ON UNIQUENESS OF BRANCHING TO FIXED POINT LIE SUBALGEBRAS

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let θ be a finite order automorphism of $\mathfrak{g}$. Let $\mathfrak{g}_0$ be the subalgebra $\{X \in \mathfrak{g} : \theta(X) = X\}$. In this article, we study for which pairs (V_1, V_2), consisting of two irreducible finite dimensional representations of $\mathfrak{g}$, we have

$$\mathrm{res}_{\mathfrak{g}_0} V_1 \simeq \mathrm{res}_{\mathfrak{g}_0} V_2.$$

In many cases, we show that V_1 and V_2 have isomorphic restrictions to $\mathfrak{g}_0$, if and only if V_1 is isomorphic to V_2^{σ}, for some outer automorphism σ of $\mathfrak{g}$.

1. Introduction

Let θ be a finite order automorphism of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}_0$ be the Lie algebra of fixed points, i.e., $\{X \in \mathfrak{g} : \theta(X) = X\}$. We ask the question of determining all the pairs (V_1, V_2) consisting of two finite dimensional irreducible representations of $\mathfrak{g}$ such that

$$\mathrm{res}_{\mathfrak{g}_0} V_1 \simeq \mathrm{res}_{\mathfrak{g}_0} V_2.$$

In this article, after a few preliminary results on the general case, we answer this question for many pairs $(\mathfrak{g}, \theta)$, including the cases where $\mathfrak{g}$ is any simple Lie algebra.

When $\mathfrak{g} = \mathfrak{t} \times \mathfrak{t} \times \cdots \times \mathfrak{t}$, taken n-times for some simple lie algebra $\mathfrak{t}$, and θ is a cyclic permutation of the simple factors this question is completely answered by Rajan in the article [Raj04]. Note that $\mathfrak{g}_0$ is the diagonal embedding of $\mathfrak{t}$ in $\mathfrak{g}$. Let $(V_1, \dots, V_n)$ and $(W_1, \dots, W_n)$ be two n-tuples of irreducible finite dimensional representations of the simple Lie algebra $\mathfrak{t}$. Rajan showed that two irreducible representations $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ and $W_1 \otimes W_2 \otimes \cdots \otimes W_n$ of $\mathfrak{g}$ have isomorphic restriction to $\mathfrak{g}_0$ if and only if there exists a permutation σ of the set $\{1, 2, \dots, n\}$ such that $V_i = W_{\sigma(i)}$ (see section 3.1 for details).

Rajan's result for $n = 2$ and $\mathfrak{t} = \mathfrak{sl}_r$ was proved using Tableaux by Purbhoo and Willigenburg in their article [PvW08]. For general n and for highest weights corresponding to rectangular tableaux Bandlow, Schilling and Thiéry showed Rajan's result in the article [BST10]. Later Venkatesh and Vishwanath proved the unique factorisation of tensor products result for irreducible integrable modules in category $\mathcal{O}$ of an indecomposable Kac-Moody Lie algebra (see [VV12]). The analogue of unique factorisation of tensor product for plethystic product is considered by Bowman and Paget in the article [BP20]. In the article [GL16], Guilhot and Lecouvey made some conjectures on the uniqueness of induced modules from Levi subalgebras of complex semisimple Lie algebras and they have proved some of these conjectures in the case of classical Lie algebras (see [GGL21] as well).

If $\mathfrak{t}$ has a non-trivial outer automorphism, then the group of outer automorphisms of $\mathfrak{t}^n$ is strictly bigger than S_n. In the context of unique decomposition of tensor product, the relevant outer automorphism σ come from S_n-the group of permutations of the simple factors. Given a complex semisimple Lie algebra $\mathfrak{g}$ and a finite order automorphism θ, there exists a θ-stable Cartan subalgebra $\mathfrak{h}$ and a θ-stable Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$ (see section 2.1). Let Φ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\{X_\alpha : \alpha \in \Phi\}$ be a Chevalley basis of $\mathfrak{g}$. Let $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha, \alpha \in \Phi\})$ be the set of automorphisms of $\mathfrak{g}$ which fix $\mathfrak{h}$, $\mathfrak{b}$ and $\{X_\alpha : \alpha \in \Phi\}$. We define H_θ a subgroup of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha, \alpha \in \Phi\})$ (see Section 5), such that $\mathrm{res}_{\mathfrak{g}_0} V \simeq \mathrm{res}_{\mathfrak{g}_0} V^{\sigma}$,

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for all $\sigma \in H_\theta$ and for all irreducible finite dimensional representations V of $\mathfrak{g}$; under certain hypothesis (see Hypothesis 5.1, 5.2) we prove that if

$$\mathrm{res}_{\mathfrak{g}_0} V_1 \simeq \mathrm{res}_{\mathfrak{g}_0} V_2$$

then $V_2 = V_1^\sigma$, for some $\sigma \in H_\theta$.

In order to state the main results of this article, we recall some notations. Let $\mathfrak{g}$ and θ be as above. An ideal I of $\mathfrak{g}$ is said to be an indecomposable θ-stable ideal if I is stable under θ and I is not a direct sum of non-zero θ-stable ideals of $\mathfrak{g}$. Note that the simple factors of an indecomposable θ-stable ideal of $\mathfrak{g}$ are isomorphic to each other; and we call this ideal of type X if the corresponding simple factors are of type X. We prove the following result (see Theorem 5.1 for more details).

Theorem 1.1. *Let $\mathfrak{g}$ be a complex semsimple Lie algebra and let θ be a finite order automorphism of $\mathfrak{g}$. Let any A_{2n}-type indecomposable θ-stable ideal of $\mathfrak{g}$ be simple, and let any D_k, A_{2k+1} and E_6-type θ-stable indecomposable ideal of $\mathfrak{g}$ have length at most 2, then for any two irreducible finite dimensional representations V_1 and V_2 of $\mathfrak{g}$, if $\mathrm{res}_{\mathfrak{g}_0} V_1 \simeq \mathrm{res}_{\mathfrak{g}_0} V_2$, then $V_1 \simeq V_2^\sigma$, for some $\sigma \in H_\theta$.*

To illustrate the above theorem in a concrete example, let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_{2n} \times \mathfrak{sl}_{2n}$, and let θ be the automorphism

$$(X, Y) \mapsto (Y, -X^t),$$

where X^t is the transpose of X. Then we get that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{so}_{2n}$. Let V_1, V_2, W_1 and W_2 be four irreducible finite dimensional representations of $\mathfrak{sl}_{2n}$. The above theorem implies that

$$\mathrm{res}_{\mathfrak{g}_0} V_1 \otimes V_2 \simeq \mathrm{res}_{\mathfrak{g}_0} W_1 \otimes W_2$$

if and only if there exists a $\sigma \in S_2$ such that

$$V_i = W^*_{\sigma(i)} \text{ or } V_i = W_{\sigma(i)},$$

for $i \in \{1, 2\}$.

We sketch the main ideas of the proof of Theorem 5.1. We begin with a complex semisimple Lie algebra $\mathfrak{g}$ with a finite order automorphism θ. We first reduce the proof to the case where θ is induced by a diagram automorphism of the Dynkin-diagram associated with $\mathfrak{g}$ (see Section 6.1). Then we show that it is sufficient to prove the main result for the case where $\mathfrak{g} = \mathfrak{k}^n$, for some simple Lie algebra $\mathfrak{k}$ and here, $\mathfrak{g}_0$ is the image of the diagonal embedding of a fixed point subalgebra $\mathfrak{k}_0$ of $\mathfrak{k}$ with respect to some non-trivial diagram automorphism of $\mathfrak{k}$ (see Section 6.2). To prove the main result in this case, we compare the Weyl characters. We use the cofactor expansion of the Weyl numerator similar to Rajan's ideas in the unique factorisation of tensor products (see Section 6.4.1 and also for the case where $\mathfrak{g}$ is simple). We hence generalise Rajan's theorem, but we use the main theorem in [Raj04] in the proofs.

To use the cofactor expansions in comparing Weyl characters one uses linear independence of certain polynomials. In our case, it is easy to see that such linear independence results fail. Hence, using some additional hypothesis on $\mathfrak{g}$ (see Hypotheses 5.1, 5.2) we prove the main theorem. In particular, we either restrict to the case where the number of simple factors of an indecomposable θ-stable ideal of $\mathfrak{g}$ is less than or equal to two or assume some generic hypothesis on the heighest weights of the representations V_1 and V_2. We think the result can be proved in greater generality, we give some evidence in Proposition 6.3. Some of the Hypotheses are indeed essential and we refer to Section 3 for a discussion on related examples.

We briefly explain the contents of each section. Section 2 contains preliminaries on finite order automorphisms, Weyl Characters and their cofactor expansions. Section 3 gives a few illustrative examples which will be useful to understand the hypotheses in the main result. Section 5 is the statement of the main theorem. In Section 6, we give a proof of the main theorem and the proof is divided into subsections. In subsection 6.1, we reduce the proof to the case where θ is an outer automorphism. In subsection 6.2, we reduce the proof to a θ-stable indecomposable ideal of $\mathfrak{g}$. The Weyl Character comparisons are done in subsection 6.4; for the case where $\mathfrak{g}$ is simple, see subsection 6.4.1. In subsection 6.4.2, we consider the case where $\mathfrak{g}$ is a θ-stable ideal of type

A_{2n+1}, D_n and E_6. In subsection 6.4.3, we consider the case where $\mathfrak{g}$ is an indecomposable ideal of type A_{2n}. In subsection 6.5, we prove a proposition which is a variant of the main result, with a slightly different hypothesis on the weights.

