb{F}_q)$-isomorphism (see Subsection 2.2)

$$\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\rho_\ell)) \simeq r_\ell(\rho)^{(\ell)}.$$

Since the restriction $\mathrm{res}_{P(\mathbb{F}_{q^\ell})}(r_\ell(\Pi))$ is isomorphic to $r_\ell(\rho_\ell)$, the space $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))$, being $P(\mathbb{F}_q)$ isomorphic to $r_\ell(\rho)^{(\ell)}$, is an irreducible representation of $P(\mathbb{F}_q)$, and hence of $\mathrm{Sp}_4(\mathbb{F}_q)$. This shows the lemma. □

7.2.3. Next, we show that $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))$ is isomorphic to $r_\ell(\pi)^{(\ell)}$. Since, we have established that both these representations are irreducible, we show that the characters as ℓ-modular representations are the same. For this, we need to understand the restriction of θ_{10} to the Klingen parabolic subgroup.

Let Q be the Klingen parabolic subgroup of Sp_4 fixing the flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle$. Let L be a Levi subgroup of Q, consisting of block diagonal matrices, and let H be the unipotent radical of Q. Then $L(\mathbb{F}_q)$ is isomorphic to $\mathbb{F}_q^\times \times \mathrm{SL}_2(\mathbb{F}_q)$, and $H(\mathbb{F}_q)$ consists of matrices of the form

$$h = \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; x, y, z \in \mathbb{F}_q.$$

Let R be one of the fields $\overline{\mathbb{F}}_\ell$ or $\mathcal{K}$. Let $\psi : \mathbb{F}_q \to R^\times$ be a non-trivial additive character. Let $\xi_{q,\psi}$ be the irreducible representation of $H(\mathbb{F}_q)$ on the space $R[\mathbb{F}_q]$ consisting of R-valued functions on $\mathbb{F}_q$, defined by

$$(\xi_{q,\psi}(h)\varphi)(t) = \psi(z + 2ty + xy)\, \varphi(x + t), \tag{7.3}$$

for $h \in H(\mathbb{F}_q)$ and $\varphi \in \Lambda[\mathbb{F}_q]$.

Let $A(\mathbb{F}_q)$ be the subgroup $(D\times \mathrm{SL}_2(\mathbb{F}_q))H(\mathbb{F}_q)$, where $D=\{\pm 1\}$, and $D\times \mathrm{SL}_2(\mathbb{F}_q)$ is considered as a subgroup of the Levi subgroup $L(\mathbb{F}_q)$. Let ϵ be the element $(-1, I_2)$, where I_2 is the identity element of $\mathrm{SL}_2(\mathbb{F}_q)$. The representation $\xi_{q,\psi}$ is extended to a representation of $A(\mathbb{F}_q)$ where the actions of ϵ and $\mathrm{SL}_2(\mathbb{F}_q)$ are defined as (see [Gol78, equations (1) and (2)]) following:

$$(\xi_{q,\psi}(\epsilon)\varphi)(t) = -\varphi(-t), \varphi\in R[\mathbb{F}_q], \tag{7.4}$$

$$\left(\xi_{q,\psi}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\varphi\right)(t) = \frac{\mu_q(\gamma)}{G(\mu_q,\psi)}\sum_{t_1\in\mathbb{F}_q}\psi\left(\frac{\alpha t^2+\delta t_1^2-2tt_1}{\gamma}\right)\varphi(t_1),\ \ \varphi\in R[\mathbb{F}_q], \tag{7.5}$$

$$\left(\xi_{q,\psi}\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix}\varphi\right)(t) = \mu(\alpha)\psi(\alpha\beta t^2)\varphi(\alpha t), \varphi\in R[\mathbb{F}_q]. \tag{7.6}$$

In the above equality, μ_q is the unique non-trivial quadratic character of $\mathbb{F}_q^\times$ and $G(\mu_q,\psi)$ is the Gauss sum of μ with respect to the additive character ψ.

