Indian Institute of Technology Kanpur
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A First Course in Linear Algebra (MTH 201A)
Exercise Set #1

Notation: In what follows, we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

(1) (a) Show that the following are equivalent for any $A \in M_n(\mathbb{F})$:
   (i) $AB = O_n$ for some nonzero $B \in M_n(\mathbb{F})$; and
   (ii) $BA = O_n$ for some nonzero $B \in M_n(\mathbb{F})$.
 (b) We say that a matrix $A \in M_n(\mathbb{F})$ is a divisor *zero divisor* in $M_n(\mathbb{F})$ if it satisfies one (equivalently both) of the conditions stated above in (1(a)i)-(1(a)ii). Find all zero divisors in $M_n(\mathbb{F})$.

(2) Let $m, n \in \mathbb{N}, A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{n\times m}(\mathbb{F})$. Show the following:
 (a) $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.
 (b) AB cannot be invertible if $m > n$.
 (c) If $n = m$ the find a necessary and sufficient condition for AB to be invertible.

(3) Let $m, n \in \mathbb{N}$. Show that the following are equivalent for a system of linear equations $A\mathbf{x} = B$, where $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{m\times 1}(\mathbb{F})$:
 (a) it does not have a unique solution; and
 (b) it has infinitely many solutions.

(4) Let A be an $m \times n$ matrix with integer entries. Show that, if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial complex solution then it has a nontrivial integral solution.

(5) Let $m, n \in \mathbb{N}$. For $A, B \in M_{m\times n}(\mathbb{F})$, we say that $A \sim B$ if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q such that $PAQ = B$. Show that this defines an equivalence relation on $M_{m\times n}(\mathbb{F})$ and find the number of distinct equivalence classes under this equivalence relation.

(6) Let $m, n \in \mathbb{N}$ and $A, B \in M_{m\times n}(\mathbb{F})$. Show that, if the homogeneous systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have same solutions then A and B are row equivalent.

(7) Show that an upper triangular matrix with entries in $\mathbb{F}$ is invertible if and only if its all diagonal entries are nonzero.