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2. Preliminaries

2.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a finite order automorphism θ. Let $\mathfrak{g}_0$ be the subalgebra $\{X \in \mathfrak{g} : \theta(X) = X\}$. The Lie algebra $\mathfrak{g}_0$ is reductive [Kac90, Lemma 8.1(c)]. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. The $\mathfrak{g}$-centraliser of the algebra $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$ (see [Kac90, Lemma 8.1(b)]). Let $Z_{\mathfrak{g}}(\mathfrak{h}_0) = \mathfrak{h}$; clearly $\mathfrak{h}$ is θ-stable. Let x be a regular element in $\mathfrak{h}_0$ and let $\mathfrak{n}$ be the span of all those root spaces $\mathfrak{g}_\alpha$ such that $\alpha(x) > 0$. The algebra $\mathfrak{h} + \mathfrak{n}$ is a θ-stable Borel subalgebra (see [Kac90, Section 8.1] for details). Hence, we may (and do) fix a θ-stable Borel subalgebra $\mathfrak{b}$, and a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{b}$.

Let Φ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let W be the Weyl group of the root system Φ. Let Φ^+ (resp. Φ^-) be the set of positive (resp. negative) roots of Φ with respect to $\mathfrak{b}$. We denote by Δ (resp. $\Delta^\vee$) the set of simple roots (resp. simple coroots) in Φ^+ (resp. $(\Phi^\vee)^+$). Let P be the weight lattice, and let Q be the root lattice. We denote by P^+ (resp. P^{++}) the set of dominant (resp. strongly dominant) with respect to $\mathfrak{b}$. For $\alpha \in \Phi$, let $s_\alpha \in W$ be the reflection on $\mathfrak{h}^*$ given by

$$\lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \ \lambda \in \mathfrak{h}^*.$$

For any subset $I \subseteq \Delta$, let Φ^I be the subroot system of Φ spanned by the set I. We denote by $\mathfrak{g}^I$ the Levi subalgebra $\mathfrak{h} \oplus \{\bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha\}$. Let W_I be the subgroup of W generated by the set $\{s_\alpha : \alpha \in I\}$. Let $s^1 s^2 \cdots s^r = w$ be a reduced expression for w as an element of the Coxeter group $(W, \{s_\alpha : \alpha \in \Delta\})$. We denote by $I(w)$ the set $\{\alpha \in \Delta : s_\alpha \in \{s^i : 1 \leq i \leq r\}\}$; the set $I(w)$ is independent of the reduced expression for w. We denote by $l(w)$ the length of a reduced expression of w in the Coxeter group $(W, \{s_\alpha : \alpha \in \Delta\})$.

2.2. Let $\mathrm{Aut}(\mathfrak{g})$ be the automorphism group of the Lie algebra $\mathfrak{g}$. We denote by $\mathrm{Inn}(\mathfrak{g})$ the group of inner automorphisms of $\mathfrak{g}$, the group of automorphisms generated by $\exp(\mathrm{ad}_X)$ such that ad_X is nilpotent. For any finite dimensional representation (π, V) of $\mathfrak{g}$, and for any $\theta \in \mathrm{Aut}(\mathfrak{g})$, we denote by (π^θ, V), the representation $(\pi \circ \theta, V)$. If $\theta \in \mathrm{Inn}(\mathfrak{g})$, then we have $\pi \simeq \pi^\theta$. We also use the notation V^θ for the representation π^θ. We have the following split exact sequence

$$1 \to \mathrm{Int}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}) \xrightarrow{f} \mathrm{Out}(\mathfrak{g}) \to 1, \tag{2.1}$$

where a splitting is given by the choice of a Cartan subalgebra $\mathfrak{h}$, a Borel subalgebra $\mathfrak{b}$ and a Chevalley basis $\{X_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Phi)\}$. The group of automorphisms which fix $\mathfrak{h}, \mathfrak{b}$ and the set $\{X_\alpha : \alpha \in \Phi\}$, denoted by $\mathrm{Aut}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$, is isomorphic to $\mathrm{Out}(\mathfrak{g})$ via the map f (see [Bou75, VIII, 5, $n^o 3$, Corrollaire 1]). Let $\mathfrak{b}$ be a θ-stable Borel subalgebra and let $\mathfrak{h}$ be a θ-stable Cartan subalgebra as fixed in the above subsection. Thus, we have $\theta = \iota\theta_0$ where ι is an inner automorphism and θ_0 belongs to $\mathrm{Aut}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$. Observe that ι is a finite order inner automorphism which fixes the Cartan subalgebra $\mathfrak{h}$ pointwise (see [Kac90, Proposition 8.1])).

Assume that $\mathfrak{g}$ is indecomposable for the action of θ. Let $\mathrm{X} \in \{\mathrm{D}_n, \mathrm{A}_n, \mathrm{E}_6\}$ and let $\mathfrak{g}$ be isomorphic to $\mathfrak{t}_1 \times \cdots \times \mathfrak{t}_n$ such that $\mathfrak{t}_i$ is a simple Lie algebra of the same type X, for all $i \in [n]$. Let $\mathfrak{s}_i$ be the Cartan subalgebra $\mathfrak{h} \cap \mathfrak{t}_i$. Let $\mathfrak{b}_i$ be the Borel subalgebra $\mathfrak{b} \cap \mathfrak{t}_i$. Let Δ_i be the simple roots of $\mathfrak{t}_i$ with respect to the choice of $\mathfrak{b}_i$ and $\mathfrak{s}_i$. Let γ_i be the unique root of Δ_i fixed by all automorphisms in $\mathrm{Aut}(\mathfrak{t}_i, \mathfrak{b}_i, \mathfrak{s}_i, \{X_\alpha\})$ and with the maximal number of neighbours in the Dynkin graph on Δ_i. The set of roots $\{\gamma_i : 1 \leq i \leq n\}$ are called *folding roots* of the pair $(\mathfrak{g}, \theta)$ with respect to the choice of $\mathfrak{b}$ and $\mathfrak{h}$.

2.3. Let V be a finite dimensional representation of $\mathfrak{g}$. Recall that the formal character of V is given by the formal sum:

$$\mathrm{ch}_V = \sum_{\mu \in \mathfrak{h}^*} m(\mu, V) e^{\mu},$$

where $m(\mu, V)$ is the multiplicity of μ in $\mathrm{res}_{\mathfrak{h}} V$. For $\lambda \in P^+$, we denote by $L(\lambda)$ the highest weight module associated to λ, and $\mathrm{ch}_{L(\lambda)}$ will be denoted by ch_λ. Recall the Weyl character formula:

$$\mathrm{ch}_\lambda = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}. \tag{2.2}$$

If V_1 and V_2 are two finite dimensional representations of $\mathfrak{g}$, then $V_1 \simeq V_2$ if and only if $\mathrm{ch}_{V_1} = \mathrm{ch}_{V_2}$. We refer to [Bou75, VII, 7-9] and [Hum72] for further details.

3. Examples

Before we begin to formulate the main theorem it is instructive to consider a few examples.

3.1. Let $\mathfrak{k}$ be a simple Lie algebra and let $\mathfrak{g}$ be the n-fold product $\mathfrak{k} \times \cdots \times \mathfrak{k}$. Let θ be the following automorphism on $\mathfrak{g}$:

$$\theta((X_1, X_2, \dots, X_n)) = (X_2, \dots, X_n, X_1),\ X_i \in \mathfrak{k},\ i \in [n].$$

Note that $\mathfrak{g}_0$ is the algebra $\Delta\mathfrak{k}$. Any irreducible representation of $\mathfrak{g}$ is of the form

$$L(\lambda) = L(\lambda_1) \otimes L(\lambda_2) \otimes \cdots \otimes L(\lambda_n),\ \lambda_i \in P^+.$$

Now, Rajan's theorem ([Raj04, Theorem 1]) says that $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$ if and only if there exists a permutation $\sigma \in S_n$ such that

$$\lambda_i = \mu_{\sigma(i)}.$$

The permutation σ can be considered as an automorphism in $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$. If $\mathrm{Out}(\mathfrak{k})$ is non-trivial, then we note that the group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$ is strictly bigger than S_n. Our aim in this article is to show, in the general case where $\mathfrak{g}$ is any semisimple complex Lie algebra and θ is a finite order automorphism of $\mathfrak{g}$, that if $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$, then there exists $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$ such that $L(\lambda) \simeq L(\mu)^\sigma$. For this purpose, in section 5, we define a subgroup H_θ of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$ which captures all the relevant automorphisms σ.