Since ℓ is odd, for R is either $\overline{\mathbb{F}}_\ell$ or $\mathcal{K}$, the $\mathrm{res}_{\mathrm{SL}_2(\mathbb{F}_q)}(\xi_{q,\psi})$ module splits into two irreducible representations $\xi_{q,\psi}^+$ and $\xi_{q,\psi}^-$ acting on the spaces of even and odd functions in $R[\mathbb{F}_q]$, respectively. We further extend the representation $\xi_{q,\psi}^-$ to the group $A(\mathbb{F}_q)$ by setting $H(\mathbb{F}_q)$ and $I_{1,2}$ to act trivially. Let $\widetilde{\xi}_{q,\psi} = \xi_{q,\psi}\otimes\xi_{q,\psi}^-$ be the representation of $A(\mathbb{F}_q)$. Then the induced representation $\mathrm{Ind}_{A(\mathbb{F}_q)}^{Q(\mathbb{F}_q)}(\widetilde{\xi}_{q,\psi})$ is irreducible, which we denote by π_1. Similarly, we get an irreducible representation

$$\pi_{1,\tau} = \mathrm{Ind}_{A(\mathbb{F}_q)}^{Q(\mathbb{F}_q)}(\widetilde{\xi}_{q,\psi_\tau}),$$

and

$$\mathrm{Res}(\pi)\simeq \pi_1\oplus\pi_{1,\tau}.$$

Let η be the additive character $\psi\circ\mathrm{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}$ of $\mathbb{F}_{q^\ell}$. We denote by Π_1 to the representation $\mathrm{Ind}_{A(\mathbb{F}_{q^\ell})}^{Q(\mathbb{F}_{q^\ell})}(\widetilde{\xi}_{q^\ell,\eta})$ and $\Pi_{1,\tau}$ to be the representation $\mathrm{Ind}_{A(\mathbb{F}_{q^\ell})}^{Q(\mathbb{F}_{q^\ell})}(\widetilde{\xi}_{q^\ell,\eta_\tau})$. We then have

$$\mathrm{Res}_{Q(\mathbb{F}_{q^\ell}}(\Pi)\simeq \Pi_1\oplus\Pi_{1,\tau}.$$

Clearly, $\widetilde{\xi}_{q^\ell,\eta}$ extends as a representation of $A(\mathbb{F}_{q^\ell})\rtimes\Gamma$, by setting

$$\sigma(f_1\otimes f_2) = (\sigma f_1)(\sigma f_2),$$

where σf is defined by setting $(\sigma f)(x) = f(\sigma(x))$, for $x\in\mathbb{F}_{q^\ell}$. The representation $\widetilde{\xi}_{q^\ell,\eta}$ admits a natural Λ-lattice. We consider the lattice $\mathcal{L}_{q^\ell,\eta} = \Lambda[\mathbb{F}_{q^\ell}]$ in $\xi_{q^\ell,\eta}$ and $\mathcal{L}_{q^\ell,\eta}^-$ to be the odd functions in $\Lambda[\mathbb{F}_{q^\ell}]$. From the equalities (7.4), (7.5) and (7.6), the lattice $\widetilde{\mathcal{L}}_{q^\ell,\eta} = \mathcal{L}_{q^\ell,\eta}\otimes\mathcal{L}_{q^\ell,\eta}^-$ is stable under the action of $A(\mathbb{F}_{q^\ell})\rtimes\Gamma$. Note that

$$\widehat{\mathrm{H}}^1(\Gamma,\widetilde{\mathcal{L}}_{q^\ell,\eta}) = \{0\}. \tag{7.7}$$

We also have from [Bor98, Corollary 2.2]

$$\widehat{\mathrm{H}}^0(\Gamma,\widetilde{\mathcal{L}}_{q^\ell,\eta}) = \widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}_{q^\ell,\eta}^-)\otimes\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}_{q^\ell,\eta}^-).$$

Again from the equalities (7.4), (7.5) and (7.6) together with the restriction and reduction mod-ℓ map

$$\Lambda[\mathbb{F}_{q^\ell}]^\sigma\to\overline{\mathbb{F}}_\ell[\mathbb{F}_q],\ \varphi\mapsto\mathrm{Res}_{\mathbb{F}_q}\overline{\varphi}$$

we get that $\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}_{q^\ell,\eta}^-)$ is isomorphic to $r_\ell(\xi_{q,\psi})^{(\ell)}$ and $\widehat{\mathrm{H}}^0(\Gamma,\mathcal{L}_{q^\ell,\eta}^-)$ is isomorphic to $r_\ell(\xi_{q,\psi}^-)^{(\ell)}$; we verify this isomorphism by cheking the compatibility of (7.5) and the others are immediate. Let $\varphi\in\Lambda[\mathbb{F}_{q^\ell}]^\sigma$ and let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{SL}_2(\mathbb{F}_q)$, with $\gamma\neq 0$. We have