3.2. In the next example, we assume that $\mathfrak{g}$ is a simple Lie algebra of type D_4, and let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with the following Dynkin diagram:

$$\begin{tikzcd}[arrows=-]
\alpha_1 \arrow[dr] & & \\
& \alpha_2 \arrow[r] & \alpha_3 \\
\alpha_4 \arrow[ur] & &
\end{tikzcd}$$

Let θ be the triality automorphism which fixes α_2 and takes α_1 to α_3. The Lie-algebra $\mathfrak{g}_0$ is a simple Lie-algebra of the type G_2. Note that $\mathfrak{h}_0^*$ is spanned by

$$\beta_1 = \frac{\alpha_1 + \alpha_3 + \alpha_4}{3},\ \beta_2 = \alpha_2.$$

Note that $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\}) \simeq S_3$ acts trivially on $\mathfrak{h}_0$. This shows that $\mathrm{res}_{\mathfrak{g}_0} L(\lambda)^\sigma \simeq \mathrm{res}_{\mathfrak{g}_0} L(\lambda)$, for all $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$. From the main theorem we also get the converse, i.e., if $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$, then there exists a $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$ such that $\sigma(\lambda) = \mu$.

Continuing with the same example, consider the case where θ is the order 2 automorphism of $\mathfrak{g}$ with $\theta(\alpha_1) = \alpha_4$, and $\theta(\alpha_2) = \alpha_2$, and $\theta(\alpha_3) = \alpha_3$. In this case $\mathfrak{h}_0^*$ is spanned by the elements

$$\beta_1 = \frac{\alpha_1 + \alpha_4}{2},\ \beta_2 = \alpha_2, \beta_3 = \alpha_3.$$

Note that the subgroup of automorphisms of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$ which act trivially on $\mathfrak{h}_0^*$ is equal to $\langle\theta\rangle$. If is obvious that $\mathrm{res}_{\mathfrak{g}_0} L(\lambda)^\theta \simeq \mathrm{res}_{\mathfrak{g}_0} L(\lambda)$. Conversely, if (λ, μ) is a pair of dominant weights such that $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$ then from the main theorem we get that $\sigma(\lambda) = \mu$, for some $\sigma \in \langle\theta\rangle$.

3.3. In this subsection, we would like to point out another interesting class of examples: Let $\mathfrak{t}$ be a simple Lie algebra and let θ_0 be a non-trivial finite order automorphism of $\mathfrak{t}$. We denote by $\mathfrak{t}_0$ the Lie algebra $\{X \in \mathfrak{t} : \theta_0(X) = X\}$. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{t}\times\mathfrak{t}$, and let θ be the automorphism of $\mathfrak{g}$ defined by the map:

$$(X, Y) \mapsto (\theta_0(Y), X),\ \ X, Y \in \mathfrak{t}.$$

Note that $\mathfrak{g}_0 = \Delta\mathfrak{t}_0$. Our main result describes the set of pairs (λ, μ) such that $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$. Note that $\Delta\mathfrak{t}_0$ is properly contained in $\Delta\mathfrak{t}$, and in this sense, we extend the theorem of Rajan on unique factorisation. Under some hypothesis (see Hypothesis 5.2) we also show the n-fold analogue, i.e., when $\mathfrak{g} = \mathfrak{t}^n$. This example is the essential content of this article.

3.4. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_3 \times \mathfrak{sl}_3 \times \mathfrak{sl}_3$, and let $\mathfrak{d}$ be the Cartan subalgebra consisting of diagonal matrices of $\mathfrak{sl}_3$. Similarly, let $\mathfrak{b}$ be the Borel subalgebra consisting of upper-triangular matrices in $\mathfrak{sl}_3$. Let Δ_0 be the set of simple roots of $\mathfrak{sl}_3$ for the choice of $\mathfrak{b}$ and $\mathfrak{d}$. Let θ_0 be a non-trivial automorphism in $\mathrm{Aut}(\mathfrak{sl}_3, \mathfrak{d}, \mathfrak{b}, \{X_\alpha\})$. Let $\theta \in \mathrm{Aut}(\mathfrak{g})$ be the following automorphism:

$$(X_1, X_2, X_3) \mapsto (\theta_0(X_3), X_1, X_2).$$

Let $\mathrm{ad} : \mathfrak{sl}_2 \to \mathfrak{sl}_3$ be the adjoint embedding. Observe that $\mathfrak{g}_0$ is the Lie algebra

$$\{(\mathrm{ad}(X), \mathrm{ad}(X), \mathrm{ad}(X)) : X \in \mathfrak{sl}_2\}.$$

One can construct many examples of pairs $(L(\lambda), L(\mu))$ consisting of two irreducible representations of $\mathfrak{g}$ such that $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$, but $\sigma(\lambda) \neq \mu$, for all $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{d}^3)$. As an example, take

$$\lambda = \varpi_{11} + \varpi_{12} + 9\varpi_{21} + 9\varpi_{22} + 11\varpi_{31} + 11\varpi_{32}$$

and

$$\mu = \varpi_{11} + 9\varpi_{12} + \varpi_{21} + 9\varpi_{22} + 3\varpi_{31} + 19\varpi_{32},$$

where $\{\varpi_{i1}, \varpi_{i2}\}$ is the set of fundamental weights for each copy of $\mathfrak{sl}_3$. It is easy to check that $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$. This is the reason for the hypothesis in the main theorem that any A_{2n}-type θ-stable indecomposable ideal of $\mathfrak{g}$ is simple.

4. Reduction of Weyl Character

In this section, we prove some technical lemmas required in the proof of the main theorem. For $w \in W$ and $\alpha \in \Delta$, we define $c_{w,\alpha} : \mathfrak{h}^* \to \mathbb{C}$, by the equation:

$$(\lambda + \rho) - w(\lambda + \rho) = \sum_{\alpha\in\Delta} c_{w,\alpha}(\lambda)\alpha. \tag{4.1}$$

It is well known that $c_{w,\alpha}(\lambda) \neq 0$ if and only if $\alpha \in I(w)$, and $c_{w,\alpha}(\lambda) \geq \langle\lambda + \rho, \alpha^\vee\rangle$, for all $\alpha \in I(w)$ (see [VV12, Lemma 2]). We will need the following basic results on the function $c_{w,\alpha}$.

Lemma 4.1. *Let $\lambda \in P^+$, and let $w \in W$ be an element such that $I(w) \subseteq \Delta\backslash\{\alpha\}$. Then $c_{s_\alpha w,\alpha}(\lambda) > \langle\lambda + \rho, \alpha^\vee\rangle$ if and only if there exists a $\beta \in I(w)$ such that $\langle\alpha, \beta^\vee\rangle \neq 0$.*

Proof. We have

$$\begin{aligned}(\lambda + \rho) - s_\alpha w(\lambda + \rho) =&(\lambda + \rho) - s_\alpha(\lambda + \rho) + s_\alpha\{(\lambda + \rho) - w(\lambda + \rho)\} \\ =&\langle\lambda + \rho, \alpha^\vee\rangle\alpha + \sum_{\beta\in I(w)} c_{w,\beta}(\lambda)\beta - \sum_{\beta\in I(w)} c_{w,\beta}(\lambda)\langle\beta, \alpha^\vee\rangle\alpha.\end{aligned}$$

As $I(w) \subseteq \Delta\backslash\{\alpha\}$, from the above equation we get that

$$c_{s_\alpha w,\alpha}(\lambda) = \langle\lambda + \rho, \alpha^\vee\rangle - \sum_{\beta\in I(w)} c_{w,\beta}(\lambda)\langle\beta, \alpha^\vee\rangle.$$

Since $\langle \beta, \alpha^\vee \rangle \leq 0$, we get the lemma. $\square$

Lemma 4.2. *Let λ be a dominant weight, and let $w_1, w_2 \in W$ be two elements such that $l(w_1 w_2) = l(w_1) + l(w_2)$. Then we have $c_{w_2,\alpha}(\lambda) \leq c_{w_1 w_2,\alpha}(\lambda)$ for all $\alpha \in \Delta$, and the inequality is strict if and only if $\alpha \in I(w_1)$.*

Proof. Let $w \in W$ be such that $l(s_\alpha w) = l(w) + 1$. Observe that

$$\begin{aligned}(\lambda+\rho) - s_\alpha w(\lambda+\rho) =&(\lambda+\rho) - w(\lambda+\rho) + w(\lambda+\rho) - s_\alpha w(\lambda+\rho) \\ =& \sum_{\beta \in I(w)} c_{w,\beta}(\lambda)\beta + \langle w(\lambda+\rho), \alpha^\vee \rangle \alpha.\end{aligned}$$

Since $s_\alpha w > w$, we have $s_\alpha w(\lambda+\rho) \leq w(\lambda+\rho)$. Hence, we get that $\langle w(\lambda+\rho), \alpha^\vee \rangle \geq 0$. Moreover, we have $\lambda + \rho \in P^{++}$. Hence, $\langle w(\lambda+\rho), \alpha^\vee \rangle$ is non-zero. Therefore, we get that $c_{s_\alpha w,\alpha}(\lambda) > c_{w,\alpha}(\lambda)$, for all $\lambda \in P^+$. If $\alpha \notin I(w_1)$, then $c_{w_2,\alpha} = c_{w_1 w_2,\alpha}$ and if $\alpha \in I(w_1)$, then the lemma follows using induction on the length of w_1. $\square$

Lemma 4.3. *Let $\lambda \in P$ and $\alpha \in \Delta$. The constant $c_{w,\alpha}(\lambda)$ is a non-negative integral linear combination of the set of integers $\{\langle \lambda+\rho, \alpha^\vee \rangle : \alpha \in \Delta\}$.*

Proof. We prove this by induction on $l(w)$. If $l(w) = 1$, then $c_{w,\alpha}(\lambda)$ is equal to $\langle \lambda+\rho, \alpha^\vee \rangle$. Assume that $w = w' s_\alpha$ such that $l(w) = l(w') + 1$ and that the lemma holds for w'. Then we have

$$(\lambda+\rho) - w' s_\alpha(\lambda+\rho) = (\lambda+\rho) - w'(\lambda+\rho) + \langle \lambda+\rho, \alpha^\vee \rangle w'(\alpha).$$

Since $w'(\alpha) \in \Phi^+$, we get the result using induction hypothesis. $\square$

4.1. Consider a pair $(\mathfrak{g}, \mathfrak{h})$ consisting of a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ and a similar subpair $(\mathfrak{g}', \mathfrak{h}')$ of $(\mathfrak{g}, \mathfrak{h})$. Let $p : \mathfrak{h}^* \to (\mathfrak{h}')^*$ be the restriction map. Using the theory of highest weight modules, we get that $p(P) \subset P'$, where P' is the weight lattice for the pair $(\mathfrak{g}', \mathfrak{h}')$.

We return to our setting defined in the subsection 2.2 and recall θ_0, the diagram automorphism as defined there. Let $I \subseteq \Delta$ be a θ_0-stable subset. Let $\mathfrak{g}^I$ be the Lie algebra generated by $\{X_\alpha, X_{-\alpha} : \alpha \in I\}$ and let $\mathfrak{h}^I$ be its Cartan subalgebra $\mathfrak{h} \cap \mathfrak{g}^I$. Let $\mathfrak{g}_0^I$ be the fixed point subalgebra $(\mathfrak{g}^I)^{\theta_0}$, and let $\mathfrak{h}_0^I := \mathfrak{h}^I \cap \mathfrak{g}_0^I$ be the Cartan subalgebra of $\mathfrak{g}^I$. Let P^I and P_0^I be the weight lattices of $(\mathfrak{g}^I, \mathfrak{h}^I)$ and $(\mathfrak{g}_0^I, \mathfrak{h}_0^I)$ respectively. Let $r^I : \mathfrak{h}^* \to (\mathfrak{h}^I)^*$, $r_0^I : \mathfrak{h}_0^* \to (\mathfrak{h}_0^I)^*$ $p : \mathfrak{h}^* \to (\mathfrak{h}_0)^*$ and $p^I : (\mathfrak{h}^I)^* \to (\mathfrak{h}_0^I)^*$ be the restriction maps. By abuse of notation we use r^I, r_0^I p, and p^I for the corresponding maps induced on the group algebra of the respective weight lattices. We then have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Z}[P] & \xrightarrow{r^I} & \mathbb{Z}[P^I] \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p^I} \\ \mathbb{Z}[P_0] & \xrightarrow{r_0^I} & \mathbb{Z}[P_0^I] \end{array}$$

Notation 4.1.1. *For $\lambda \in P^+$, let U_λ be the normalised Weyl numerator*

$$e^{-(\lambda+\rho)}\Big(\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}\Big).$$

Note that $U_\lambda \in \mathbb{Z}[Q]$. We use the notation u_λ for $p(U_\lambda)$.

Lemma 4.4. *Let $\lambda_1, \lambda_2 \in P^+$. Then the following are equivalent:*

(1) The representations $\mathrm{res}_{\mathfrak{g}_0} L(\lambda_1) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\lambda_2)$,
(2) $p(\mathrm{ch}_{\lambda_1}) = p(\mathrm{ch}_{\lambda_2})$,
(3) $u_{\lambda_1} = u_{\lambda_2}$.

Proof. It is straightforward to see that (1) is equivalent to (2). From our choice of $\mathfrak{h}$, the set Δ is stable under θ. Let $\Delta = \coprod_i^k \Delta_i$ be the orbit decomposition of Δ under θ. We set $\beta_i = p(\alpha)$, where $\alpha \in \Delta_i$. Note that $\{\beta_1, \dots, \beta_k\}$ is a basis for the space $\mathfrak{h}_0^*$. Let C^+ be the set $\{\sum_{i=1}^k n_i\beta_i : n_i \in \mathbb{Z}_{\geq 0}\}$. Now, observe that $p(U_{\lambda_i}) \in \mathbb{Z}[C^+]$. Since $\lambda_i \in P^+$, we get that $p(e^{\lambda_i}) \in C^+$, for all $i \in \{1,2\}$. Now the lemma follows from the unique factorisation in the polynomial ring. This gives (2) if and only if (3). □

Notation 4.1.2. *Let $\{\varpi_\alpha^I\}$ be the set of fundamental weights of the root system Φ^I spanned by the simple roots in I. For $\lambda \in P^+$, we denote by λ^I the weight*

$$\sum_{\alpha\in I}\langle \lambda+\rho, \alpha^\vee\rangle \varpi_\alpha^I.$$

Let ρ^I be the weight $\sum_{\alpha\in I}\varpi_\alpha^I$. Note that $\lambda^I - \rho^I \in (P^I)^+$.

Lemma 4.5. *For $\lambda \in P^+$, we have $r^I(U_\lambda) = U_{\lambda^I-\rho^I}$.*

Proof. We have

$$U_\lambda = \sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-(\lambda+\rho)}.$$

Note that $r^I(e^{w(\lambda+\rho)-(\lambda+\rho)})$ is equal to zero, if $w \notin W_I$ and $r^I(e^{w(\lambda+\rho)-(\lambda+\rho)})$ is equal to $e^{w(\lambda^I)-(\lambda^I)}$, for $w \in W_I$. For $w \in W_I$ and $\alpha \in I$, we have

$$(\lambda+\rho) - s_\alpha w(\lambda+\rho) = \langle\lambda+\rho,\alpha^\vee\rangle\alpha + (\lambda+\rho) - w(\lambda+\rho) - \langle(\lambda+\rho)-w(\lambda+\rho),\alpha^\vee\rangle$$

and

$$(\lambda^I) - s_\alpha w(\lambda^I) = \langle\lambda^I,\alpha^\vee\rangle\alpha + \lambda^I - w\lambda^I - \langle\lambda^I - w\lambda^I,\alpha^\vee\rangle.$$

Using the definition of λ^I, we get that

$$\langle\lambda+\rho,\alpha^\vee\rangle = \langle\lambda^I,\alpha^\vee\rangle,\ \alpha\in I.$$

Hence using induction on the length of w, we get that

$$(\lambda+\rho) - w(\lambda+\rho) = (\lambda^I) - w(\lambda^I),\ w \in W_I.$$

Hence, we prove the lemma. □

Let I' be a θ_0-stable subset of Δ with a unique folding root α_0. We set $I = I'\backslash\{\alpha_0\}$. Let γ^I be the weight $\sum_{\alpha\in I}\langle\alpha,\alpha_0^\vee\rangle\varpi_\alpha^I$. Then we have the following lemma.

Lemma 4.6. *For any $w \in W_I$, we get that $e^{ws_{\alpha_0}(\lambda+\rho)-(\lambda+\rho)}$ is equal to*

$$e^{-\langle\lambda+\rho,\alpha_0^\vee\rangle\alpha_0}e^{w(\lambda^I+\langle\lambda+\rho,\alpha_0^\vee\rangle\gamma^I-\rho^I)-(\lambda^I+\langle\lambda+\rho,\alpha_0^\vee\rangle\gamma^I-\rho^I)}$$

Proof. We have

$$(\lambda+\rho) - ws_{\alpha_0}(\lambda+\rho) = \langle\lambda+\rho,\alpha_0^\vee\rangle\alpha_0 + (\lambda+\rho) - w(\lambda+\rho) + \langle\lambda+\rho,\alpha_0^\vee\rangle(w\alpha_0-\alpha_0)$$

Using Lemma 4.5, we get that $(\lambda+\rho) - w(\lambda+\rho)$ is equal to $\lambda^I - w\lambda^I$. Now the lemma follows from the observation that

$$w\alpha_0 - \alpha_0 = \gamma^I - w\gamma^I,\ w\in W_I.$$

□

5. Main Theorem

Let $\mathfrak{g}$ be a semisimple Lie algebra with a finite order automorphism θ. Let $\mathfrak{b}$ be a θ-stable Borel subalgebra and let $\mathfrak{h}$ be a θ-stable Cartan subalgebra contained in $\mathfrak{b}$. Let $\{X_\alpha : \alpha \in \Phi\}$ be a Chevalley basis. We set

$$H_\theta := \{g \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\}) : gX = X,\ \forall\ X \in \mathfrak{h}_0\}. \tag{5.1}$$

For any $\lambda \in P^+$, using the Weyl character formula we get that $p(\mathrm{ch}_\lambda) = p(\mathrm{ch}_{\sigma\lambda})$, for all $\sigma \in H_\theta$. Hence, we get that

$$\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\sigma\lambda).$$

We consider the following hypothesis on the tuple $(\mathfrak{g}, \theta, \lambda, \mu)$.

Hypothesis 5.1.