$$\begin{aligned}\xi_{q^\ell,\eta}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\varphi(t) &= \frac{\mu_{q^\ell}(\gamma)}{G(\mu_{q^\ell},\eta)}\sum_{t_1\in\mathbb{F}_{q^\ell}}\eta\left(\frac{\alpha t^2+\delta t_1^2-2tt_1}{\gamma}\right)\varphi(t_1)\\ &= \frac{\mu_{q^\ell}(\gamma)}{G(\mu_{q^\ell},\eta)}\sum_{t_1\in\mathbb{F}_q}\psi\circ\mathrm{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}\left(\frac{\alpha t^2+\delta t_1^2-2tt_1}{\gamma}\right)\varphi(t_1)\\ &\quad+\frac{\mu_{q^\ell}(\gamma)}{G(\mu_{q^\ell},\eta)}\sum_{t_1\in\mathbb{F}_{q^\ell}\setminus\mathbb{F}_q}\psi\circ\mathrm{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}\left(\frac{\alpha t^2+\delta t_1^2-2tt_1}{\gamma}\right)\varphi(t_1).\end{aligned}\tag{7.8}$$

We then have

$$\psi\circ\mathrm{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}\left(\frac{\alpha t^2+\delta (t_1^\sigma)^2-2tt_1^\sigma}{\gamma}\right)\varphi(t_1^\sigma)=\psi\circ\mathrm{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}\left(\frac{\alpha t^2+\delta t_1^2-2tt_1}{\gamma}\right)\varphi(t_1),$$

Where t^σ is the element $\sigma(t)$. Thus the expression in (7.8) mod-ℓ is equal to

$$\frac{\overline{\mu_q(\gamma)}^\ell}{G(\overline{\mu_q},\overline{\psi})^\ell}\sum_{t_1\in\mathbb{F}_q}\overline{\psi}\left(\frac{\alpha t^2+\delta t_1^2-2tt_1}{\gamma}\right)^\ell\overline{\varphi(t_1)}=\left(r_\ell(\xi_{q,\psi})^{(\ell)}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\overline{\varphi}\right)(t).$$

where $\overline{\mu_q}$, $\overline{\psi}$ and $\overline{\varphi}$ are the mod-ℓ reductions of μ_q, ψ and φ, respectively. Lemma 5.2.1 gives the equality of following characters:

$$\overline{\chi_{\widetilde{\xi}_{q^\ell,\eta}}(g)}=\overline{\chi_{\widetilde{\xi}_{q,\psi}}(g)}^\ell \text{ and } \overline{\chi_{\widetilde{\xi}_{q^\ell,\eta_\tau}}(g)}=\overline{\chi_{\widetilde{\xi}_{q,\psi_\tau}}(g)}^\ell\tag{7.9}$$

for all $g\in A(\mathbb{F}_q)$. We now prove the following theorem.

Theorem 7.2.3. *Let ℓ be an odd prime. Let Γ be the Galois group $\mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$, and let σ be the Frobenius element in Γ. Let π and Π be the unipotent cuspidal $\mathcal{K}$-representation of $\mathrm{Sp}_4(\mathbb{F}_q)$ and $\mathrm{Sp}_4(\mathbb{F}_{q^\ell})$, respectively. Then the Tate cohomology group $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))$ is isomorphic to the Frobenius twist $r_\ell(\pi)^{(\ell)}$.*

Proof. Let χ_Π and χ_π be the characters of Π and π respectively. We will show that

$$\overline{\chi_\Pi(g)}=\overline{\chi_\pi(g)}^\ell,$$

for all $g\in\mathrm{Sp}_4(\mathbb{F}_q)$. Let $g=su=us$ be the Jordan decomposition of g. Note that either g is contained in a parabolic subgroup of $\mathrm{Sp}_4(\mathbb{F}_q)$ or u is the identity element and s is regular. If s is a regular semisimple element, we have $\chi_\Pi(s)=1=\chi_\pi(s)$. Assume that g belongs to a parabolic subgroup $P(\mathbb{F}_q)$ or $Q(\mathbb{F}_q)$. If $g\in P(\mathbb{F}_q)$, then it follows from the arguments of Lemma 7.2.2 that

$$\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))\simeq r_\ell(\pi)^{(\ell)}$$

as a representation of $P(\mathbb{F}_q)$. Thus we have

$$\overline{\chi_\Pi(g)}=\overline{\chi_\pi(g)}^\ell$$

for all $g\in P(\mathbb{F}_q)$.