(1) Any indecomposable θ-stable ideal of $\mathfrak{g}$ of type A_{2n} is simple.
(2) Any indecomposable θ-stable ideal $\mathfrak{g}'$ of $\mathfrak{g}$ of the type A_{2n+1}, D_n and E_6 is isomorphic to $\mathfrak{t} \times \mathfrak{t}$, where $\mathfrak{t}$ is a simple factor of $\mathfrak{g}'$.

Hypothesis 5.2. *Let $\mathfrak{g}$ be equal to $\prod_{i=1}^{r} \mathfrak{g}_i$, where $\mathfrak{g}_i$ is a θ-stable indecomposable ideal of $\mathfrak{g}$.*

(1) Any indecomposable θ-stable ideal $\mathfrak{g}_i$ of the type A_{2n} is simple.
(2) Let $\mathfrak{g}_i$ be an indecomposable θ-stable ideal of type $\{\mathrm{A}_{2n+1}, \mathrm{D}_n, \mathrm{E}_6\}$. Let $S_i \subset \Delta$ be the set of folding roots of the pair $(\mathfrak{g}_i, \theta)$. Then we assume that $\langle \lambda + \rho, \alpha^\vee \rangle \neq \langle \lambda + \rho, \beta^\vee \rangle$ and $\langle \mu + \rho, \alpha^\vee \rangle \neq \langle \mu + \rho, \beta^\vee \rangle$, for all $\alpha \neq \beta \in S_i$.

Theorem 5.1. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and let θ be a finite order automorphism of $\mathfrak{g}$. Let $\lambda, \mu \in P^+$ be two weights such that the tuple $(\mathfrak{g}, \theta, \lambda, \mu)$ satisfies Hypothesis 5.1 or 5.2. If $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$, then, there exists a $\sigma \in H_\theta$ such that $\sigma\lambda = \mu$.*

6. Proof of Main theorem

6.1. Recall from subsection 2.2 that we have $\theta = \iota\theta_0$, with $\iota \in \mathrm{Inn}(\mathfrak{g})$ and $\theta_0 \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$. Let $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$ be the Cartan subalgebra of $\mathfrak{g}_0$. We denote by $\mathfrak{g}^{\theta_0}$ the Lie algebra $\{X \in \mathfrak{g} : \theta_0(X) = X\}$. Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{g}^{\theta_0}$ containing $\mathfrak{h}_0$. Since $Z_{\mathfrak{g}}(\tilde{\mathfrak{h}}) \subseteq Z_{\mathfrak{g}}(\mathfrak{h}_0) = \mathfrak{h}$, we get that $Z_{\mathfrak{g}}(\tilde{\mathfrak{h}}) = \mathfrak{h}$. Since $\mathfrak{g}^{\theta_0}$ is reductive, the algebra $\tilde{\mathfrak{h}}$ is abelian. Thus, we get that $\tilde{\mathfrak{h}} \subseteq \mathfrak{h} \cap \mathfrak{g}^{\theta_0} = \mathfrak{h}_0$ and hence, $\mathfrak{h}_0 = \tilde{\mathfrak{h}}$ i.e., $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}^{\theta_0}$. We then observe that

$$\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$$

if and only if

$$\mathrm{res}_{\mathfrak{g}^{\theta_0}} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}^{\theta_0}} L(\mu).$$

Since, H_θ is equal to H_{θ_0}, we may assume that $\iota = \mathrm{id}$. Hence, it is enough to prove the theorem for $\theta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \{X_\alpha\})$.

6.2. Let $\mathfrak{g} = \prod_\Gamma \mathfrak{g}^\Gamma$, where $\mathfrak{g}^\Gamma$ is maximal among all semisimple ideals with all its simple factors are of type Γ. Let $\mathfrak{h} = \prod_\Gamma \mathfrak{h}^\Gamma$ be the fixed θ-stable Cartan subalgebra of $\mathfrak{g}$. Then θ is equal to $\prod_\Gamma \theta^\Gamma$, where $\theta^\Gamma \in \mathrm{Aut}(\mathfrak{g}^\Gamma, \mathfrak{b}^\Gamma, \mathfrak{h}^\Gamma, \{X_\alpha : \alpha \in \Phi^\Gamma\})$. Here, Φ^Γ is the root system of $\mathfrak{g}^\Gamma$ with respect to $\mathfrak{h}^\Gamma$. We have $\Delta = \coprod_\Gamma \Delta^\Gamma$, where Δ^Γ is a basis for the root system Φ^Γ corresponding to the choice of $\mathfrak{b}^\Gamma$. Note that Δ^Γ is a θ-stable subset of Δ. Observe that

$$H_\theta = \prod_\Gamma H_{\theta^\Gamma}.$$

Let $\otimes_\Gamma L(\lambda^\Gamma)$ and $\otimes_\Gamma L(\mu^\Gamma)$ be two irreducible representations of $\mathfrak{g} = \prod_\Gamma \mathfrak{g}^\Gamma$ such that

$$\mathrm{res}_{\mathfrak{g}_0} \otimes_\Gamma L(\lambda^\Gamma) \simeq \mathrm{res}_{\mathfrak{g}_0} \otimes_\Gamma L(\mu^\Gamma). \tag{6.1}$$

Applying Lemma 4.4 to the above equality we get that

$$p(\prod_\Gamma U_{\lambda^\Gamma}) = p(\prod_\Gamma U_{\mu^\Gamma})$$

We apply the map r_0^I and we obtain

$$r_0^I p(\prod_{\Gamma} U_{\lambda^\Gamma}) = p^I(r^I(\prod_{\Gamma} U_{\lambda^\Gamma})) = r_0^I p(\prod_{\Gamma} U_{\mu^\Gamma}) = p^I(r^I(\prod_{\Gamma} U_{\mu^\Gamma})). \tag{6.2}$$

Since, $r^I(\prod_\Gamma U_{\lambda^\Gamma}) = U_{\lambda^\Gamma}$, it is enough to prove the theorem for the case where $\mathfrak{g} = \mathfrak{g}^\Gamma$.

6.3. Let $\mathfrak{g} = \mathfrak{t}_1 \times \cdots \times \mathfrak{t}_n$, where $\mathfrak{t}_i$ are simple Lie algebras of same type. Let $\mathfrak{h} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_n$ be a θ-stable Cartan subalgebra, where $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{t}_i$. Let $\mathfrak{b} = \mathfrak{b}_1 \times \cdots \times \mathfrak{b}_n$ be a θ-stable Borel subalgebra containing $\mathfrak{h}$. We denote by Δ_i the set of simple roots of $\mathfrak{t}_i$ with respect to the choice of $\mathfrak{b}_i$, and let $\Delta = \Delta_1 \coprod \cdots \coprod \Delta_n$. Let $\mathfrak{g} = \prod_{j=1}^r \mathfrak{g}_j$, where $\mathfrak{g}_j$ is an ideal which cannot be decomposed as a product of θ-stable ideals of $\mathfrak{g}$ (indecomposable θ-stable ideal). Let $\theta = \prod_{i=1}^r \theta_i$, where $\theta_i \in \mathrm{Aut}(\mathfrak{g}_i, \mathfrak{h}_i, \mathfrak{b}_i, \{X_\alpha\})$. Since H_θ fixes $\mathfrak{h}_0$ pointwise, we get that

$$H_\theta = H_{\theta_1} \times H_{\theta_2} \times \cdots \times H_{\theta_r}.$$

Using similar arguments as in Subsection 6.2, we may assume that $j = 1$. We then observe that

$$\mathfrak{g}_0 = \{(X, \sigma_2(X), \ldots, \sigma_n(X)) : X \in \mathfrak{t}_1^{\sigma_1},\ \sigma_1 \in \mathrm{Aut}(\mathfrak{t}_1), \sigma_i \in \mathrm{Isom}(\mathfrak{t}_1, \mathfrak{t}_i),\ 1 < i \leq n\}$$

and

$$\mathfrak{h}_0 = \{(X, \sigma_2(X), \ldots, \sigma_n(X)) : X \in \mathfrak{h}_1^{\sigma_1},\ \sigma_1 \in \mathrm{Aut}(\mathfrak{t}_1), \sigma_i \in \mathrm{Isom}(\mathfrak{t}_1, \mathfrak{t}_i),\ 1 < i \leq n\}$$

If $\sigma_1 = \mathrm{id}$, then $H_\theta \simeq S_n$, the group of permutations of the simple factors of $\mathfrak{g}$. Let σ_1 be a non-trivial automorphism. In the case where $\mathfrak{t}_i$ are of type A_k or E_6 and in the case where $\mathfrak{t}_i$ is of type D_4 and θ is a triality automorphism, the group H_θ is equal to the full group of outer automorphisms induced by the diagram automorphisms of the Dynkin-diagram associated to Δ. In the case where $\mathfrak{t}_i$ are of type D_4 and σ_1 is an order 2 automorphism we note that H_θ is a proper subgroup of the group of outer automorphisms of $\mathfrak{g}$ generated by the group of permutations of the simple factors of $\mathfrak{g}$ and the automorphisms $\{(\sigma_1, \mathrm{id}, \ldots, \mathrm{id}), (\mathrm{id}, \ldots, \mathrm{id}, \sigma_i\sigma_1\sigma_i^{-1}, \mathrm{id}, \ldots, \mathrm{id}) : 2 \leq i \leq n\}$.

6.4. We consider the case where σ_1 is non-trivial. We note that the case of $\sigma_1 = \mathrm{id}$ follows from the work of Rajan. Hence, we assume that $\mathfrak{t}$ is of type A_k, D_k and E_6. We have the following numbering on Dynkin diagram on the set of simple roots Δ_1 of $\mathfrak{t}_1$.

A_k: $\alpha_1 - \alpha_2 - \cdots - \alpha_k$

D_k $(k \geq 4)$: α_1 and α_3 both joined to α_2; $\alpha_2 - \alpha_4 \cdots \alpha_k$

E_6: $\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$, with α_2 joined to α_4

Fig.1: Labelling of Dynkin diagrams.

We fix the numbering on Δ_i, for $i > 1$, by using the map $\tilde{\sigma}_i : \Delta_1 \to \Delta_i$ induced by the isomorphism $\sigma_i : \mathfrak{t}_1 \to \mathfrak{t}_i$. We use the notation $\{\alpha_{ij}\}$ for the set of simple roots Δ_i, with $\alpha_{ij} = \tilde{\sigma}_i(\alpha_{1j})$.

6.4.1. In this subsection we take a slight digression and recall some character expansions when $\mathfrak{g}$ is a simple Lie algebra. Let $\mathfrak{g}$ be a simple Lie algebra of type X where $\mathrm{X} \in \{\mathrm{A}_{2k+1}, \mathrm{D}_k, \mathrm{E}_6\}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let P^+ be the set of dominant weights with respect to a choice of a base Δ of the root system Φ, and let $\lambda \in P^+$. Let θ be an automorphism induced by a diagram automorphism. Let α be the folding root of Δ and set $I = \Delta \backslash \{\alpha\}$. Let λ' be the weight $\lambda^I - \rho^I$ as defined in Notation 4.1.2 and let λ'' be the weight

$$\lambda^I + \langle \lambda + \rho, \alpha^\vee \rangle \gamma^I - \rho^I$$

as defined in Lemma 4.6. We set $X_\alpha = e^{-\alpha}$, for all $\alpha \in \Delta$. Recall the notation U_λ for the normalised Weyl numerator as defined in 4.1.1. Note that $U_\lambda \in \mathbb{C}[X_\beta : \beta \in \Delta\backslash\{\alpha\}][X_\alpha]$. Using Lemma 4.1 we get that $U_{\lambda'}$ is the constant term of U_λ as a polynomial in X_α. Using Lemma 4.2 we have $c_{w,\alpha} \geq \langle \lambda + \rho, \alpha^\vee \rangle$ and the equality holds if and only if $w = w' s_\alpha$, where $s_\alpha \notin I(w')$. Hence applying Lemma 4.6 for $I = \Delta\backslash\{\alpha\}$ and θ, we get that

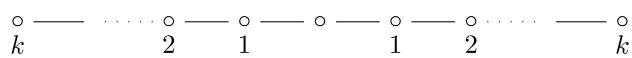
$$U_\lambda = U_{\lambda'} + U_{\lambda''} X_\alpha^{\langle \lambda+\rho, \alpha^\vee \rangle} + \text{ (higher degree terms in } X_\alpha)$$

Lemma 6.1. *Let λ and μ be two dominant weights of a simple Lie algebra $\mathfrak{g}$ of the type A_{2k+1}, D_k and E_6. Let $\theta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$ and let α be a folding root. Assume that $\langle \lambda + \rho, \alpha \rangle$ is equal to $\langle \mu + \rho, \alpha \rangle$. If $u_{\lambda''}/u_{\lambda'} = u_{\mu''}/u_{\mu'}$, then there exists a $\sigma \in H_\theta$ such that $\sigma(\lambda) = \mu$. In particular, $u_\lambda = u_\mu$.*

Proof. We begin with the case where $\mathfrak{g}$ is of Type A_{2k+1}. Let I be the set $\Delta\backslash\{\alpha_{k+1}\}$. The set I is stable under any diagram automorphism of Δ. Let $\{\varpi_i' : i \in [k]\}$ be the set of fundamental weights of $\mathfrak{g}_0^I$ with respect to $\mathfrak{h}_0^I$ and $\mathfrak{b}_0^I$. The numbering on the set $\{\varpi_i' : i \in [k]\}$ is chosen to satisfy the equality

$$p(\lambda^I) = \sum_{i=1}^{k} (\langle \lambda + \rho, \alpha_{k+1-i}^\vee \rangle + \langle \lambda + \rho, \alpha_{k+1+i}^\vee \rangle) \varpi_i'.$$

The following figure illustrates the numbering of ϖ_i'.



We set

$$\lambda_1 = \sum_{i=1}^{k} \langle \lambda + \rho, \alpha_{k+1-i}^\vee \rangle \varpi_i'$$

and

$$\lambda_2 = \sum_{i=1}^{k} \langle \lambda + \rho, \alpha_{k+1+i}^\vee \rangle) \varpi_i'.$$

We then have

$$u_{\lambda'} = U_{\lambda_1 - \rho'} U_{\lambda_2 - \rho'}$$

and

$$u_{\lambda''} = U_{\lambda_1 + \langle \lambda+\rho, \alpha_{k+1}^\vee \rangle \varpi_1' + \rho'} U_{\lambda_2 + \langle \lambda+\rho, \alpha_{k+1}^\vee \rangle \varpi_1' + \rho'},$$

where $\rho' = \sum_{i=1}^{k} \varpi_i'$.

Consider the case where $\mathfrak{g}$ is of the Type D_k and θ is an order two automorphism. Without loss of generality we may assume that $\theta(\alpha_1) = \alpha_3$ and $\theta(\alpha_4) = \alpha_4$. Let I be the set $\{\alpha_4, \alpha_5, \dots, \alpha_k\}$. Let $p : \mathfrak{h}^* \to (\mathfrak{h}_0)^*$ be the restriction map. We set $t = p(X_{\alpha_3}) = p(X_{\alpha_1})$. We then have

$$u_{\lambda'} = (1 - t^{\langle \lambda+\rho, \alpha_1^\vee \rangle})(1 - t^{\langle \lambda+\rho, \alpha_3^\vee \rangle}) U_{\lambda^I - \rho^I}$$

and

$$u_{\lambda''} = (1 - t^{\langle \lambda+\rho, \alpha_1^\vee \rangle + \langle \lambda+\rho, \alpha_2^\vee \rangle})(1 - t^{\langle \lambda+\rho, \alpha_3^\vee \rangle + \langle \lambda+\rho, \alpha_2^\vee \rangle}) U_{\lambda^I - \rho^I + \langle \lambda+\rho, \alpha_2^\vee \rangle \varpi_4^I}.$$

Here, ϖ_4^I is the weight $\varpi_{\alpha_4}^I$ in the Notations 4.1.2.

Consider the case where $\mathfrak{g}$ is a Lie algebra of Type D_4 and θ be a triality automorphism. Let $t = p(X_{\alpha_1}) = p(X_{\alpha_3}) = p(X_{\alpha_4})$. We set $n_i = \langle \lambda + \rho, \alpha_i^\vee \rangle$. We then have

$$u_{\lambda'} = (1 - t^{n_1})(1 - t^{n_3})(1 - t^{n_4})$$

and

$$u_{\lambda''} = (1 - t^{n_1+n_2})(1 - t^{n_3+n_2})(1 - t^{n_4+n_2}).$$

Now consider the case where $\mathfrak{g}$ is of Type E_6. Let $I=\{\alpha_1,\alpha_3,\alpha_5,\alpha_6\}$, $J=\{\alpha_2\}$ and $K=I\cup J$. Note that $\theta(\alpha_1)=\alpha_6$ and $\theta(\alpha_3)=\alpha_5$. We set $n_i=\langle\lambda+\rho,\alpha_i^\vee\rangle$, for $i\in[6]$. Let ϖ_1 and ϖ_2 be two fundamental weights of the root system of $\mathfrak{g}_0^I$ with respect to the choice of $\mathfrak{h}_0^I$ and $\mathfrak{b}_0^I$ such that

$$p(\lambda^I)=(n_1+n_6)\varpi_2+(n_3+n_5)\varpi_1.$$

Let $\lambda_1=n_1\varpi_1+n_3\varpi_2$ and $\lambda_2=n_6\varpi_1+n_5\varpi_2$. We set $\rho'=\varpi_1+\varpi_2$. Let $p(X_{\alpha_2})=t$. We then have

$$u_{\lambda'}=U_{\lambda_1-\rho'}U_{\lambda_2-\rho'}(1-t^{n_2})$$

and

$$u_{\lambda''}=U_{\lambda_1-\rho'+n_4\varpi_1}U_{\lambda_2-\rho'+n_4\varpi_1}(1-t^{n_2+n_4}).$$

We set $N=\langle\lambda+\rho,\alpha^\vee\rangle$, where α is the unique folding root in Δ. Assume that $u_{\lambda''}/u_{\lambda'}=u_{\mu''}/u_{\mu'}$ for two dominant weights λ and μ. In the case of A_{2k+1}, we have

$$U_{\lambda_1-\rho'}U_{\lambda_2-\rho'}U_{\mu_1-\rho'+N\varpi_1'}U_{\mu_2-\rho'+N\varpi_1'}=U_{\mu_1-\rho'}U_{\mu_2-\rho'}U_{\lambda_1-\rho'+N\varpi_1'}U_{\lambda_2-\rho'+N\varpi_1'}.$$

Now using Rajan's theorem we get that

$$\{\lambda_1,\lambda_2,\mu_1+N\varpi_1',\mu_2+N\varpi_1'\}=\{\mu_1,\mu_2,\lambda_1+N\varpi_1',\lambda_2+N\varpi_1'\}. \tag{6.3}$$

For $\kappa\in(P^I)^+$, let $n_1(\kappa)$ be the coefficient of ϖ_1' in κ. Let $n_1(\lambda_1)$ be the minimum value in the set $\{n_1(\lambda_1),n_1(\lambda_2),n_1(\mu_1),n_1(\mu_2)\}$. The equality (6.3) implies that either $\lambda_1=\mu_1$ or $\lambda_1=\mu_2$. Hence we get the equality of multisets $\{\lambda_1,\lambda_2\}=\{\mu_1,\mu_2\}$. This implies that $\lambda=\mu$ or $\lambda=\theta(\mu)$.