Suppose $g\in Q(\mathbb{F}_q)$. Note that $\chi_\Pi=\chi_{\Pi_1}+\chi_{\Pi_{1,\tau}}$ and $\chi_\pi=\chi_{\pi_1}+\chi_{\pi_{1,\tau}}$. Using character formula for induced representations, we have

$$\chi_{\Pi_1}(g)=\sum_{y\in Y_\ell,\, ygy^{-1}\in A(\mathbb{F}_{q^\ell})}\chi_{\widetilde{\xi}_{q^\ell,\eta}}(ygy^{-1})\tag{7.10}$$

and

$$\chi_{\pi_1}(g)=\sum_{y\in Y,\, ygy^{-1}\in A(\mathbb{F}_q)}\chi_{\widetilde{\xi}_{q,\psi}}(ygy^{-1}),\tag{7.11}$$

where Y_ℓ and Y denote the right coset spaces $A(\mathbb{F}_{q^\ell})\backslash Q(\mathbb{F}_{q^\ell})$ and $A(\mathbb{F}_q)\backslash Q(\mathbb{F}_q)$ respectively. We can choose Y (resp. Y_ℓ) to be a subset of representatives of the form $\{(a, I_2) : a \in \mathbb{F}_q^\times/\{\pm 1\}\}$ in $L(\mathbb{F}_q)$ (resp. $L(\mathbb{F}_{q^\ell})$). Thus, we observe that $Y_\ell^\sigma = Y$. Moreover, the support of χ_{Π_1} and $\chi_{\Pi_{1,\tau}}$ (resp. χ_{π_1} and $\chi_{\pi_{1,\tau}}$) is contained in $A(\mathbb{F}_{q^\ell})$ (resp. $A(\mathbb{F}_q)$). For $g \in A(\mathbb{F}_{q^\ell})$, we have $ygy^{-1} \in A(\mathbb{F}_{q^\ell})$, for all $y \in Q(\mathbb{F}_{q^\ell})$. We also observe that $ygy^{-1} = g$, for all $g \in L(\mathbb{F}_q)$ and y is one of the representatives. Since Γ acts freely on $Y_\ell \setminus Y$, there exists a subset $\mathcal{U} \subseteq Y_\ell \setminus Y$ such that $Y_\ell \setminus Y$ is the disjoint union of $\sigma^i\mathcal{U}$, $0 \leq i \leq \ell - 1$. Also, note that

$$\chi_{\widetilde{\xi}_{q^\ell,\eta}}(\sigma(y)g\sigma(y)^{-1}) = \chi_{\xi_{q^\ell,\eta}}(\sigma(y)g\sigma(y)^{-1})\chi_{\xi^-_{q^\ell,\eta}}(g) = \chi_{\widetilde{\xi}_{q^\ell,\eta}}(ygy^{-1}).$$

for all $g \in A(\mathbb{F}_{q^\ell})$. Then the identity (7.10) becomes

$$\chi_{\Pi_1}(g) = \sum_{y\in Y,\, ygy^{-1}\in A(\mathbb{F}_q)} \chi_{\widetilde{\xi}_{q^\ell,\eta}}(ygy^{-1}) \;+\; \ell \sum_{u\in\mathcal{U},\, ugu^{-1}\in A(\mathbb{F}_{q^\ell})} \chi_{\widetilde{\xi}_{q^\ell,\eta}}(ugu^{-1}).$$

Using the relations (7.9) and taking mod-ℓ reduction to the above equation, we get

$$\overline{\chi_{\Pi_1}(g)} = \sum_{y\in Y,\, ygy^{-1}\in A(\mathbb{F}_q)} \overline{\chi_{\widetilde{\xi}_{q,\psi}}(ygy^{-1})}^{\ell}.$$

Now, taking mod-ℓ reduction to the relation (7.11) and comparing it with the above identity, we get

$$\overline{\chi_{\Pi_1}(g)} = \overline{\chi_{\pi_1}(g)}^{\ell}.$$

Similarly, we have

$$\overline{\chi_{\Pi_{1,\tau}}(g)} = \overline{\chi_{\pi_{1,\tau}}(g)}^{\ell}.$$

Thus, we get the character identity

$$\overline{\chi_{\Pi}(g)} = \overline{\chi_{\pi}(g)}^{\ell},$$

for all $g \in \mathrm{Sp}_4(\mathbb{F}_q)$. Using [Lam01, Corollary 7.21], we conclude that the Tate cohomology group $\widehat{\mathrm{H}}^0(\Gamma, r_\ell(\Pi))$ is isomorphic to $r_\ell(\pi)^{(\ell)}$. □

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