We now consider the case D_k with θ is of order 2. We set $n_i=\langle\lambda+\rho,\alpha_i^\vee\rangle$ and $m_i=\langle\mu+\rho,\alpha_i^\vee\rangle$. Then we have

$$(1-t^{n_1+n_2})(1-t^{n_3+n_2})(1-t^{m_1})(1-t^{m_3})=(1-t^{m_1+m_2})(1-t^{m_3+m_2})(1-t^{n_1})(1-t^{n_3})$$

and

$$U_{\mu^I-\rho^I+N\varpi_4^I}U_{\lambda^I-\rho^I}=U_{\lambda^I-\rho^I+N\varpi_4^I}U_{\mu^I-\rho^I}.$$

Note that $m_2=n_2=N$. Hence, we get that $\lambda^I=\mu^I$ and $\{n_1,n_3\}=\{m_1,m_3\}$. This implies that either $\lambda=\mu$ or $\lambda=\theta(\mu)$. The proofs in the cases D_4 with triality automorphism and E_6 with order two automorphism are similar to that of A_{2k+1} and D_k. □

6.4.2. We continue with the reduction that $\mathfrak{g}=\mathfrak{k}^n$ is indecomposable for the action of θ. Let S be the set of folding roots in Δ. Let $p:\mathfrak{h}^*\to(\mathfrak{h}_0)^*$ be the restriction map; recall that we also use the same notation p for the induced reduction map $p:\mathbb{Z}[P]\to\mathbb{Z}[P_0]$. We set $t_\alpha=p(X_\alpha)$ for all $\alpha\in\Delta$. Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ and $\mu=(\mu_1,\mu_2,\dots,\mu_n)$ be two dominant weights of $\mathfrak{g}=\mathfrak{k}^n$, with respect to $\mathfrak{h}$ and $\mathfrak{b}$, such that

$$\mathrm{res}_{\mathfrak{g}_0}L(\lambda_1)\otimes L(\lambda_2)\otimes\cdots\otimes L(\lambda_n)\simeq\mathrm{res}_{\mathfrak{g}_0}L(\mu_1)\otimes L(\mu_2)\otimes\cdots\otimes L(\mu_n).$$

Using Lemma 4.4 we get that

$$u_{\lambda_1}u_{\lambda_2}\cdots u_{\lambda_n}=u_{\mu_1}u_{\mu_2}\cdots u_{\mu_n}, \tag{6.4}$$

where $u_\lambda=p(U_\lambda)$. We then get that

$$\prod_{i=1}^n(u_{\lambda_i'}+u_{\lambda_i''}t_\alpha^{\langle\lambda_i+\rho,\alpha^\vee\rangle}+\text{ (higher degree terms in } t_\alpha))=$$
$$\prod_{i=1}^n(u_{\mu_i'}+u_{\mu_i''}t_\alpha^{\langle\mu_i+\rho,\alpha^\vee\rangle}+\text{ (higher degree terms in } t_\alpha)).$$

Comparing the constant term we get that

$$u_{\lambda_1'}\cdots u_{\lambda_n'}=u_{\mu_1'}\cdots u_{\mu_n'}. \tag{6.5}$$

Setting X_α to zero for all $\alpha\in\Delta\backslash S$, we get the following equality of multisets:

$$\{\langle\lambda+\rho,\alpha^\vee\rangle:\alpha\in S\}=\{\langle\mu+\rho,\alpha^\vee\rangle:\alpha\in S\} \tag{6.6}$$

Assume that all the entries in the multisets in the equality (6.6) are distinct (Hypothesis 5.2). We set

$$N = \min\{\langle \lambda + \rho, \alpha^\vee \rangle : \alpha \in S\}.$$

Comparing the coefficient of t_α^N on the both sides of the equality (6.4), we get that

$$u_{\lambda_i''}/u_{\lambda_i'} = u_{\mu_j''}/u_{\mu_j'}$$

for some i and j. Using Lemma 6.1 we get that $u_{\lambda_i} = u_{\mu_j}$. Using induction on the integer n, we get the equality of multisets

$$\{u_{\lambda_i} : 1 \le i \le n\} = \{u_{\mu_i} : 1 \le i \le n\}.$$

Again applying Lemma 6.1, there exists a $\sigma \in H_\theta$ such that $\sigma(\lambda) = \mu$. This completes the proof of the theorem in the case where the elements of the multisets in the equality (6.6) are distinct.

We consider the Hypothesis 5.1. Here, we have $n \le 2$. Lemma 6.1 gives the proof in the case where $n = 1$. Now assume that $n = 2$. If the elements of the multiset

$$\{\langle \lambda + \rho, \alpha^\vee \rangle : \alpha \in S\}$$

are distinct then we are reduced to the above case. Otherwise, comparing the coefficient of $t_\alpha^{\langle \lambda + \rho, \alpha^\vee \rangle}$, for $\alpha \in S$, in the equality (6.4), we get that

$$u_{\lambda_1''} u_{\lambda_2'} + u_{\lambda_2''} u_{\lambda_1'} = u_{\mu_1''} u_{\mu_2'} + u_{\mu_2''} u_{\mu_1'}.$$

Now using the equality (6.5), we get

$$\frac{u_{\lambda_1''}}{u_{\lambda_1'}} + \frac{u_{\lambda_2''}}{u_{\lambda_2'}} = \frac{u_{\mu_1''}}{u_{\mu_1'}} + \frac{u_{\mu_2''}}{u_{\mu_2'}}.$$

Since λ'' is determined by λ', the equality (6.5) gives us that

$$\frac{u_{\lambda_1''}}{u_{\lambda_1'}} \frac{u_{\lambda_2''}}{u_{\lambda_2'}} = \frac{u_{\mu_1''}}{u_{\mu_1'}} \frac{u_{\mu_2''}}{u_{\mu_2'}}.$$

From the above equality we get that the equality of multisets

$$\{\frac{u_{\lambda_1''}}{u_{\lambda_1'}}, \frac{u_{\lambda_2''}}{u_{\lambda_2'}}\} = \{\frac{u_{\mu_1''}}{u_{\mu_1'}}, \frac{u_{\mu_2''}}{u_{\mu_2'}}\}.$$

Hence, we complete the proof of the main theorem when $\mathfrak{t}$ is of type A_{2k+1} or D_k or E_6.

Remark 6.2. For $n \ge 3$, we can find examples of dominant weights $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of $\mathfrak{g} = \mathfrak{t}^n$ such that

$$\{\langle \lambda + \rho, \alpha^\vee \rangle : \alpha \in S\} = \{\langle \mu + \rho, \alpha^\vee \rangle : \alpha \in S\}$$

and

$$\sum_{i=1}^n \frac{u_{\lambda_i''}}{u_{\lambda_i'}} = \sum_{i=1}^n \frac{u_{\mu_i''}}{u_{\mu_i'}}$$

but $\lambda \neq \sigma(\mu)$ for any $\sigma \in H_\theta$. However, under certain additional hypothesis on λ and μ we can still prove the main result.

6.4.3. Let us consider the case where $\mathfrak{g} = \mathfrak{t}^n$, where $\mathfrak{t}$ is a simple Lie algebra of Type A_{2k}. Under the Hypotheses 5.1 and 5.2 we assume that $n \le 2$ in the A_2 case and $n = 1$ in the A_{2k} case, for $k > 1$. Let $p : \mathfrak{h}^* \to (\mathfrak{h}_0)^*$ be the restriction map.

Let us first consider the case where $\mathfrak{t}$ is of type A_2. We set $p(X_{\alpha_{i1}}) = p(X_{\alpha_{i2}}) = t$, for $i \in \{1, 2\}$. Let u_λ be the polynomial $p(U_\lambda)$. First consider the case where $n = 1$. We drop the subscript i. We have

$$\begin{aligned} u_\lambda = & \sum_{w \in W} (-1)^{l(w)} t^{c_{w,\alpha_1}(\lambda) + c_{w,\alpha_2}(\lambda)} \\ = & (1 - t^{n_1})(1 - t^{n_2})(1 - t^{n_1 + n_2}). \end{aligned}$$

If $u_\lambda = u_\mu$, then we immediately get that

$$\{\langle \lambda + \rho, \alpha_1^\vee \rangle, \langle \lambda + \rho, \alpha_2^\vee \rangle\} = \{\langle \mu + \rho, \alpha_1^\vee \rangle, \langle \mu + \rho, \alpha_2^\vee \rangle\}.$$

Let us consider the case $n=2$. Let (λ_1,λ_2) and (μ_1,μ_2) be two dominant weights of $\mathfrak{g}$, such that
$$u_{\lambda_1}u_{\lambda_2}=u_{\mu_1}u_{\mu_2},$$
where $u_{\lambda_i}=p(U_{\lambda_i})$ and $u_{\mu_i}=p(U_{\mu_i})$. We set $n_{ij}=\langle\lambda_i+\rho,\alpha_{ij}^{\vee}\rangle$ and $m_{ij}=\langle\mu_i+\rho,\alpha_{ij}^{\vee}\rangle$, for $i,j\in\{1,2\}$. Then, we get that
$$\{n_{11},n_{12},n_{11}+n_{22},n_{21},n_{22},n_{21}+n_{22}\}=\{m_{11},m_{12},m_{11}+m_{22},m_{21},m_{22},m_{21}+m_{22}\}.$$
From the above equality we deduce that
$$\{\{n_{11},n_{12}\},\{n_{21},n_{22}\}\}=\{\{m_{11},m_{12}\},\{m_{21},m_{22}\}\}.$$
This proves the theorem in this case.

We now consider the A_{2k+2}-case. Here, we have $\mathfrak{k}=\mathfrak{g}$. Let $I_1=\{\alpha_i:1\le i\le k\}$ and let $I_2=\{\alpha_i:k+3\le i\le 2k+2\}$. Let I be the set $I_1\cup I_2$. We set $J=\{\alpha_{k+1},\alpha_{k+2}\}$. Let $\{\varpi_i':i\in[k]\}$ be the fundamental weights of $\mathfrak{g}_0^I$ with respect to $\mathfrak{h}_0^I$ and $\mathfrak{b}_0^I$. The numbering of $\{\varpi_i':i\in[k]\}$ is chosen such that
$$p(\lambda^I)=\sum_{i=1}^{k}(\langle\lambda+\rho,\alpha_{k+1-i}\rangle+\langle\lambda+\rho,\alpha_{k+2+i}\rangle)\varpi_i'.$$
We set $\rho'=\sum_{i=1}^{k}\varpi_i'$. We define
$$\lambda_1=\sum_{i=1}^{k}\langle\lambda+\rho,\alpha_{k+1-i}\rangle\varpi_i'$$
and
$$\lambda_2=\sum_{i=1}^{k}\langle\lambda+\rho,\alpha_{k+2+i}\rangle\varpi_i'.$$
Let $u_\lambda=p(U_\lambda)$ and let $p(X_{\alpha_{k+1}})=p(X_{\alpha_{k+2}})=t$. Using Lemmas 4.5 and 4.6 we get that
$$u_\lambda=P_0+t^{\min\{n_{k+1},n_{k+2}\}}P_1+\text{higher order terms in }\ t,$$
where $P_0=U_{\lambda_1-\rho'}U_{\lambda_2-\rho'}$ and
$$P_1=\begin{cases}U_{\lambda_1+n_{k+1}\varpi_1'-\rho'}U_{\lambda_2-\rho'} \text{ if } n_{k+1}<n_{k+2},\\ U_{\lambda_2+n_{k+2}\varpi_1'-\rho'}U_{\lambda_1-\rho'} \text{ if } n_{k+2}<n_{k+1},\\ U_{\lambda_1+n_{k+1}\varpi_1'-\rho'}U_{\lambda_2-\rho'}+U_{\lambda_2+n_{k+2}\varpi_1'-\rho'}U_{\lambda_1-\rho'} \text{ if } n_{k+2}=n_{k+1}.\end{cases}\tag{6.7}$$
Assume that $u_\lambda=u_\mu$ for two dominant weights $\lambda,\mu\in P^+$. Setting X_α to 0, for all$\alpha\in I$, we get that
$$\{n_{k+1},n_{k+2}\}=\{m_{k+1},m_{k+2}\}.$$
Setting $t=0$, we then get that
$$U_{\lambda_1-\rho'}U_{\lambda_2-\rho'}=U_{\mu_1-\rho'}U_{\mu_2-\rho'}.\tag{6.8}$$
From Rajan's theorem we get that
$$\{\lambda_1,\lambda_2\}=\{\mu_1,\mu_2\}$$
If $n_{k+1}=n_{k+2}$, then $m_{k+1}=m_{k+2}$ and in this case we get that $\lambda=\theta(\mu)$ or $\lambda=\mu$. Now, we assume that $n_{k+1}<n_{k+2}$ and without loss of generality we may assume that $m_{k+1}<m_{k+2}$. Thus, we get that $n_{k+1}=m_{k+1}$ and $n_{k+1}=m_{k+2}$. We then get that
$$U_{\lambda_1+n_{k+1}\varpi_1'-\rho'}U_{\lambda_2-\rho'}=U_{\mu_1+m_{k+1}\varpi_1'-\rho'}U_{\mu_2-\rho'}\tag{6.9}$$
Using Equations (6.8) and (6.9) and Rajan's theorem, we get that
$$\{\lambda_1-\rho',\mu_1+m_{k+1}\varpi_1'-\rho'\}=\{\mu_1-\rho',\lambda_1+n_{k+1}\varpi_1'-\rho'\}.$$
This implies that $\lambda_1=\mu_1$, and hence, we get that $\lambda_2=\mu_2$. Hence $\lambda=\mu$ or $\lambda=\theta(\mu)$. This completes the proof of the theorem in A_{2k+2} case. With this we conclude the proof of the main theorem.

6.5. In this subsection, we prove a proposition which is a variant of the main result, with a slightly different hypothesis on the weights different from hypothesis 5.1 and 5.2. We continue to use the same notations as above. In Hypothesis 5.2 we assume that the integers $\langle \lambda+\rho, \alpha^\vee\rangle$ are all distinct where α varies over the set of folding roots. The following proposition shows that the main theorem holds true in many more cases than the generic Hypothesis 5.2.

Proposition 6.3. *Let $\mathfrak{g} = \mathfrak{k}^n$ where $\mathfrak{k}$ a simple Lie algebra of type A_{2k+1} or D_k or E_6. Let θ be a diagram automorphism of $\mathfrak{g}$ such that $\mathfrak{g}$ does not have any non-zero proper θ-stable ideals. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be two dominant weights such that*

$$\langle \lambda+\rho, \beta^\vee\rangle > n \max\{\langle \lambda+\rho, \alpha^\vee\rangle : \alpha \in S\},\ \beta \in \Delta \backslash S. \tag{6.10}$$

If $\mathrm{res}_{\mathfrak{g}_0} L(\lambda) \simeq \mathrm{res}_{\mathfrak{g}_0} L(\mu)$, then there exists a $\sigma \in H_\theta$ such that $\sigma(\lambda) = \mu$.

Proof. Using equality (6.6) we get the equality of multisets:

$$\{\langle \lambda+\rho, \beta^\vee\rangle : \beta \in \Delta\backslash S\} = \{\langle \mu+\rho, \beta^\vee\rangle : \beta \in \Delta\backslash S\}$$

and

$$\{\langle \lambda+\rho, \beta^\vee\rangle : \beta \in S\} = \{\langle \mu+\rho, \beta^\vee\rangle : \beta \in S\}.$$

Using Lemma 4.3 and the above inequality (6.10) we get that $c_{w,\alpha}(\lambda) > n\langle \lambda+\rho, \alpha\rangle$ for all $\alpha \in S$. Let $I \subseteq [n]$ be such that $\langle \lambda_i+\rho, \alpha^\vee\rangle$ is equal to $\min\{\langle \lambda+\rho, \alpha^\vee\rangle : \alpha \in S\}$, for all $i \in I$. Similarly, we define $J \subseteq [n]$ such that $\langle \mu_j+\rho, \alpha^\vee\rangle$ is equal to $\min\{\langle \mu+\rho, \alpha^\vee\rangle : \alpha \in S\}$, for all $j \in J$. Observe that $|I| = |J|$. Now, comparing the coefficient of $t^{r\langle \lambda_i+\rho,\alpha\rangle}$, for $i \in I$, in the equality (6.4) we get that

$$E_r(\{u_{\lambda_i''}/u_{\lambda_i'}, \ldots, u_{\lambda_i''}/u_{\lambda_i'} : i \in I\}) = E_r(\{u_{\mu_i''}/u_{\mu_i'}, \ldots, u_{\mu_i''}/u_{\mu_i'} : i \in J\}),\ r \in [|I|] = [|J|],$$

where E_r is the r^{th}-elementary symmetric polynomial in $|I| = |J|$-variables. This shows that the following equality of multisets

$$\{u_{\lambda_i''}/u_{\lambda_i'} : i \in I\} = \{u_{\mu_j''}/u_{\mu_j'} : j \in J\}.$$

Using Lemma 6.1, we get that

$$\{u_{\lambda_i} : i \in [n]\} = \{u_{\mu_i} : i \in [n]\}.$$

□

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